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On the equivalence of Hille's and Robinson's functional equations

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Abstract. This paper gives a proof of the fact that Hille's functional equation is equivalent to Robinson's functional equation.

1. Hille [1] solved the following functional equation:

$$|f(s+it)|^2 = |f(s)|^2+|f(it)|^2, \tag{1}$$

where $f = f(z)$ is an entire function of a complex variable z and s, t are real variables.

Robinson [4] solved the following functional equation:

$$|f(s+it)| = |f(s)+f(it)|, \tag{2}$$

where $f = f(z)$ is an entire function of z.

The purpose of this note is to prove that (1) is equivalent to (2). To this end we introduce the function g defined as

$$g(z) = \overline{f(\bar{z})}. \tag{3}$$

We see that g is an entire function since f is an entire function.

2. Proof that (1) implies (2).

By (1) we have

$$|f(s+it)|^2 = f(s+it)\overline{f(s+it)} = f(s)\overline{f(s)}+f(it)\overline{f(it)}. \tag{4}$$

By (3), (4) we have

$$f(s+it)g(s-it) = f(s)g(s)+f(it)g(-it). \tag{5}$$

By (5) and by the Identity Theorem we have for all complex x, y

$$f(x+y)g(x-y) = f(x)g(x)+f(y)g(-y). \tag{6}$$

Putting $y = -x$ in (6) and using the fact that $f(0) = 0$ (which follows from (1)) yields

$$\big(f(x)+f(-x)\big)g(x) = 0. \tag{7}$$

We may assume that $f \not\equiv 0$. Then, by (3), we also have $g \not\equiv 0$. Since the ring of entire functions has no divisors of zero, we see by (7) that f is an odd function.

Hence, by (1), we have

$$|f(s-it)|^2 = |f(s)|^2+|f(-it)|^2 = |f(s)|^2+|f(it)|^2. \tag{8}$$

By (1), (3), (8) we have for all complex z

$$|f(z)| = |g(z)|. \tag{9}$$

Since f, g are entire functions, by (9) and by the Maximum Modulus Principle we see that for all complex z

$$g(z) = Cf(z), \tag{10}$$

where C is a complex constant of modulus 1.

By (3), (10), in view of the fact that f, and consequently by (3) also g, is an odd function we have

$$\overline{f(s)} = g(s) = Cf(s), \tag{11}$$

$$\overline{f(it)} = g(-it) = -g(it) = -Cf(it). \tag{12}$$

By (4), (11), (12) we have

$$|f(s+it)|^2 = C\big(f(s)^2-f(it)^2\big), \tag{13}$$

and

$$\begin{aligned} |f(s)+f(it)|^2 &= \big(f(s)+f(it)\big)\big(\overline{f(s)}+\overline{f(it)}\big) \\ &= \big(f(s)+f(it)\big)\big(Cf(s)-Cf(it)\big) \\ &= C\big(f(s)^2-f(it)^2\big). \end{aligned} \tag{14}$$

By (13), (14) we have (2).

3. Proof that (2) implies (1).

Squaring both sides of (2) and using (3) yields

$$f(s+it)g(s-it) = \big(f(s)+f(it)\big)\big(g(s)+g(-it)\big). \tag{15}$$

By (15) and by the Identity Theorem we have for all complex x, y

$$f(x+y)g(x-y) = \big(f(x)+f(y)\big)\big(g(x)+g(-y)\big). \tag{16}$$

Putting $y=-x$ in (16) and using the fact that $f(0)=0$, which follows from (2), we get (7), hence we see that f is an odd function, provided $f \not\equiv 0$.

Thus, by (16), we have

$$f(x+y)g(x-y) = \big(f(x)+f(y)\big)\big(g(x)-g(y)\big). \tag{17}$$

Differentiating both sides of (17) with respect to y, putting $y = x$ and noting that $g(0) = 0$, we obtain

$$g'(0)f(2x) = 2f(x)g'(x). \tag{18}$$

Differentiating both sides of (17) with respect to y, putting $y = -x$ and using the facts that $f(0) = 0$ and, the function f being odd, so is g, by (3), and the function f' is even we get

$$f'(0)g(2x) = 2f'(x)g(x). \tag{19}$$

Differentiating both sides of (17) with respect to y and putting $y = 0$ gives

$$f'(x)g(x)-f(x)g'(x) = f'(0)g(x)-g'(0)f(x). \tag{20}$$

By (18), (19), (20) we have

$$f'(0)g(2x)-g'(0)f(2x) = 2\big(f'(0)g(x)-g'(0)f(x)\big). \tag{21}$$

Since $C(x) = f'(0)g(x)-g'(0)f(x)$ is an entire function of x and, by (21), satisfies, for all complex x, the functional equation $C(2x) = 2C(x)$, we have $C(x) = Kx$ [2, 3], where K is a complex constant, i. e.,

$$f'(0)g(x)-g'(0)f(x) = Kx. \tag{22}$$

Differentiating both sides of (22) with respect to x and putting $x = 0$ yields $K = 0$.

Hence

$$f'(0)g(x)-g'(0)f(x) = 0. \tag{23}$$

But (3) yields $g'(0) = \overline{f'(0)}$ and so

$$|g'(0)| = |f'(0)|. \tag{24}$$

We shall prove that

$$f'(0) \neq 0. \tag{25}$$

Assume the contrary. Then, by (19), we have for all complex x

$$f'(x)g(x) = 0. \tag{26}$$

Since the ring of entire functions has no divisors of zero, by (26) either f is a complex constant and so, by (2), $f \equiv 0$ or $g \equiv 0$ and so, by (3), $f \equiv 0$. This is contrary to our assumption that $f \not\equiv 0$. Consequently (25) holds.

By (3), (23), (24), (25) we have for all complex z

$$|f(\bar{z})| = |g(z)| = |f(z)|. \tag{27}$$

By (2), the oddness of f and the Parallelogram Law we have

(28)

$$|f(s+it)|^2+|f(s-it)|^2 = |f(s)+f(it)|^2+|f(s)-f(it)|^2 = 2|f(s)|^2+2|f(it)|^2,$$

where s, t are real variables.

By (27), (28) we have (1).

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References

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