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A classification of dynamical systems

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Abstract. The present paper deals with some properties of dynamical systems (X, G, π) satisfying conditions called in the paper assumption (A). We discuss how many—in the sense of Baire category—Lagrange stable (Lagrange unstable, Poisson unstable, completely unstable, dispersive or parallelizable) systems can be defined on a given topological space. We also study dynamical systems in $\mathbf{R}^n$ with regard to stability of sets.

Among the important properties of dynamical systems are parallelizability, dispersivity, complete unstability, Poisson unstability and Lagrange unstability. Our paper tries to answer the following question: how many (in the sense of Baire category) systems with one of these properties can be defined on a given topological space.

A classification of dynamical systems with regard to stability of sets is also given.

In the second and third parts of this paper the above problems are discussed in the spaces $\mathbf{R}^n$.

Most of the definitions adopted here are taken from [1] and [4]. We consider dynamical systems (X, G, π) satisfying the following assumption:

(A) X is a non-empty first countable Hausdorff topological space, $(G, +, \prec)$ is a topological, ordered, abelian group with neutral element 0 and with topology induced by the order relation which does not admit first and last elements.

I. Dynamical systems in compact spaces. Let X be a topological space and let G satisfy assumption (A). Let us recall the well-known definitions (see for instance [1], [4]).

DEFINITION 1.1. A triple (X, G, π) is said to be a *dynamical system* iff π is a mapping from $G \times X$ into X such that for all $t, s \in G$, $x \in X$

$$\pi(0, x) = x, \tag{1.1}$$

$$\pi(t, \pi(s, x)) = \pi(t+s, x), \tag{1.2}$$

$$\pi \text{ is continuous.} \tag{1.3}$$

Definition 1.2. The sets

$$\pi(x) := \pi^+(x) \cup \pi^-(x),$$
$$\pi^+(x) := \{\pi(t, x);\ t \in G,\ 0 \prec t\},$$
$$\pi^-(x) := \{\pi(t, x):\ t \in G,\ t \prec 0\}$$

are called the *trajectory*, the *positive semi-trajectory* and the *negative semi-trajectory* of x respectively.

Definition 1.3. The sets

$$\Lambda_\pi(x) := \Lambda_\pi^+(x) \cup \Lambda_\pi^-(x),$$
$$\Lambda_\pi^+(x) := \bigcap \{\overline{\pi^+(\pi(t, x))}:\ t \in G\},$$
$$\Lambda_\pi^-(x) := \bigcap \{\overline{\pi^-(\pi(t, x))}:\ t \in G\}$$

are called the *limit set*, the *positive limit set* and the *negative limit set* of x respectively.

In a similar way some other notions are introduced. For example:

Definition 1.4. The sets

$$D_\pi^+(x) := \bigcap \{\overline{\pi^+(V)}:\ V \text{ in a basis of neighbourhoods of } x\},$$
$$J_\pi^+(x) := \bigcap \{D_\pi^+(\pi(t, x)):\ t \in G\}$$

are called the *first positive prolongation* and the *positive prolongational limit set* of x respectively.

We define the first negative prolongation and the negative prolongational limit set of x similarly.

Remark 1.1. $D_\pi^+(x) = \pi^+(x) \cup J_\pi^+(x)$ for every $x \in X$.

For dynamical systems satisfying assumption (A) we have

Theorem 1.1 (see [4]). *Let* $x \in X$. *Then*

(1.4) $\Lambda_\pi^+(x) = \{y \in X:\ \exists t_n \in G,\ t_n \to \infty \text{ such that } \pi(t_n, x) \to y\},$

(1.5) $D_\pi^+(x) = \{y \in X:\ \exists t_n \in G,\ 0 \prec t_n,\ \exists x_n \in X,\ x_n \to x \text{ such that } \pi(t_n, x_n) \to y\},$

(1.6) $J_\pi^+(x) = \{y \in X:\ \exists t_n \in G,\ t_n \to \infty,\ \exists x_n \in X,\ x_n \to x \text{ such that } \pi(t_n, x_n) \to y\},$

where $t_n \to \infty$ *if for every* $s \in G$ *there is* $n_0 \in N$ *such that* $s \prec t_n$ *for every* $n \geqslant n_0$.

Definition 1.5. A set $M \subset X$ is called *invariant* if

$$\pi(t, x) \in M \quad \text{for all } x \in M \text{ and } t \in G.$$

DEFINITION 1.6. A non-empty set $M \subset X$ is called *minimal* if it is closed, invariant, and no proper subset of M has these properties, i.e. $\bar{M} = M$, $M = \pi(M) := \bigcup\{\pi(x)\colon x \in M\}$ and

$$(\emptyset \neq C \subset M,\ \bar{C} = C,\ \pi(C) = C) \Rightarrow C = M.$$

THEOREM 1.2 [4] (see also [1] for another form).

(i) *A non-empty set $M \subset X$ is minimal if and only if $\overline{\pi(x)} = M$ for every $x \in M$.*

(ii) *Every non-empty compact invariant set contains a minimal set.*

(iii) *If $M \neq \emptyset$ is compact, then*

$$M \text{ is minimal} \Leftrightarrow (\Lambda_\pi(x) = M \text{ for every } x \in M).$$

DEFINITION 1.7. A dynamical system (X, G, π) is called

(i) *parallelizable* if there exists a non-empty set $S \subset X$ and a homeomorphism $h\colon X \to G \times S$ such that

(i.1) $\bigcup\{\pi(t, S)\colon t \in G\} = X$,

(i.2) $h(\pi(t, x)) = (t, x)$ for all $t \in G$, $x \in S$,

where $\pi(t, S) := \{\pi(t, x)\colon x \in S\}$,

(ii) *dispersive* if for all $x, y \in X$ there exist neighbourhoods U_x of x and U_y of y such that there exists $T \in G_*$ with $U_x \cap \pi(t, U_y) = \emptyset$ for each $t \in G \setminus [-T, T]$, where $G_* := \{t \in G\colon 0 \prec t\}$,

(iii) *completely unstable* if every $x \in X$ is wandering, i.e. $x \notin J_\pi^+(x)$ for every $x \in X$,

(iv) *Poisson unstable* if each $x \in X$ is Poisson unstable, i.e. $x \notin \Lambda_\pi(x)$ for every $x \in X$,

(v) *Lagrange unstable* if $\overline{\pi(x)}$ is not compact for every $x \in X$.

Unless otherwise stated, we assume throughout the paper that the pair (X, G) satisfies assumption (A).

THEOREM 1.3 (see [1], [4]).

(i) *A dynamical system (X, G, π) is dispersive if and only if $J_\pi^+(x) = \emptyset$ for each $x \in X$.*

(ii) *Every parallelizable dynamical system $(X, \boldsymbol{R}, \pi)$ is dispersive.*

(iii) *Let X be a metric separable locally compact space. Then every dispersive dynamical system $(X, \boldsymbol{R}, \pi)$ is parallelizable.*

COROLLARY 1.1. *If X is a metric separable locally compact space (e.g. $X = \boldsymbol{R}^n$), then a dynamical system $(X, \boldsymbol{R}, \pi)$ is parallelizable if and only if it is dispersive.*

We will study the families of all maps π for which the corresponding dynamical systems are respectively parallelizable, dispersive, completely unstable, Poisson unstable or Lagrange unstable.

Let $\mathscr{C}(Y, X)$ denote the family of all continuous maps from Y to X. For any fixed X, G satisfying assumption (A) we set

(1.7) $\mathscr{C} := \{\pi \in \mathscr{C}(G \times X, X)$: (X, G, π) is a dynamical system satisfying (A)$\}$,

(1.8) $\mathscr{D} := \{\pi \in \mathscr{C}$: (X, G, π) is dispersive$\}$,

(1.9) $\mathscr{K} := \{\pi \in \mathscr{C}$: (X, G, π) is completely unstable$\}$,

(1.10) $\tilde{\mathscr{P}} := \{\pi \in \mathscr{C}$: (X, G, π) is Poisson unstable$\}$,

(1.11) $\tilde{\mathscr{L}} := \{\pi \in \mathscr{C}$: (X, G, π) is Lagrange unstable$\}$.

THEOREM 1.4. $\mathscr{D} \subset \mathscr{K} \subset \tilde{\mathscr{P}} \subset \tilde{\mathscr{L}}$.

The proof follows easily from the definitions. We only prove the last inclusion. Suppose that $\tilde{\mathscr{P}} \not\subset \tilde{\mathscr{L}}$, i.e. there is $\pi \in \tilde{\mathscr{P}} \setminus \tilde{\mathscr{L}}$. So there is $x \in X$ such that $\overline{\pi(x)}$ is compact. In view of Theorem 1.2 there is a compact minimal set M contained in $\overline{\pi(x)}$ such that $y \in \Lambda_\pi(y) = M$ for each $y \in M$. Thus $\pi \notin \tilde{\mathscr{P}}$, a contradiction which completes the proof.

THEOREM 1.5. *Let X be a compact space. Then* $\mathscr{D} = \mathscr{K} = \tilde{\mathscr{P}} = \tilde{\mathscr{L}} = \emptyset$.

In order to prove the last equality, take $\pi \in \mathscr{C}$. Since X is a non-empty compact invariant set, by Theorem 1.2 there is a compact minimal set $M \subset X$, so $\overline{\pi(x)} = M$ for each $x \in M$. Thus the dynamical system (X, G, π) is Lagrange unstable for no $\pi \in \mathscr{C}$, i.e. $\tilde{\mathscr{L}} = \emptyset$, and Theorem 1.4 gives the assertion.

COROLLARY 1.2. *Let X be a compact space. No dynamical system $(X, \boldsymbol{R}, \pi)$ with $\pi \in \mathscr{C}$ is parallelizable.*

Remark 1.2. The local compactness of X is not sufficient for the above assertion to hold.

EXAMPLE 1.1. Let $(\boldsymbol{R}^2, \boldsymbol{R}, \pi)$ be the dynamical system given by the differential equations

$$dx_1/dt = 1, \qquad dx_2/dt = 0.$$

For every $x \in \boldsymbol{R}^2$, $J_\pi(x) = \emptyset$, so $\pi \in \mathscr{D} \neq \emptyset$.

II. A classification of dynamical systems with regard to stability of sets. Let (X, d) be a metric space.

DEFINITION 2.1. A non-empty set $M \subset X$ is called

(i) BH-*stable* (Bhatia–Hajek-stable) in (X, G, π) if for all $x \notin M$ and $y \in M$ there are δ, $\eta \in \boldsymbol{R}_+$ such that $B(x, \delta) \cap \pi^+(B(y, \eta)) = \emptyset$,

(ii) *uniformly stable* in (X, G, π) if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\pi^+(B(M, \delta)) \subset B(M, \varepsilon)$.

Here $B(x, \delta) := \{y \in X\colon d(x, y) < \delta\}$ and $B(M, \delta) := \bigcup\{B(x, \delta)\colon x \in M\}$.

THEOREM 2.1 (see [4]). *If X is a locally compact metric space and $\emptyset \neq M \subset X$, then*

(i) *M is* BH-*stable in $(X, \boldsymbol{R}, \pi) \Leftrightarrow D_\pi^+(M) = M$.*

(ii) *If M is compact then*

$$(M \text{ is uniformly stable in } (X, \boldsymbol{R}, \pi)) \Leftrightarrow D_\pi^+(M) = M.$$

For every non-empty set X we set

$$2^X := \{A \subset X\}, \quad \mathscr{C}l(X) := \{A \in 2^X\colon \bar{A} = A\}. \tag{2.1}$$

We define the function $\tilde{d}\colon 2^X \times 2^X \to \bar{\boldsymbol{R}}$ by

$$\begin{aligned} &\tilde{d}(\emptyset, A) := \begin{cases} 0 & \text{for } A = \emptyset, \\ \infty & \text{for } A \in 2^X \setminus \{\emptyset\}, \end{cases} \\ &\tilde{d}(A, B) := \max\big(\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\big) \quad \text{for } A, B \in 2^X \setminus \{\emptyset\} \end{aligned} \tag{2.2}$$

where $d(x, B) := \inf_{y \in B} d(x, y)$, i.e. $\tilde{d}$ is the Hausdorff metric in $\mathscr{C}l(X) \setminus \{\emptyset\}$ (see [2]).

Remark 2.1. $(\mathscr{C}l(X), d_1)$ is a metric space, where

$$d_1(A, B) := \min(1, \tilde{d}(A, B)) \quad \text{for } A, B \in \mathscr{C}l(X).$$

In the sequel we shall consider dynamical systems in $X = \boldsymbol{R}^n$.

Let $\mathscr{C}$ be the set introduced in (1.7) and let $\varrho\colon \mathscr{C} \times \mathscr{C} \to \bar{\boldsymbol{R}}$ be defined by

$$\varrho(\varphi, \pi) := \sup\{d(\varphi(t, x), \pi(t, x))\colon (t, x) \in G \times X\}. \tag{2.3}$$

Remark 2.2. $(\mathscr{C}, \varrho_1)$ is a metric space where

$$\varrho_1(\varphi, \pi) := \min(1, \varrho(\varphi, \pi)) \quad \text{for } \varphi, \pi \in \mathscr{C}. \tag{2.4}$$

For $X = \mathscr{R}^n$ the following theorems are true.

THEOREM 2.2. *For all $\varphi, \pi \in \mathscr{C}$ and $\eta \in \boldsymbol{R}$ we have the implication*

$$\varrho(\varphi, \pi) \leqslant \eta \Rightarrow \tilde{d}(W_\varphi(x), W_\pi(x)) \leqslant \eta \quad \textit{for each } x \in X, \tag{2.5}$$

where $W = \Lambda^+, D^+, J^+$.

We prove this theorem for $W = J^+$, for example. We shall consider two cases.

First, suppose that there is $y \in J_\pi^+(x) \neq \emptyset$. In view of (1.6) there are sequences $x_k \to x$, $t_k \to \infty$ such that $\pi(t_k, x_k) \to y$. So there is $r > 0$ such that $\pi(t_k, x_k) \in B(y, r)$ for all k. For every $\varphi \in \mathscr{C}$ such that $\varrho(\varphi, \pi) \leqslant \eta$ we have

$$d(\varphi(t_k, x_k), y) \leqslant d(\varphi(t_k, x_k), \pi(t_k, x_k)) + d(\pi(t_k, x_k), y) \leqslant \eta + r,$$

i.e. the sequence $\varphi(t_k, x_k) \in \boldsymbol{R}^n$ is bounded. Thus there is a subsequence $\varphi(t_{k_s}, x_{k_s})$ converging to $z \in J_\varphi^+(x)$ and $J_\varphi^+(x) \neq \emptyset$.

We estimate the distance between y and z:

$$d(y, z) \leqslant d(y, \pi(t_{k_s}, x_{k_s})) + d(\pi(t_{k_s}, x_{k_s}), \varphi(t_{k_s}, x_{k_s})) + d(\varphi(t_{k_s}, x_{k_s}), z).$$

Hence

$$\forall \varepsilon > 0: \quad d(y, z) \leqslant \varepsilon + \eta, \quad \text{i.e.} \quad d(y, z) \leqslant \eta.$$

So, for every $y \in J_\pi^+(x)$ there is $z \in J_\varphi^+(x)$ such that

$$d(y, J_\varphi^+(x)) \leqslant d(y, z) \leqslant \eta \quad \text{for every } y \in J_\pi^+(x).$$

Analogously changing the roles of π and φ we obtain

$$d(z, J_\pi^+(x)) \leqslant d(z, y) \leqslant \eta \quad \text{for every } z \in J_\varphi^+(x).$$

The above inequalities give

$$\tilde{d}(J_\pi^+(x), J_\varphi^+(x)) \leqslant \eta.$$

Now, let $J_\pi^+(x) = \emptyset$. Suppose that there is $\varphi \in \mathscr{C}$ such that $\varrho(\varphi, \pi) \leqslant \eta$ and $J_\varphi^+(x) \neq \emptyset$. From the first part of the proof we have $J_\pi^+(x) \neq \emptyset$, which gives a contradiction and finishes the proof.

The proof for the case $W = \Lambda^+$ is analogous (see (1.4)). Hence in view of Remark 1.1 we obtain our assertion for the case $W = D^+$. The proof of the theorem is finished.

It is known (see [1], [4]) that $W_\pi(x) \in \mathscr{C}l(X)$ for each $\pi \in \mathscr{C}$ and $x \in X$, where $W = \Lambda^+, D^+, J^+$. Therefore from (2.5) we get

THEOREM 2.3. *For each $x \in X$ the map*

$$W(x): \mathscr{C} \ni \pi \to W_\pi(x) \in \mathscr{C}l(X) \tag{2.6}$$

is uniformly continuous from $(\mathscr{C}, \varrho_1)$ to $(\mathscr{C}l(X), d_1)$.

Analogous theorems hold for $W = \Lambda^-, D^-, J^-$ and of course for $W = \Lambda$, D, J.

Let $M \subset X$. Set

$$W_\pi(M) := \bigcup_{x \in M} W_\pi(x). \tag{2.7}$$

It is easy to prove the implication

$$\tilde{d}(W_\pi(x), W_\varphi(x)) \leqslant \eta \text{ for every } x \in M \;\Rightarrow\; \tilde{d}(W_\pi(M), W_\varphi(M)) \leqslant \eta,$$

which, for $W = \Lambda^+, D^+, J^+$, instantly gives

COROLLARY 2.1. *For all $M \subset X$, $\eta \in \mathbf{R}$, $\pi, \varphi \in \mathscr{C}$ we have*

$$\varrho(\pi, \varphi) \leqslant \eta \;\Rightarrow\; \tilde{d}(W_\pi(M), W_\varphi(M)) \leqslant \eta. \tag{2.8}$$

Applying the above results we have

THEOREM 2.4. *For every non-empty set $M \in \mathscr{C}l(X)$ the set*

$$S_M := \{\varphi \in \mathscr{C}\colon\ D_\varphi^+(M) = M\}$$

is closed in $\mathscr{C}$ with the topology of uniform convergence.

Proof. Let $\varphi_k \in S_M$ such that $\varphi_k \to \pi$ in $\mathscr{C}$. Then from (2.8) we obtain

$$\forall \varepsilon > 0\ \exists n_0 \in N\ \forall k \geqslant n_0\colon\ \ \tilde{d}(D_{\varphi_k}^+(M),\, D_\pi^+(M)) \leqslant \varepsilon,$$

hence for every $\varepsilon > 0$, $\tilde{d}(M,\, D_\pi^+(M)) \leqslant \varepsilon$. This means that $D_\pi^+(M) = M$, which gives the assertion.

Let $\pi \in S_M$. Then

$$\overline{\pi^+(x)} \subset \overline{\pi^+(M)} \subset D_\pi^+(M) = M \qquad \text{for every } x \in M.$$

Remark 2.3. If a non-empty set $M \subset X$ is compact, then $S_M \subset \mathscr{C} \setminus \tilde{\mathscr{L}} \neq \mathscr{C}$.

As a simple corollary of Theorem 2.4 and 2.1 we get

THEOREM 2.5. *Let $G = \boldsymbol{R}$ and a non-empty set $M \subset \boldsymbol{R}^n$. Then*

(i) *if $\bar{M} = M$ such that $S_M \neq \mathscr{C}$, then $\{\varphi \in \mathscr{C}\colon M$ is not* BH-*stable$\}$ is of the second Baire category in $\mathscr{C}$,*

(ii) *if M is compact then $\{\varphi \in \mathscr{C}\colon M$ is not uniformly stable$\}$ is of the second Baire category in $\mathscr{C}$.*

Remark 2.4. It may happen that for a given set $M \subset X$ and a map π, M is uniformly stable for π, but in every neighbourhood of π there is π' such that M is not uniformly stable for π'.

EXAMPLE 2.1. Let $(\boldsymbol{R}^n, \boldsymbol{R}, \pi_k)$, $k = 0, 1, 2, \ldots$, be the sequence of dynamical systems given by the differential equations

$$dx_1/dt = f_k(x_1),\ dx_2/dt = 0,\ \ldots,\ dx_n/dt = 0,$$

where $f_0 \equiv 0$ and

$$f_k(x_1) = \begin{cases} \dfrac{1}{k\pi}\cos^2(k\pi x_1 - \tfrac{1}{2}\pi), & x_1 \in (0,\, 1/k), \\ 0, & x_1 \notin (0,\, 1/k) \end{cases}$$

for $k = 1, 2, \ldots$ We then have

$$\pi_k(t, (x_1, \ldots, x_n)) = \begin{cases} \left(\dfrac{1}{k\pi}\big(\tfrac{1}{2}\pi + \arctan(t + \tan(k\pi x_1 - \tfrac{1}{2}\pi))\big),\ x_2, \ldots, x_n\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x_1 \in (0,\, 1/k), & \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\ \ \text{for } x_1 \notin (0,\, 1/k). & \end{cases}$$

We notice that, as $k \to \infty$, $(f_k, 0, \ldots, 0)$ and π_k are convergent to $(f_0, 0, \ldots, 0)$ and π_0 in the spaces $\mathscr{C}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ and $\mathscr{C}(\boldsymbol{R} \times \boldsymbol{R}^n, \boldsymbol{R}^n)$ respectively, where the spaces are equipped with the topology of uniform convergence.

For these dynamical systems we have

$$W_{\pi_k}^{+}(x_1, \ldots, x_n) = \begin{cases} \{(1/k, x_2, \ldots, x_n)\}, & x_1 \in (0, 1/k), \\ \{(x_1, \ldots, x_n)\}, & x_1 \notin (0, 1/k), \end{cases}$$

$$W_{\pi_k}^{-}(x_1, \ldots, x_n) = \begin{cases} \{(0, x_2, \ldots, x_n)\}, & x_1 \in (0, 1/k), \\ \{(x_1, \ldots, x_n)\}, & x_1 \notin (0, 1/k), \end{cases}$$

where $W = \Lambda, J$.

Let $M := \{(0, \ldots, 0)\} \subset \boldsymbol{R}^n$. For every neighbourhood of $\pi_0 \in S_M$ there is $\pi_k \notin S_M$, because

$$D_{\pi_k}^{+}(M) = \{(x_1, 0, \ldots, 0)\colon\ x_1 \in [0, 1/k]\} \supsetneq M = D_{\pi_0}^{+}(M).$$

III. A classification of dynamical systems in $\boldsymbol{R}^n$. Let Y be a topological space and let (X, d) be a metric space. In a set $\mathscr{C} \subset \mathscr{C}(Y, X)$ we introduce the following equivalence relation S: if $\varphi, \pi \in \mathscr{C}$, then

$$(3.1) \qquad (\varphi, \pi) \in S \overset{\mathrm{df}}{\Leftrightarrow} \sup_{y \in Y} d\big(\varphi(y), \pi(y)\big) < \infty.$$

We write $\mathscr{C}/S := \{C_\pi\colon\ \pi \in \mathscr{C}\}$ for the set of equivalence classes.

We define the function

$$(3.2) \qquad \varrho\colon\ \mathscr{C} \times \mathscr{C} \ni (\varphi, \pi) \to \sup_{y \in Y} d\big(\varphi(y), \pi(y)\big) \in \bar{\boldsymbol{R}}$$

and its restriction

$$(3.3) \qquad \varrho_\pi := \varrho|_{C_\pi}.$$

THEOREM 3.1. *If Y is a topological space and (X, d) is a (complete) metric space, then for every $\pi \in \mathscr{C}$*

(i) *(C_π, ϱ_π) is a (complete) metric space,*

(ii) *$C_\pi = \bigcup_{r=1}^{\infty} B(\pi, r)$, where $B(\pi, r) := \{\varphi \in \mathscr{C}\colon\ \varrho(\varphi, \pi) < r\}$.*

Proof. Obviously $\varrho_\pi\colon\ C_\pi \times C_\pi \to \boldsymbol{R}$. Also it is very easy to prove that (C_π, ϱ_π) is a metric space. We now show that this space is complete if so is (X, d).

Let $\{\varphi_k\} \subset C_\pi$ be a Cauchy sequence, i.e.

$$\forall \varepsilon > 0\ \exists n_0 \in N\ \forall k, m \geqslant n_0\colon\ \ \varrho_\pi(\varphi_k, \varphi_m) < \varepsilon.$$

By the definition of ϱ_π we see that $\{\varphi_k(y)\} \subset X$ is a Cauchy sequence for every $y \in Y$. Thus the completeness of (X, d) implies that φ_k is pointwise convergent to some $\varphi\colon\ Y \to X$. Since

$$\forall \varepsilon > 0\ \exists n_0 \in N\ \ \forall k, m \geqslant n_0\ \ \forall y \in Y:\ \ d(\varphi_k(y), \varphi_m(y)) < \varepsilon,$$

letting $m \to \infty$ we obtain the uniform convergence of φ_k to φ. Thus $\varphi \in \mathscr{C}(Y, X)$ and $\varphi \in C_{\varphi_k} = C_\pi$, which completes the proof.

Remark 3.1 If Y is a topological space, (X, d) is a metric space, $C_* \subset \mathscr{C}(Y, X)$ and $\varrho_* := \varrho|_{C_*}$ gives a metric in C_*, then $C_* \subset C_\pi$ for every $\pi \in C_*$. This is obvious, because for every $\pi \in C_*$, if $\varphi = C_*$ then $\varrho(\varphi, \pi) < \infty$, so $\varphi \in C_\pi$.

The above means that S-equivalence classes are biggest subsets (in the sense of inclusion) for which the restriction of ϱ is a metric.

THEOREM 3.2. *If Y is a topological space and (X, d) is a (complete) metric space, then $(\mathscr{C}(Y, X), \varrho_1)$ is a (complete) metric space* (see (2.4)).

Let $\mathscr{C} \subset \mathscr{C}(Y, X)$ and $\{\chi_x \subset \mathscr{C}: x \in X\}$ be a family of subsets of $\mathscr{C}$ satisfying the condition

(C) $$(\pi \in \chi_x \Rightarrow C_\pi \subset \chi_x) \quad \text{for every } x \in X.$$

In view of Remark 3.1 we have

LEMMA 3.1. *Let (C_*, ϱ_*) be a metric space $(C_* \subset \mathscr{C})$. Then $C_* \cap \chi_x \neq \emptyset \Leftrightarrow C_* \subset \chi_x$.*

LEMMA 3.2. *Let $\pi \in \mathscr{C}$. Then*

(i) $$\pi \notin \bigcap_{x\in X} \chi_x \Leftrightarrow C_\pi \cap \bigcap_{x\in X} \chi_x = \emptyset,$$

(ii) $$\pi \notin \bigcup_{x\in X} \chi_x \Leftrightarrow C_\pi \cap \bigcup_{x\in X} \chi_x = \emptyset.$$

COROLLARY 3.1. *$\bigcap_{x\in X} \chi_x$, $\bigcup_{x\in X} \chi_x$ are closed and open sets in $(\mathscr{C}, \varrho_1)$.*

This is evident by virtue of Theorem 3.1 and Lemma 3.2.

Remark 3.2. In the quotient set $\mathscr{C}/S$ we can introduce the following equivalence relations:

(3.4) $$C_\pi(\chi)C_\varphi \Leftrightarrow \{x \in X:\ \pi \in \chi_x\} = \{x \in X:\ \varphi \in \chi_x\},$$

(3.5) $$C_\pi(\tilde{\chi})C_\varphi \Leftrightarrow \pi, \varphi \in \bigcup_{x\in X} \chi_x \vee \pi, \varphi \notin \bigcup_{x\in X} \chi_x.$$

According to Lemma 3.2 these relations are well defined, i.e. their definitions are independent of the choice of representatives of the classes C_π, C_φ.

Now, we take the set $\mathscr{C}$ defined by (1.7).

Example 2.1 shows that the families

$$\mathscr{P}_x = \{\varphi \in \mathscr{C}:\ x \in \Lambda_\varphi(x)\}, \quad x \in X,$$

$$\mathscr{C} \setminus \mathscr{K}_x = \{\varphi \in \mathscr{C}:\ x \in J^+_\varphi(x)\}, \quad x \in X,$$

do not satisfy condition (C), because there is $x \in \boldsymbol{R}^n$ such that $\pi_0 \in \mathscr{P}_x \cap (\mathscr{C} \setminus \mathscr{K})$, $\pi_k \in C_{\pi_0}$ and $\pi_k \notin \mathscr{P}_x \cup (\mathscr{C} \setminus \mathscr{K}_x)$ for $k \in N$.

The properties of the families $\mathscr{P}_x$ and $\mathscr{C} \setminus \mathscr{K}_x$ presented above give the following:

Remark 3.3. Let $x \in X$.

(i) The set of functions π for which x is nonwandering (the trajectory of x is Poisson stable) is not an open set.

(ii) The limit of a uniformly convergent sequence of functions π for each of which x is wandering (the trajectory of x is Poisson unstable) does not necessarily satisfy this condition.

Now, we shall give two examples of families satisfying condition (C) with $X = \boldsymbol{R}^n$

$$\mathscr{A}_x := \{\pi \in \mathscr{C}\colon \ \overline{\pi(x)} \text{ compact}\}, \tag{3.6}$$

$$\mathscr{B}_x := \{\pi \in \mathscr{C}\colon \ J_\pi^+(x) \neq \emptyset\}. \tag{3.7}$$

We take any $\pi \in \mathscr{A}_x$ and $\varphi \in C_\pi$. So $\overline{\pi(x)}$ is compact, i.e. there is $\eta \in \boldsymbol{R}_+$ such that $\overline{\pi(x)} \subset B(x, \eta)$ and there is $r \in \boldsymbol{R}_+$ such that $\varrho(\varphi, \pi) = r$. Therefore

$$d(\varphi(t, x), x) \leqslant d(\varphi(t, x), \pi(t, x)) + d(\pi(t, x), x) < r + \eta$$

for $t \in G$, i.e. $\varphi(x)$ is bounded. This is equivalent to the compactness of $\overline{\varphi(x)}$ in $\boldsymbol{R}^n$, so the family $\mathscr{A}_x$ satisfies (C).

The family $\mathscr{B}_x$ satisfies condition (C) in view of Theorem 2.2.

From the above there is a family satisfying condition (C), for which $\emptyset \neq \bigcup_{x \in X} \chi_x \neq \mathscr{C}$, and with the use of Corollary 3.1, we get the following theorem:

THEOREM 3.3. *The space $\mathscr{C}$ with the uniform convergence topology is not connected.*

We now apply the results presented at the beginning of Section II to the families $\mathscr{A}_x$ and $\mathscr{B}_x$. In this way we obtain some important theorems on the families $\mathscr{D}$, $\mathscr{K}$, $\tilde{\mathscr{P}}$, $\tilde{\mathscr{L}}$ for $X = \boldsymbol{R}^n$ (see (1.8)–(1.11)).

THEOREM 3.4. *Let $C_* \subset \mathscr{C}$ and suppose that $\varrho_* := \varrho|_{C_*}$ gives a metric in C_*. Then*

$$(\textit{there is } \pi \in C_* \textit{ satisfying } (*)) \Leftrightarrow (\textit{every } \pi \in C_* \textit{ satisfies } (*))$$

where $(*)$ is one of the conditions:

(i) *(X, G, π) is dispersive,*
(ii) *(X, G, π) is Lagrange unstable,*
(iii) *(X, G, π) is not dispersive,*
(iv) *(X, G, π) is not Lagrange unstable.*

COROLLARY 3.2. *Subsets C_* of the class $C_\pi \subset \tilde{\mathscr{P}}$ are the only possible metric spaces (C_*, ϱ_*) in which we can try to find a function π for which the dynamical system is dispersive, completely unstable or Poisson unstable.*

COROLLARY 3.3. *If in a metric space (C_*, ϱ_*) there is a function π for which the dynamical system satisfies one of the following conditions:*

(i) *(X, G, π) is not Lagrange unstable,*
(ii) *(X, G, π) is not Poisson unstable,*
(iii) *(X, G, π) is not completely unstable,*
(iv) *(X, G, π) is not dispersive,*

then no dynamical system, for functions belonging to C_*, is dispersive.

The proof of Theorem 3.4 follows from the equalities

$$\mathscr{C}\setminus\tilde{\mathscr{P}} = \bigcup_{x\in X} \mathscr{A}_x, \quad \mathscr{C}\setminus\mathscr{D} = \bigcup_{x\in X} \mathscr{B}_x. \tag{3.8}$$

We can define the set

$$\mathscr{L} := \{\pi\in\mathscr{C}\colon (X, G, \pi) \text{ is Lagrange stable}\}, \tag{3.9}$$

where (X, G, π) is Lagrange stable, when $\overline{\pi(x)}$ is compact for every $x \in X$ (see [1], [4]).

It is easy to observe that

$$\mathscr{L} = \bigcap_{x\in X} \mathscr{A}_x. \tag{3.10}$$

Remark 3.4. Theorem 3.4 is also true when $(*)$ is one of the conditions:

(v) (X, G, π) is Lagrange stable,
(vi) (X, G, π) is not Lagrange stable.

Equalities (3.8), (3.10) and the lemmas on families satisfying (C) also imply:

THEOREM 3.5. *The sets $\mathscr{D}$, $\tilde{\mathscr{P}}$ and $\mathscr{L}$ are both closed and open in the space $(\mathscr{C}, \varrho_1)$.*

Since $\mathscr{D} \neq \mathscr{K}$ we have

COROLLARY 3.4. *$\mathscr{D}$ is not dense in $\mathscr{K}$.*

The set $\mathscr{C}$ is closed in $(\mathscr{C}(G\times X, X), \varrho_1)$, so $(\mathscr{C}, \varrho_1)$ is the complete metric space (see Theorem 3.2). In virtue of theorems of Baire category theory (see [3]) we get

THEOREM 3.6. *The sets $\mathscr{D}$, $\mathscr{K}$, $\tilde{\mathscr{P}}$, $\tilde{\mathscr{P}}$ and $\mathscr{L}$ are of the second Baire category in $(\mathscr{C}, \varrho_1)$ but they are not residual sets in this space.*

Remark 3.5. The relations (χ) and $(\bar{\chi})$ defined in (3.4), (3.5) applied to the families $\mathscr{A}_x$ and $\mathscr{B}_x$ give partitions of the metric spaces C_π.

For example, if in the case $\chi_x := \mathscr{A}_x$ we have $C_\pi (\tilde{\chi}) C_\varphi$, then the dynamical systems are Lagrange unstable for all functions belonging to C_π and C_φ spaces, or are not Lagrange unstable for all those fnctions. These dynamical systems are or are not dispersive simultaneously, when $\chi_x := \mathscr{B}_x$.

Remark 3.6. In the quotient set $\mathscr{C}/S$ the relation

$$C_\pi \sim C_\varphi \Leftrightarrow \pi,\ \varphi \in \mathscr{L} \vee \pi,\ \varphi \in \tilde{\mathscr{L}} \vee \pi,\ \varphi \notin \mathscr{L} \cup \tilde{\mathscr{L}} \tag{3.11}$$

is also a well defined equivalence relation (see Lemma 3.2 and (3.8), (3.10)).

The above results can be applied to dynamical systems given by differential equations.

THEOREM 3.7. *Let* $(\boldsymbol{R}^n, \boldsymbol{R}, \pi_f)$ *and* $(\boldsymbol{R}^n, \boldsymbol{R}, \pi_g)$ *be the dynamical systems given by systems of differential equations* $x' = f(x)$ *and* $x' = g(x)$ *respectively. If there is a set* $K \subset \boldsymbol{R}^n$ *bounded and invariant in* $(\boldsymbol{R}^n, \boldsymbol{R}, \pi_f)$ *such that*

$$\{x \in \boldsymbol{R}^n:\ g(x) \neq f(x)\} \subset K, \tag{3.12}$$

then $\pi_g \in C_{\pi_f}$.

Proof. We consider two cases. First, let $x_0 \notin K$. From the invariance of $\boldsymbol{R}^n \setminus K$ it follows that $\pi_f(t, x_0) \notin K$ for every $t \in \boldsymbol{R}$. Hence and from (3.12) we get

$$\frac{d}{dt}\pi_f(t, x_0) = f\big(\pi_f(t, x_0)\big) = g\big(\pi_f(t, x_0)\big)$$

for every $t \in \boldsymbol{R}$. In view of the uniqueness of solutions of the Cauchy problem

$$x' = g(x), \qquad x(0) = x_0$$

we have

$$\pi_f(t, x_0) = \pi_g(t, x_0) \quad \text{for every } t \in \boldsymbol{R}.$$

Now, let $x_0 \in K$. K is invariant, i.e. $\pi_f(t, x_0) \in K$ for every $t \in \boldsymbol{R}$. Suppose that there is $t_0 \in \boldsymbol{R}$ such that $\pi_g(t_0, x_0) \notin K$ and put $y_0 := \pi_g(t_0, x_0)$. From the first part of the proof we get $x_0 = \pi_g(-t_0, y_0) = \pi_f(-t_0, y_0) \notin K$, a contradiction. This proves $\pi_g(t, x_0) \in K$ for every $t \in \boldsymbol{R}$. By the boundedness of K there is $M \in \boldsymbol{R}_+$ such that $\|x\| \leqslant M$ for every $x \in K$, so

$$d\big(\pi_g(t, x_0), \pi_f(t, x_0)\big) \leqslant \|\pi_g(t, x_0)\| + \|\pi_f(t, x_0)\| \leqslant 2M$$

for every $x_0 \in K$ and $t \in \boldsymbol{R}$.

In this way we have shown that $\varrho(\pi_g, \pi_f) \leqslant 2M$, i.e. $\pi_g \in C_{\pi_f}$.

COROLLARY 3.5. *If f and g satisfy the assumptions of the above theorem, then the dynamical systems given by f and g have the same sets of Lagrange stable points, i.e.*

$$\{x \in \boldsymbol{R}^n:\ \overline{\pi_f(x)} \text{ compact}\} = \{x \in \boldsymbol{R}^n:\ \overline{\pi_g(x)} \text{ compact}\},$$

$$\{x \in \boldsymbol{R}^n:\ J_{\pi_f}(x) = \emptyset\} = \{x \in \boldsymbol{R}^n:\ J_{\pi_g}(x) = \emptyset\}.$$

Obviously we can also apply other methods in order to verify whether two dynamical systems given by differential equations belong to the same class.

Remark 3.7. If the set of partial limits of

$$\int_0^\alpha (f(s)-g(s))\,ds \quad \text{as } \alpha\to\infty \text{ and } \alpha\to-\infty$$

is bounded, then $\pi_g \in C_{\pi_f}$.

For example, if $f(s)-g(s) = (\sin s, \cos s)$ for $s\in \boldsymbol{R}$, then the dynamical systems $(\boldsymbol{R}^2, \boldsymbol{R}, \pi_f)$ and $(\boldsymbol{R}^2, \boldsymbol{R}, \pi_g)$ have this property.

References

[1] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, Springer, Berlin 1970.
[2] R. Engelking, *General Topology*, PWN, Warszawa 1977.
[3] R. C. Haworth and R. A. McCoy, *Baire spaces*, Dissertationes Math. 141 (1977).
[4] A. Pelczar, *General Dynamical Systems* (in Polish), Monographs of the Jagellonian University, No. 293, Kraków 1978.

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