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Estimates of derivatives and Schwarz derivatives for generalized Aharonov pairs and determination of the totality of extremal pairs

by KAZIMIERZ WŁODARCZYK (Łódź)

Abstract. In the present paper new results are obtained for pairs of functions univalent in the unit disc and satisfying the Aharonov condition there: the results concern the problem of maximization of certain functionals dependent on the derivatives and Schwarz derivatives of these functions taken at any two points.

Introduction. The investigation of pairs of functions univalent in $\Delta = \{z: |z| < 1\}$ with disjoint images has been initiated by Lavrent'ev [14] and continued in the papers of Kufarev [11], Kufarev and Falles [12], [13], De Temple [4], Jenkins [10], Lebedev [15] and Seiler [18]. It seems that the class A of pairs introduced by Aharonov [1], i.e. pairs (F, G) of functions F and G analytic and univalent in Δ, such that $F(0) = G(0) = 0$ and satisfying the condition $F(z)\,G(\zeta) \neq 1$ for all $(z, \zeta) \in \Delta \times \Delta$, is especially interesting. Moreover, Aharonov pairs in the case of $G = F$, $G = \bar{F}$ or $G = -\bar{F}$, where $\bar{F}(z) = \overline{F(\bar{z})}$, generate Bieberbach–Eilenberg ([3], [5]) functions, bounded functions and Grunsky–Shah ([7], [19]) functions, respectively. The class A was examined by Hummel [8], Hummel and Schiffer [9].

The present paper concerns the class $C(a, b)$, $ab \in C\setminus\{1\}$, of pairs (F, G), analytic and univalent in Δ, of functions F and G of the form

$$F(z) = a + a_1 z + a_2 z^2 + \ldots, \qquad G(z) = b + b_1 z + b_2 z^2 + \ldots,$$

satisfying the condition $F(z)\,G(\zeta) \neq 1$ for all $(z, \zeta) \in \Delta \times \Delta$.

Note that $C(0, 0) = A$. If $G = F$, or $G = \bar{F}$ and $|a| < 1$, or $G = -\bar{F}$, then the class $C(a, b)$ reduces to one of the classes defined by Gromova and Labedev [6].

The Schwarz derivative $\{h;\ \}$ of a function h is defined as follows:

$$\{h; z\} = 6\left[\frac{\partial^2}{\partial z\,\partial\zeta}\log\frac{h(z)-h(\zeta)}{z-\zeta}\right]_{\zeta=z} = \left(\frac{h''(z)}{h'(z)}\right)' - \frac{1}{2}\left(\frac{h''(z)}{h'(z)}\right)^2.$$

The role of the Schwarz derivatives in the study of univalent functions is well known (see e.g., Aleksandrov [2], Schober [17], Pommerenke [16], Lebedev [15]).

In this paper we solve a problem of maximization of certain functionals dependent on the derivatives $F'(z_1)$, $G'(z_2)$ and Schwarz derivatives $\{F; z_1\}$, $\{G; z_2\}$ of functions F and G for pairs $(F, G) \in C(a, b)$, where $z_1, z_2 \in \Delta$. We also determine all extremal pairs and give a full characterization of them.

The main results of the paper are formulated in Section 1, while their proofs are given in Section 2. In Section 3 we consider some special cases concerning, among others, Aharonov pairs and Bieberbach–Eilenberg, bounded and Grunsky–Shah functions.

1. Main results. Let $z, z_1, z_2 \in \Delta$ and let

$$p(z) = \frac{z+z_1}{1+\bar{z}_1 z}, \quad q(z) = \frac{z+z_2}{1+\bar{z}_2 z}, \quad \hat{p}(z) = \frac{z-z_1}{1-\bar{z}_1 z}, \quad \hat{q}(z) = \frac{z-z_2}{1-\bar{z}_2 z}.$$

We shall use the following

LEMMA. *If $(F, G) \in C(a, b)$ and $z_1, z_2 \in \Delta$, then $(\hat{F}, \hat{G}) \in C(a, b)$, where*

$$\text{(A)} \quad \begin{gathered} \hat{F}(z) = \frac{a+f(z)}{1+bf(z)}, \quad f(z) = \frac{F[p(z)]-F(z_1)}{1-F[p(z)]\,G(z_2)}, \\ \hat{G}(z) = \frac{b+g(z)}{1+ag(z)}, \quad g(z) = \frac{G[q(z)]-G(z_2)}{1-G[q(z)]\,F(z_1)}, \end{gathered}$$

and

$$\text{(B)} \quad \begin{gathered} F(z) = \frac{a[1-\hat{f}(0)\,\hat{g}(0)]-(1-ab)\hat{f}(0)+[1-ab\hat{f}(0)\,\hat{g}(0)]\,\hat{f}(z)}{1-ab\hat{f}(0)\,\hat{g}(0)+\{b[1-\hat{f}(0)\,\hat{g}(0)]-(1-ab)\,\hat{g}(0)\}\,\hat{f}(z)}, \\ \hat{f}(z) = \frac{\hat{F}[\hat{p}(z)]-a}{1-b\hat{F}[\hat{p}(z)]}, \\ G(z) = \frac{b[1-\hat{f}(0)\,\hat{g}(0)]-(1-ab)\,\hat{g}(0)+[1-ab\hat{f}(0)\,\hat{g}(0)]\,\hat{g}(z)}{1-ab\hat{f}(0)\,\hat{g}(0)+\{a[1-\hat{f}(0)\,\hat{g}(0)]-(1-ab)\hat{f}(0)\}\,\hat{g}(z)}, \\ \hat{g}(z) = \frac{\hat{G}[\hat{q}(z)]-b}{1-a\hat{G}[\hat{q}(z)]}. \end{gathered}$$

The function h_ε, $|\varepsilon| = 1$, is defined by $h_\varepsilon(z) = h(\varepsilon z)$.

The following theorems hold.

THEOREM 1. *If $(F, G) \in C(a, b)$ and $z_1, z_2 \in \Delta$, then*

$$|F'(z_1)\,G'(z_2)| \leqslant \frac{|1-F(z_1)\,G(z_2)|^2}{(1-|z_1|^2)(1-|z_2|^2)} \tag{1.1}$$

and equality holds only for the pairs $(F_{\varepsilon_1}, G_{\varepsilon_2})$ defined by the pairs $(\hat{F}_{\varepsilon_1}, \hat{G}_{\varepsilon_2})$,

$|\varepsilon_1| = |\varepsilon_2| = 1$, *by means of formulae* (**B**), *where the functions* $\hat{F}$ *and* $\hat{G}$ *satisfy the equations*

$$\hat{F}(z) = \frac{\alpha z + a}{1 + b\alpha z}, \quad \hat{G}(z) = \frac{\beta z + b}{1 + a\beta z}, \quad |\alpha\beta| = 1. \tag{1.2}$$

THEOREM 2. *If* $z_1, z_2 \in \Delta$, $(F, G) \in C(a, b)$, $ab \neq 0$, *then*

$$\left| \{F; z_1\} \frac{G'(z_2)}{F'(z_1)} a^2 + \{G; z_2\} \frac{F'(z_1)}{G'(z_2)} b^2 + \frac{12ab\, F'(z_1)\, G'(z_2)}{[1 - F(z_1)\, G(z_2)]^2} \right| \tag{1.3}$$

$$\leqslant \frac{6|a^2|}{(1-|z_1|^2)^2} \left|\frac{G'(z_2)}{F'(z_1)}\right| + \frac{6|b^2|}{(1-|z_2|^2)^2} \left|\frac{F'(z_1)}{G'(z_2)}\right|$$

and, in the case of $ab \in (0; 1) \cup (1; \infty)$, *equality holds only for the pairs* $(F_{\varepsilon_1}, G_{\varepsilon_2})$ *defined by the pairs* $(\hat{F}_{\varepsilon_1}, \hat{G}_{\varepsilon_2})$, $|\varepsilon_1| = |\varepsilon_2| = 1$, *by means of formulae* (**B**), *where the functions* $\hat{F}$, $\hat{G}$,

$$\hat{F}(z) = a + \hat{a}_1 z + \hat{a}_2 z^2 + \ldots, \qquad \hat{G}(z) = b + \hat{b}_1 z + \hat{b}_2 z^2 + \ldots,$$

satisfy the equations

$$\frac{\sqrt{ab} + 1/\sqrt{ab} - [\sqrt{b/a}\,\hat{F}(z) + \sqrt{a/b}/\hat{F}(z)]}{\sqrt{b/a}\,\hat{F}(z) - \sqrt{a/b}/\hat{F}(z)} = \frac{\hat{a}_1}{a} \frac{z}{(z-\zeta_1)(z+\bar{\zeta}_1)}, \quad \frac{\hat{a}_1}{a} > 0, \tag{1.4}$$

$$\frac{\sqrt{ab} + 1/\sqrt{ab} - [\sqrt{a/b}\,\hat{G}(z) + \sqrt{b/a}/\hat{G}(z)]}{\sqrt{a/b}\,\hat{G}(z) - \sqrt{b/a}/\hat{G}(z)} = \frac{\hat{b}_1}{b} \frac{z}{(z-\zeta_2)(z+\bar{\zeta}_2)}, \quad \frac{\hat{b}_1}{b} > 0,$$

with $|\zeta_k| = 1$, $k = 1, 2$,

$$-2\,\mathrm{Im}\,\{\zeta_k\}\, i = \frac{\hat{a}_2}{\hat{a}_1} + \frac{b\hat{a}_1}{1-ab} \qquad |\mathrm{Im}\,\{\zeta_k\}| \leqslant 1 - \hat{a}_1 \sqrt{b/a}/(1-ab), \qquad k = 1,$$

$$= \frac{\hat{b}_2}{\hat{b}_1} + \frac{a\hat{b}_1}{1-ab}, \qquad \leqslant 1 - \hat{b}_1 \sqrt{a/b}/(1-ab), \qquad k = 2.$$

For each $\hat{a}_1$, $\hat{b}_1$, the functions F, G belong to one-parameter families, where the parameters are $\hat{a}_2$, $\hat{b}_2$ or ζ_1, ζ_2. The extremal domains possess two slits changing their lengths and also the initial points.

2. Proofs. The proofs of Theorems 1, 2 are a simple consequence of Lemma and Theorems 2, 6 of [20].

Proof of Lemma. The fact that $(\hat{F}, \hat{G}) \in C(a, b)$ is verified in a direct way.

From (A), after replacing z by $\hat{p}(z)$ or $\hat{q}(z)$, respectively, we get

$$F(z) = \frac{F(z_1)+\hat{f}(z)}{1+\hat{f}(z)\,G(z_2)}, \qquad G(z) = \frac{G(z_2)+\hat{g}(z)}{1+\hat{g}(z)\,F(z_1)}.$$

Hence, setting $z = 0$, we find $F(z_1)$ and $G(z_2)$ and, in consequence, we obtain (B).

Proof of Theorem 1. By assumption and Lemma, $(\hat{F}, \hat{G}) \in C(a, b)$, and consequently, in virtue of Theorem 2 [20],

$$|\hat{a}_1 \hat{b}_1| \leqslant |1-ab|^2, \tag{2.1}$$

with equality holding only for the pairs $(\hat{F}_{\varepsilon_1}, \hat{G}_{\varepsilon_2})$ defined by equations (1.2). Inequality (1.1) is obtained immediately from (2.1) by substituting the coefficients $\hat{a}_1$, $\hat{b}_1$ determined from (A).

Proof of Theorem 2. In virtue of Theorem 6 [20], for the pairs $(\hat{F}, \hat{G}) \in C(a, b)$, $ab \neq 0$, we have

$$\left|\frac{\hat{A}}{(\hat{a}_1 b)^2}+\frac{\hat{B}}{(\hat{b}_1 a)^2}\right| \leqslant \frac{1}{|\hat{a}_1 b|^2}+\frac{1}{|\hat{b}_1 a|^2}, \tag{2.2}$$

where

$$\hat{A} = \frac{\hat{a}_3}{\hat{a}_1}-\left(\frac{\hat{a}_2}{\hat{a}_1}\right)^2+\frac{b}{a}\,\frac{\hat{a}_1^2}{(1-ab)^2}, \qquad \hat{B} = \frac{\hat{b}_3}{\hat{b}_1}-\left(\frac{\hat{b}_2}{\hat{b}_1}\right)^2+\frac{a}{b}\,\frac{\hat{b}_1^2}{(1-ab)^2};$$

in the case of $ab \in (0;1) \cup (1;\infty)$ equality holds only for the pairs $(\hat{F}_{\varepsilon_1}, \hat{G}_{\varepsilon_2})$ defined by equations (1.4). Inequality (1.3) follows from inequality (2.2) and from (A) after applying the lemma.

3. Special cases. Putting $b = a$ and letting $a \to 0$, from Theorems 1, 2 one directly obtains the corresponding results for the class A:

COROLLARY 1. *If $z_1, z_2 \in \Delta$, and (F, G) is an Aharonov pair, then inequality (1.1) is satisfied and equality holds only for the pairs $(F_{\varepsilon_1}, G_{\varepsilon_2})$ defined by the functions $(\hat{f}, \hat{g})$ by means of the formulae*

$$F(z) = \frac{\hat{f}(z)-\hat{f}(0)}{1-\hat{g}(0)\hat{f}(z)}, \qquad G(z) = \frac{\hat{g}(z)-\hat{g}(0)}{1-\hat{f}(0)\,\hat{g}(z)}, \tag{3.1}$$

where $\hat{f}(z) = \alpha\hat{p}(z)$, $\hat{g}(z) = \beta\hat{q}(z)$, $|\alpha\beta| = 1$.

COROLLARY 2. *If $z_1, z_2 \in \Delta$, $(F, G) \in A$, then*

$$\left|\{F; z_1\}\, G'^2(z_2)+\{G; z_2\}\, F'^2(z_1)+\frac{12F'^2(z_1)\,G'^2(z_2)}{[1-F(z_1)\,G(z_2)]^2}\right| \leqslant \frac{6|G'^2(z_2)|}{(1-|z_1|^2)^2}+\frac{6|F'^2(z_1)|}{(1-|z_2|^2)^2}$$

and equality holds only for the pairs $(F_{\varepsilon_1}, G_{\varepsilon_2})$ *defined by the pairs* $(\hat{F}_{\varepsilon_1}, \hat{G}_{\varepsilon_2})$, $|\varepsilon_1| = |\varepsilon_2| = 1$, *by means of formulae* (3.1), *where* $\hat{f}(z) = \hat{F}[\hat{p}(z)]$, $\hat{g}(z) = \hat{G}[\hat{q}(z)]$ *and the functions* $\hat{F}$, $\hat{G}$ *satisfy the equations*

$$\frac{\hat{F}(z)}{\hat{F}^2(z)-1} = \hat{a}_1 \frac{z}{(z-\zeta_1)(z+\bar{\zeta}_1)},$$

$$\frac{\hat{G}(z)}{\hat{G}^2(z)-1} = \hat{b}_1 \frac{z}{(z-\zeta_2)(z+\bar{\zeta}_2)}, \qquad \hat{a}_1, \hat{b}_1 > 0,$$

with $|\zeta_k| = 1$, $k = 1, 2$,

$$-2\,\mathrm{Im}\,\{\zeta_k\}\, i = \frac{\hat{a}_2}{\hat{a}_1} \qquad |\mathrm{Im}\,\{\zeta_k\}| \leqslant 1-\hat{a}_1, \qquad k=1,$$

$$= \frac{\hat{b}_2}{\hat{b}_1}, \qquad \leqslant 1-\hat{b}_1, \qquad k=2.$$

Let F be a function of the form $F(z) = a + a_1 z + a_2 z^2 + \ldots$, univalent in Δ, satisfying for all $(z, \zeta) \in \Delta \times \Delta$ any one of the following conditions:

(i) $F(z)F(\zeta) \neq 1$; (ii) $F(z)\overline{F(\zeta)} \neq 1$; (iii) $F(z)\overline{F(\zeta)} \neq -1$.

We shall say that: $F \in B(a)$, where $a^2 \in \mathbf{C} \setminus \{1\}$, if F satisfies (i) ($B(0)$ – the class of Bieberbach–Eilenberg functions); $F \in S_1(a)$, where $|a| < 1$, if F satisfies (ii) (the class $S_1(0)$, when $0 < a_1 \leqslant 1$, coincides with the known class $S(a_1)$); $F \in G(a)$, where $a \in \mathbf{C}$, if F satisfies (iii) ($G(0)$ – the class of Grunsky–Shah functions).

For $G = F$ and $z_1 = z_2$, or $G = \bar{F}$, $z_1 = \bar{z}_2$ and $|a| < 1$, or $G = -\bar{F}$ and $z_1 = \bar{z}_2$, from Theorems 1, 2 and Corollaries 1, 2 we obtain at once suitable results for the classes of generalized Bieberbach–Eilenberg functions (class $B(a)$), of generalized bounded functions (class $S_1(a)$), and of generalized Grunsky–Shah functions (class $G(a)$).

And thus, e.g., we have

COROLLARY 3. *If* $z_1 \in \Delta$ *and* F *is a Bieberbach–Eilenberg function, then*

$$\left|\{F; z_1\} + \frac{6F'^2(z_1)}{[1-F^2(z_1)]^2}\right| \leqslant \frac{6}{(1-|z_1|^2)^2}$$

and equality holds only for functions F_ε *defined by functions* $\hat{F}_\varepsilon$, $|\varepsilon| = 1$, *by means of the formulae*

$$F(z) = \frac{\hat{f}(z) - \hat{f}(0)}{1 - \hat{f}(0)\hat{f}(z)}, \qquad \hat{f}(z) = \hat{F}[\hat{p}(z)],$$

where the functions $\hat{F}$ *satisfy the equations*

$$\frac{\hat{F}(z)}{\hat{F}^2(z)-1} = \hat{a}_1 \frac{z}{(z-\zeta_1)(z+\bar{\zeta}_1)}, \quad \hat{a}_1 > 0, \tag{3.2}$$

with $|\zeta_1| = 1$, $-2\,\mathrm{Im}\,\{\zeta_1\}\,i = \hat{a}_2/\hat{a}_1$, $|\mathrm{Im}\,\{\zeta_1\}| \leqslant 1-\hat{a}_1$.

COROLLARY 4. *If* $z_1 \in \Delta$, $F \in S_1(a)$, $|a| < 1$, *then*

$$\left|\{F;\,z_1\} + \frac{6\,|F'^2(z_1)|}{[1-|F(z_1)|^2]^2}\right| \leqslant \frac{6}{(1-|z_1|^2)^2}$$

and, in the case of $a \neq 0$, *equality holds only for functions* F_ε *defined by functions* $\hat{F}_\varepsilon \in S_1(a)$, $|\varepsilon| = 1$, *by means of the formulae*

$$F(z) = \frac{a[1-|\hat{f}(0)|^2]-(1-|a|^2)\hat{f}(0)+[1-|a|^2\,|\hat{f}(0)|^2]\,\hat{f}(z)}{1-|a|^2\,|\hat{f}(0)|^2+\{\bar{a}[1-|\hat{f}(0)|^2]-(1-|a|^2)\overline{\hat{f}(0)}\}\,\hat{f}(z)},$$

$$\hat{f}(z) = \frac{\hat{F}[\hat{p}(z)]-a}{1-a\hat{F}[\hat{p}(z)]},$$

where the functions $\hat{F}$ *satisfy the equations*

$$\frac{|a|+1/|a|-[\sqrt{\bar{a}/a}\,\hat{F}(z)+\sqrt{a/\bar{a}}/\hat{F}(z)]}{\sqrt{\bar{a}/a}\,\hat{F}(z)-\sqrt{a/\bar{a}}/\hat{F}(z)} = \frac{\bar{a}_1}{a}\,\frac{z}{(z-\zeta_1)(z+\bar{\zeta}_1)}, \quad \frac{\hat{a}_1}{a} > 0,$$

with $|\zeta_1| = 1$, $-2\,\mathrm{Im}\,\{\zeta_1\}\,i = \dfrac{\hat{a}_2}{\hat{a}_1}+\dfrac{\hat{a}_1\,\bar{a}}{1-|a|^2}$, $|\mathrm{Im}\,\{\zeta_1\}| \leqslant 1-\dfrac{\hat{a}_1}{a}\,\dfrac{|a|}{1-|a|^2}$.

COROLLARY 5. *If* $z_1 \in \Delta$ *and* F *is Grunsky–Shah function, then*

$$\left|\{F;\,z_1\} - \frac{6\,|F'^2(z_1)|}{[1+|F(z_1)|^2]^2}\right| \leqslant \frac{6}{(1-|z_1|^2)^2}$$

and equality holds only for functions F_ε *defined by functions* $\hat{F}_\varepsilon$, *by means of the formulae*

$$F(z) = \frac{\hat{f}(z)-\hat{f}(0)}{1+\overline{\hat{f}(0)}\,\hat{f}(z)}, \quad \hat{f}(z) = \hat{F}[\hat{p}(z)],$$

where the functions $\hat{F}$ *satisfy equations* (3.2).

Thus, for each $\hat{a}_1$, the functions F belong to one-parameter families, where the parameters are $\hat{a}_2$ or ζ_1. The extremal domains possess two slits changing their lengths.

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