ANNALES
POLONICI MATHEMATICI
XXXVIII (1980)

On an ODE problem for equisingularities

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Abstract. A type of ODE is studied. The results are applied to algebraic geometry. An unsolved ODE problem is proposed.

1. Introduction. Let $P(x), Q_i(x), 1 \leqslant i \leqslant n$, be defined and analytic in a neighbourhood of 0 in $\boldsymbol{R}^n$. Suppose $P(0) = 0$ and $P(x) > 0$ for $x \neq 0$; we say that $P(x)$ is *positive definite* in this case. Moreover, suppose

$$\lim \frac{Q_i(x)}{P(x)} = 0 \quad \text{as } x \to 0, \ 1 \leqslant i \leqslant n.$$

Define $Q_i(0)/P(0) = 0$. Then 0 is an equilibrium point of the *continuous* vector field

$$\frac{dx_i}{dt} = \frac{Q_i(x)}{P(x)}, \quad 1 \leqslant i \leqslant n, \tag{1}$$

which is analytic except at 0.

An *analytic arc* at a point $a \in \boldsymbol{R}^n$ is a parametric arc $\alpha\colon [0, \varepsilon) \to \boldsymbol{R}^n$, written as $x_i = a_i(s), 0 \leqslant s < \varepsilon, 1 \leqslant i \leqslant n$, with $a(0) = a$, where each a_i is a convergent power series in s (cf. [6], § 3, p. 25). A *Newton–Puiseux arc* at $a \in \boldsymbol{R}^n$ is an arc $\lambda\colon [0, \varepsilon) \to \boldsymbol{R}^n$, $\lambda(0) = a$, for which there exists a substitution of the form $s = \sigma(\bar{s}) = \bar{s}^k$, k a positive integer, such that $\lambda \circ \sigma$ is an analytic arc.

For example, $x = \sum_{n=3}^{\infty} t^{n/2}$, $y = t$, is a Newton–Puiseux arc at 0 in $\boldsymbol{R}^2$. It is not analytic.

An arc $\lambda\colon x_i = \lambda_i(s)$, $1 \leqslant i \leqslant n$, is Newton–Puiseux if and only if each $\lambda_i(s)$ is a convergent fractional power series in s, i.e. a series whose powers can be written as fractions having a *same* denominator ([8], p. 97).

Let $\varphi_t(x)$ denote the trajectory of (1) satisfying the initial condition $\varphi_0(x) = x$. Let $\delta > 0$ be a given constant. Then for all x in a sufficiently small neighbourhood of 0, $\varphi_t(x)$ is defined at least for $0 \leqslant t \leqslant \delta$. Hence $\varphi_\delta\colon x \to \varphi_\delta(x)$ maps a neighbourhood of 0 into a neighbourhood of 0.

In this paper we are interested in the following

Problem. *Let λ be a Newton–Puiseux arc at 0. Is it true that $\varphi_\delta \circ \lambda$ is also a Newton–Puiseux arc at 0?*

This problem does not seem to have been considered before by differential equationists.

It turns out that, in general, the answer is *no*. An example is constructed in the following section. We then introduce in Section 3 the notion of the *distribution* of *Łojasiewicz exponents* associated with an analytic function-germ, in terms of which a sufficient condition for $\varphi_\delta \circ \lambda$ to be Newton–Puiseux is formulated; this is contained in the Main Theorem in Section 5.

The vector field (1) and the above problem arise naturally in the problem of equisingularities (cf. [3] and [4]). In the last section, we propose the notion of Newton–Puiseux trivialization and derive from the Main Theorem an existence theorem for such trivializations. This theorem applies in particular to Whitney's non-homogeneity example, an example of Kuiper, and the double cusp in Catastrophe Theory ([10]).

2. An example. Consider the vector field in $\boldsymbol{R}^2$

$$\frac{dx}{dt} = \frac{y^3(x+y^2)}{x^2+2y^4}, \quad \frac{dy}{dt} = 0. \tag{2}$$

Take $\lambda(y) = (0, y)$, $0 \leqslant y < \varepsilon$, the parameter being the variable y. Solving (2) by separable variables, we find $\varphi_\delta\big((0, y)\big) = \big(x(y), y\big)$, where $x \equiv x(y)$ is the root of the equation

$$\tfrac{1}{2}x^2 - xy^2 + 3y^4 \log\left[1 + \frac{x}{y^2}\right] = \delta y^3. \tag{3}$$

One then easily verifies that any convergent series of the form

$$x = \sum_{i=1}^{\infty} a_i y^{r_i}, \quad y > 0,\ 0 < r_1 < r_2 < \dots \tag{4}$$

is *not* a solution of (3). Here r_i can be any real numbers.

Now if $\varphi_\delta \circ \lambda$ was Newton–Puiseux, then there would exist two convergent series $x = x(s)$, $y = y(s)$, $0 \leqslant s < \varepsilon$, which satisfy (3) identically. Eliminating s, we would then find a series of the form (4) satisfying (3), where all r_i are rationals. Thus $\varphi_\delta \circ \lambda$ is *not* Newton–Puiseux.

3. Distribution of Łojasiewicz exponents. Recall first that for a fractional power series $g(t) = c_1 t^{r_1} + \dots$, $c_1 \neq 0$, its order is the degree of the first non-vanishing term: $O(g) \equiv r_1$. If $g(t) = 0$, then $O(g) = \infty$.

Let $f(x)$ be defined and analytic for x near 0 in $\boldsymbol{R}^n$, $f(0) = 0$. For a Newton–Puiseux arc at 0, λ: $x_i = \lambda_i(s)$, $0 \leqslant s < \varepsilon$, $1 \leqslant i \leqslant n$, we define

$$\mathscr{D}_f(\lambda) \equiv \frac{1}{m} O\big(f(\lambda(s))\big),$$

where $m = \min_i \big\{O\big(\lambda_i(s)\big)\big\}$, $1 \leqslant i \leqslant n$, $\lambda(s) = \big(\lambda_1(s), \dots, \lambda_n(s)\big)$.

Let $\mathscr{P}_0$ denote the set of all Newton–Puiseux arcs at 0. Then $\mathscr{D}_f$

is a map

$$\mathscr{D}_f\colon \mathscr{P}_0 \to Q^+ \cup \{\infty\},$$

where Q^+ denotes the set of all positive rational numbers. Here $\mathscr{D}_f(\lambda) = \infty$ if and only if f vanishes identically along λ.

We call $\mathscr{D}_f$ the *distribution of Łojasiewicz exponents associated to f.*

EXAMPLE. Consider $P(x, y) = x^2+2y^4$, which is the denominator in (2). Let λ be an arc of the form

$$x = a_1 y^{r_1} + \ldots, \quad r_1 > 0, \ a_1 \neq 0.$$

Then

$$\mathscr{D}_P(\lambda) = \begin{cases} 4 & \text{if} \quad r_1 \geqslant 2, \\ 2r_1 & \text{if} \quad r_1 \leqslant 2. \end{cases}$$

EXAMPLE. Consider $P(x, y) = x^2+y^2$. Then for all $\lambda \in \mathscr{P}_0$, $\mathscr{D}_P(\lambda) = 2$.

4. The horn-topology, degree of constant range. We now define a neighbourhood system $\{H_d(\lambda; w)\}$ for each $\lambda \in \mathscr{P}_0$, where $d \geqslant 1$, $w > 0$. Roughly speaking, $H_d(\lambda; w)$ consists of all $\mu \in \mathscr{P}_0$ whose order of contact with λ is at least d.

Let $\boldsymbol{R}^n$ be suitably rotated, if necessary, so that λ is tangent to the positive x_n-axis. Then λ can be parametrized by x_n, $x_n \geqslant 0$, as

$$x_i = \lambda_i(x_n), \quad O(\lambda_i) > 1, \ 1 \leqslant i \leqslant n-1 \ (\lambda_n(x_n) \equiv x_n).$$

For $\mu \in \mathscr{P}_0$, which makes an acute angle with the positive x_n-axis, we can also parametrize it by x_n, $x_n \geqslant 0$, as

$$x_i = \mu_i(x_n), \quad 1 \leqslant i \leqslant n-1 \ (\mu(x_n) \equiv x_n).$$

Here $O(\mu_i) \geqslant 1$ for each i; and $\min\limits_{1\leqslant i\leqslant n-1} \{O(\mu_i)\} > 1$ if and only if μ is tangent to the positive x_n-axis.

Define $\mu \in H_d(\lambda; w)$ if and only if $|\lambda_i(x_n) - \mu_i(x_n)| < w x_n^d$, $1 \leqslant i \leqslant n-1$, for all sufficiently small values of x_n. We do not consider any μ which is perpendicular to or makes an obtuse angle with the positive x_n-axis; it is "too far" from λ in our topology.

Observe that $H_d(\lambda; w) \subset H_{d'}(\lambda; w)$ if $d > d'$.

The above neighbourhood system gives rise to a topology on $\mathscr{P}_0$, called the *horn-topology*; and $H_d(\lambda; w)$ is called a *horn-neighbourhood* of λ (in $\mathscr{P}_0$) of degree d.

Consider the distribution $\mathscr{D}_f$ defined in the last section. If $\mathscr{D}_f(\lambda)$ is finite, then $\mathscr{D}_f$ is a constant in some horn-neighbourhood of λ. This assertion is a special case of the following stronger result.

PROPOSITION. *If $\mathscr{D}_f(\lambda)$ is finite, then there is a rational number $d_f(\lambda)$, $d_f(\lambda) \geqslant 1$, which is the smallest number having the property that $\mathscr{D}_f$ is a constant on $H_{d_f(\lambda)}(\lambda; w)$ for sufficiently small w.*

We call $d_f(\lambda)$ the *degree of constant range of $\mathscr{D}_f$ at λ.*

EXAMPLES. Let $\lambda(y) = (0, y)$ as before. If $P(x, y) = x^2+2y^4$, then $d_P(\lambda) = 2$. For $P(x, y) = x^2+y^2$, $d_P(\lambda) = 1$. And for $P(x, y)= x^4+y^6$, $d_P(\lambda) = \frac{3}{2}$.

For a proof of the proposition, consider a general arc μ, obtained by perturbing λ at degree d,

$$\mu:\ x_i = \lambda_i(x_n)+c_i x_n^d, \quad 1 \leqslant i \leqslant n-1,$$

where d is an indeterminant, c_i are sufficiently small numbers.

Write $f(\lambda(x_n))$ as

$$f(\lambda(x_n)) = a_1 x_n^{r_1}+\ldots, \quad a_1 \neq 0,\ r_1 = \mathscr{D}_f(\lambda).$$

Then

$$f(\mu(x_n)) = f(\lambda(x_n))+[g_1(c)x_n^{q_1}+\ldots],$$

where $g_1(c)$ is a polynomial in $c = (c_1, \ldots, c_n)$, $g_1(0) = 0$, and q_1 is a *linear* function of d with *rational* coefficients. Let c be restricted to a region $|c| < \eta$ in which $|g_1(c)| < |a_1|$, and $g_1(c) = 0$ only when $c = 0$. Then $\mathscr{D}_f(\mu) = \mathscr{D}_f(\lambda)$ if and only if $q_1 \geqslant r_1$.

If d is sufficiently large, say $d > \mathscr{D}_f(\lambda)$, then we find $O(f(\mu(x_n))) = \mathscr{D}_f(\lambda)$ by direct inspection. Therefore, the smallest value d_1 of d such that $q_1 \geqslant r_1$ can be found by solving the linear equation $q_1 = r_1$. Now, put $d_f(\lambda) = \mathrm{Max}\{1, d_1\}$.

5. The Main Theorem. Let us consider a more general case. Let a non-autonomous system

$$\frac{dx_i}{dt} = \frac{Q_i(x, t)}{P(x, t)}, \quad 1 \leqslant i \leqslant n, \tag{1'}$$

be given, where P, Q are defined and analytic in (x, t), for x in $\boldsymbol{R}^n$ near 0, $|t| < \eta$ in $\boldsymbol{R}$. Suppose for every *fixed* value of t, $P(x, t) = 0$ only when $x = 0$, and suppose

$$\lim \frac{Q_i(x, t)}{P(x, t)} = 0 \quad \text{as } x \to 0,\ 1 \leqslant i \leqslant n.$$

As before, let $\mathscr{P}_0$ denote the set of all Newton–Puiseux arcs at 0 in $\boldsymbol{R}^n$ (*not* those in $\boldsymbol{R}^{n+1}$!). Let $f(x, t)$ be an analytic function. For $\lambda \in \mathscr{P}_0$ and each *fixed* value of t, $\mathscr{D}_f(\lambda)$ is defined. If $\mathscr{D}_f(\lambda) < \infty$, then $d_f(\lambda)$ is also defined. Of course, they depend on t.

Let $\varphi_t(x)$ denote the trajectory of $(1')$ satisfying $\varphi_0(x) = x$.

MAIN THEOREM. *Let $\lambda \in \mathscr{P}_0$ be given. Suppose that $\mathscr{D}_P(\lambda)$ and $d_P(\lambda)$ are independent of t. Moreover, suppose*

(5) $$\mathscr{D}_{Q_i}(\mu) \geqslant \mathscr{D}_P(\lambda) + d_P(\lambda), \qquad 1 \leqslant i \leqslant n,$$

for $\mu \in H_{d_{P(\lambda)}}(\lambda; w)$.

Then $\varphi_\delta \circ \lambda \in H_{d_{P(\lambda)}}(\lambda; w)$, *for all sufficiently small* δ. *Moreover, if* $s = \sigma(\bar{s}) = \bar{s}^m$ *is a substitution for which* $\lambda(\bar{s}^m)$ *is analytic, then* $\varphi_t \circ \lambda \circ \sigma$ *is analytic in* $(\bar{s}, t)$.

Note that for the autonomous system (1), the first hypothesis is superfluous. Note also that (5) can be rewritten as

$$\mathscr{D}_{Q_i}(\mu) \geqslant \mathscr{D}_P(\mu) + d_P(\mu), \qquad \mu \in H_{d_{P(\lambda)}}(\lambda; w),$$

since

$$\mathscr{D}_P(\mu) = \mathscr{D}_P(\lambda), \qquad d_P(\mu) = d_P(\lambda).$$

In the example of Section 2, (5) is not satisfied; it was shown there that $\varphi_\delta \circ \lambda \notin \mathscr{P}_0$.

EXAMPLE. For $\frac{dx}{dt} = \frac{x^3}{x^2+y^2}$, $\frac{dy}{dt} = \frac{y^3}{x^2+y^2}$ in $\boldsymbol{R}^2$, (5) is satisfied for any λ. Hence $\varphi_\delta \circ \lambda \in \mathscr{P}_0$.

COROLLARY. *Under the same hypothesis,* $\varphi_\delta \circ \lambda$ *is of class* C^k *at* 0, *where* k *denotes the largest integer* $\leqslant d_P(\lambda)$.

This follows immediately from the fact that $\varphi_\delta \circ \lambda \in H_{d_{P(\lambda)}}(\lambda; w)$.

AN UNSOLVED PROBLEM. *Assume, instead of* (5), *that*

(6) $$\mathscr{D}_{Q_i}(\lambda) \geqslant \mathscr{D}_P(\lambda) + k \quad \textit{for all } \lambda \in \mathscr{P}_0,$$

where k *is a given integer. Is it true that* $\varphi_\delta \circ \lambda$ *is* C^k *at* 0?

Observe that in the example of Section 2, (6) holds for $k = 1$, and one can show that $\varphi_\delta \circ \lambda$ is C^1, where $\lambda(y) = (0, y)$.

6. Proof. We shall omit t in P and Q for simplicity of notation.

Let λ be parametrized by x_n as in Section 4, $x_i = \lambda_i(x_n)$, $O(\lambda_i) > 1$, $1 \leqslant i \leqslant n-1$. Consider the point set $\mathscr{R}_d(\lambda; w)$ in $\boldsymbol{R}^n$ which consists of all x satisfying the inequality

$$\sum_{i=1}^{n-1} |x_i - \lambda_i(x_n)| < w x_n^d.$$

This is a horn-shaped open set with vertex 0, but $0 \notin \mathscr{R}_d$. We call $\mathscr{R}_d(\lambda; w)$ the *horn-region of degree d around* λ. Let $\mu \in \mathscr{P}_0$ be parametrized as $x_i = \mu_i(x_n)$, $1 \leqslant i \leqslant n-1$. Then $\mu(x_n) \in \mathscr{R}_d(\lambda; w)$ for all sufficiently small x_n if and only if $\mu \in H_d(\lambda; w)$.

Write each λ_i as

$$\lambda_i(x_n) = \sum_{j=1}^{\infty} c_{ij} x_n^{r_{ij}}, \qquad 1 \leqslant i \leqslant n-1,$$

where $r_{ij} = n_{ij}/m$, m, n_{ij} are integers, $0 < m < n_{i1} < n_{i2} < \ldots$ Then consider the change of coordinate system $u = \Phi(x)$ defined for $x_n \geqslant 0$ by

$$u_i = x_i - \sum_{j=1}^{\infty} c_{ij} x_n^{r_{ij}}, \quad 1 \leqslant i \leqslant n-1,$$
$$u_n = x_n.$$

Notice that Φ maps the open half space $x_n > 0$ bianalytically onto the open half space $u_n > 0$, and is a homeomorphism of $x_n \geqslant 0$ onto $u_n \geqslant 0$. It may *not* be analytic along $x_n = 0$.

Now Φ induces a map Φ^* defined by $\mu^* = \Phi^*(\mu) = \Phi \circ \mu$. In particular, $\lambda^* = \Phi^*(\lambda)$ is the non-negative u_n-axis, and $H_d(\lambda; w)$ is mapped bijectively to $H_d(\lambda^*; w)$ under Φ^*. Moreover, Φ maps $\mathscr{R}_d(\lambda; w)$ bianalytically onto $\mathscr{R}_d(\lambda^*; w)$ in the u-space.

From now on, we shall write $d_f(\lambda)$ simply as d.

Consider a second change of coordinate system, $v = \Psi(u)$, defined for $u_n > 0$ by

$$v_i = u_n^{-d} u_i, \quad 1 \leqslant i \leqslant n-1,$$
$$v_n = u_n.$$

When $d = 1$, this is the well-known "blowing-up" process in Algebraic Geomerty. Observe that Ψ is bianalytic.

Define Ψ^{-1}, *for all v*, by

$$u_i = v_i v_n^d, \quad 1 \leqslant i \leqslant n-1,$$
$$u_n = v_n.$$

The restriction of Ψ^{-1} to $v_n > 0$ is the inverse of Ψ. The whole coordinate hyperplane $v_n = 0$ is mapped into the origin 0 in the u-space under Ψ^{-1}; a half line

$$v_i = c_i, \quad 1 \leqslant i \leqslant n-1, \ v_n \geqslant 0,$$

where c_i are constants, that is parallel to the non-negative v_n-axis, is mapped to the arc

$$u_i = c_i u_n^d, \quad u_n \geqslant 0.$$

We have to assume $u_n \geqslant 0$ in our consideration. As d is not necessarily an integer, u_n^d may not make sense in case $u_n < 0$.

Now consider the half-open cylinder $C = \{v \mid \sum_{i=1}^{n-1} v_i^2 < w^2, \ v_n > 0\}$. Under Ψ, $\mathscr{R}_d(\lambda^*; w)$ is mapped bianalytically onto C.

Let us now consider the restriction of (1′) to $\mathscr{R}_d(\lambda; w)$. Its image under $d\Phi$ is the vector field

$$\frac{du_i}{dt} = \frac{Q_i^*(u)}{P^*(u)}, \quad 1 \leqslant i \leqslant n, \tag{7}$$

in $\mathscr{R}_d(\lambda^*; w)$, where

$$Q_i^*(u) = Q_i(\Phi^{-1}(u)) - \sum_j c_i r_{ij} u_n^{r_{ij}-1} Q_n(\Phi^{-1}(u)),$$
$$Q_n^*(u) = Q_n(\Phi^{-1}(u)), \quad P^*(u) = P(\Phi^{-1}(u)).$$

For $\mu^* \in H_d(\lambda^*; w)$, we have

$$\mathscr{D}_{P^*}(\mu^*) = \mathscr{D}_P(\mu), \quad \mathscr{D}_{Q_i^*}(\mu^*) \geqslant \mathscr{D}_{Q_i}(\mu), \quad 1 \leqslant i \leqslant n.$$

The last inequality is an obvious fact on formal power series; see [8], p. 89. Hence inequality (5) of the Main Theorem remains valid for (7). But P^*, Q_i^* may not be analytic in u_n along $u_n = 0$, since they contain powers of $u_n^{1/m}$. Notice that, however, P^*, Q_i^* are analytic in $u_1, \ldots, u_{n-1}, s$, when u_n is substituted by s^m; this fact will suffice for our argument.

Next, let us consider $d\Psi$, which carries the vector field (7) into

$$(8) \qquad \begin{aligned} \frac{dv_i}{dt} &= \frac{1}{v_n^d} \cdot \frac{Q_i^{**}(v)}{P^{**}(v)} - \frac{d}{v_n} \cdot \frac{v_i Q_n^{**}(v)}{P^{**}(v)}, \quad 1 \leqslant i \leqslant n-1, \\ \frac{dv_n}{dt} &= \frac{Q_n^{**}}{P^{**}}, \end{aligned}$$

in C, where $Q_i^{**}(v) = Q_i^*(v_1 v_n^d, \ldots, v_{n-1} v_n^d, v_n)$, $1 \leqslant i \leqslant n$, $P^{**}(v) = P^*(v_1 v_n^d, \ldots, v_{n-1} v_n^d, v_n)$.

In the following, we shall use $\mathscr{D}$ as a shorthand for the number $\mathscr{D}_P(\lambda)$.

We assert that $P^{**}(v) = v_n^{\mathscr{D}} U(v)$, where $U(v)$ is a unit in the sense that $U(v_1, \ldots, v_{n-1}, s^m)$ is *analytic* in $v_1, \ldots, v_{n-1}, s$, and $U(0) \neq 0$. In fact, since $\mathscr{D}_{P^*}(\mu^*) = \mathscr{D}$ for all $\mu^* \in H_d(\lambda^*; w)$, $O(P^{**}(c_1, \ldots, c_{n-1}, v_n)) = \mathscr{D}$ for *all* sufficiently small constants c_i. Hence $P^{**}(v) = v_n^{\mathscr{D}} U(v)$, as asserted.

Similarly, each Q_i^{**}, $1 \leqslant i \leqslant n$, is divisible at least by $v_n^{\mathscr{D}+d}$.

Now, put $v_n = s^m$ in order to obtain analyticity. Then (8) is transformed into the vector field

$$(9) \qquad \begin{aligned} \frac{dv_i}{dt} &= V_i(v_1, \ldots, v_{n-1}, s), \quad 1 \leqslant i \leqslant n-1, \\ \frac{ds}{dt} &= \frac{Q_n^{**}(v_1, \ldots, v_{n-1}, s^m)}{m s^{m-1} P^{**}(v_1, \ldots, v_{n-1}, s^m)}, \end{aligned}$$

where V_i is the right-hand side of the ith equation of (8) with v_n replaced by s^m. Note that (9) is so far only defined for $s > 0$. But by what we have just proved, the numerators in (9) contain at least as many factors of s as the denominators; after cancelling these factors, the denominators become units, and so (9) can be extended, in an obvious way, into a vector field *defined* and *analytic* also for $s \leqslant 0$. Moreover, for the extended vector field, $ds/dt = 0$ when $s = 0$.

The above observation is of vital importance to our argument.

Let $\psi_t(v_1, \ldots, v_{n-1}, s)$ denote the trajectory of the extended vector field with $\psi_0(v_1, \ldots, v_{n-1}, s) = (v_1, \ldots, v_{n-1}, s)$. Let $\psi_t^{(i)}$, $1 \leqslant i \leqslant n$, denote its components. By a standard theorem in ODE (see, e.g. [1], p. 119), each $\psi_t^{(i)}$ is an analytic function of $v_1, \ldots, v_{n-1}, s$, and t. Note that $\psi_t^{(n)}(v_1, \ldots \ldots, v_{n-1}, 0) \equiv 0$ for all $v_1, \ldots, v_{n-1}$, and all t. Hence $\xi: s \to \psi_\delta(0, \ldots, 0, s)$, $0 \leqslant s < \varepsilon$, is an analytic arc whose initial point $\psi_\delta(0, \ldots, 0, 0)$ lies in the coordinate hyperplane $s = 0$. (However, $\psi_\delta(0, \ldots, 0) \neq 0$ in general.)

Let $\delta > 0$ be small enough, then ξ lies in the region $\sum_{i=1}^{n-1} v_i^2 < w^2$, $s \geqslant 0$. Define $v_i = \gamma(v_i)$, $1 \leqslant i \leqslant n-1$, $v_n = \gamma(s) = s^m$, where $s \geqslant 0$. Then $\gamma \circ \xi$ is an analytic arc in the v-space, parametrized by s, which is contained in the cylinder C, except that its initial point lies in the hyperplane $v_n = 0$. Now $\Psi^{-1} \circ \gamma \circ \xi$ is an analytic arc at 0 in the u-space, belonging to $H_d(\gamma^*; w)$. Finally, $\Phi^{-1} \circ \Psi^{-1} \circ \gamma \circ \xi \in H_d(\lambda; w)$. This last arc, when using x_n as the parameter, is just $\varphi_\delta \circ \lambda$ by our construction. The proof of the Main Theorem is now complete.

Illustrative Example. In $\boldsymbol{R}^2$, consider the autonomous system

$$\frac{dx}{dt} = \frac{y^4(x+y^2)}{x^2+2y^4}, \qquad \frac{dy}{dt} = \frac{x^2y^2}{x^2+2y^4}, \tag{10}$$

and $\lambda(y) = (0, y)$, $y \geqslant 0$.

Unlike (1), inequality (5) is satisfied by (10), with $\mathscr{D}_P(\lambda) = 4$, $d_P(\lambda) = 2$, $\mathscr{D}_{Q_1}(\mu) = 6$, $\mathscr{D}_{Q_2}(\mu) \geqslant 6$. Hence $\varphi_\delta \circ \lambda$ is Newton–Puiseux.

Now, let us examine how the general proof works through in this example. The first step, $u = \Phi(x)$, is not needed, as λ is already the non-negative y-axis. The second step, $v = \Psi(u)$, which blows-up the horn-region $\mathscr{R}_d$, reduces to

$$v_1 = y^{-2}x, \qquad v_2 = y.$$

The image of (10) under $d\Psi$ is

$$\frac{dv_1}{dt} = \frac{1+v_1-2v_1^3v_2}{v_1^2+2}, \qquad \frac{dv_2}{dt} = \frac{v_1^2v_2^2}{v_1^2+2}, \tag{11}$$

where the common factors in v_2 have been cancelled. Now, (11) is *defined* and *analytic for all* v_1, v_2. Hence $v_2 \to \psi_\delta(0, v_2)$, $v_2 \geqslant 0$, is an analytic arc, and, for this example, $\varphi_\delta \circ \lambda$ is also analytic.

7. An application to equisingularities. Let us first consider three typical examples in $\boldsymbol{R}^3$. The first is the Whitney function

$$W(x, y, t) = xy(x-y)[x-(t+2)y], \qquad |t| < 1,$$

the second is Kuiper's example,

$$K(x, y, t) = x^5 + y^5 + tx^3y^3, \qquad t \in \boldsymbol{R},$$

and the last one is the double cusp in catastrophe theory

$$D(x, y, t) = x^4 + y^4 + tx^2y^2, \qquad t \neq \pm 2.$$

In each of these examples, the singularities of the function are points on the t-axis.

More generally, let $F(x_1, \ldots, x_n, t)$ be a function defined and analytic near $0 \in \boldsymbol{R}^n \times \boldsymbol{R}$, whose singularities form an interval I of the t-axis, containing 0.

A *local trivialization* of F along I near 0 is a homeomorphism h between two neighbourhoods of 0 in $\boldsymbol{R}^n \times \boldsymbol{R}$, which preserves the t-levels, such that $h(0, t) = (0, t)$ and $F(h(x, t)) = F(x, 0)$.

One may like to require that h be of class C^k, in this case we have a C^k-trivialization. But this notion does not seem to be an interesting one. Whitney showed, in [9], p. 238, that $W(x, y, t)$ did not admit any C^1-trivialization; Kuiper, in [2], p. 206, proved that $K(x, y, t)$ had no C^2-trivialization. For $D(x, y, t)$, C^0-trivialization exists ([5], p. 152), but there is no C^∞-trivialization ([7]), nor is there a C^1-trivialization. On the other hand, some kind of analyticity would certainly be more desirable than mere differentiability. Thus we propose the following alternative.

DEFINITION. A local trivialization h is called *Newton–Puiseux* if for any analytic arc λ at 0, $x_i = \lambda_i(s)$, $1 \leqslant i \leqslant n$, lying in the coordinate hyperplane $t = 0$, $h(\lambda(s), t)$ is a Newton–Puiseux wing in the sense that there exists a substitution $s = \bar{s}^m$, m a positive integer, for which $h(\lambda(\bar{s}^m), t)$ is analytic in $(\bar{s}, t)$.

Every Newton–Puiseux trivialization is a *real* semi-analytic fibration in the sense of Whitney ([9], p. 230).

Recall that $\operatorname{Grad} F = \left(\dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n}, \dfrac{\partial F}{\partial t}\right)$, and $\operatorname{Grad} F = 0$ only when $x = 0$, $t \in I$. As before, let $\mathscr{P}_0$ denote the set of all Newton–Puiseux arcs at 0 in the hyperplane $t = 0$.

THEOREM. *Let* $P(x, t) = |\operatorname{Grad} F(x, t)|$, $Q(x, t) = \partial F/\partial t$. *Suppose for every* $\lambda \in \mathscr{P}_0$

(12) $$\mathscr{D}_P(\lambda) \text{ is independent of } t,$$

and for each fixed value of t in I,

(13) $$\mathscr{D}_Q(\lambda) \geqslant \mathscr{D}_P(\lambda) + d_P(\lambda).$$

Then $F(x, t)$ admits a Newton–Puiseux trivialization along I near 0.

One can verify easily that all the above three examples satisfy the hypothesis. Hence they admit Newton–Puiseux trivializations.

Strickly speaking, $P(x, t)$ is not analytic, hence $\mathscr{D}_P$ is not defined. This difficulty can be avoided by putting $\mathscr{D}_P \equiv \frac{1}{2}\mathscr{D}_{P^2}$, where P^2's analytic.

The first assumption (12) implies that $d_P(\lambda)$ is also independent of t.

Given any (x^*, t^*) off the t-axis, let $c = F(x^*, t^*)$, then $F(x, t) = c$ is the level surface of F passing through (x^*, t^*), and is locally a manifold there. The projection of the vector $\partial/\partial t$ to the tangent space at (x^*, t^*) of the level surface is the vector

$$X(x^*, t^*) \equiv P(x^*, t^*)^{-2} Q(x^*, t^*) \operatorname{Grad} F(x^*, t^*).$$

Then consider the vector field $\partial/\partial t - X(x^*, t^*)$, which was constructed in [3], [4] for similar purpose. Normalizing its t-component to 1, we find the vector field

$$\sum_{i=1}^{n} \tilde{P}^{-2} Q \frac{\partial F}{\partial x_i} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial t}, \tag{14}$$

where $\tilde{P} = \left\| \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right) \right\|$.

Along the t-axis, define (14) to be $\partial/\partial t$.

Let $\varphi_t(x)$ denote the trajectory of (14) with $\varphi_0(x) = (x, 0)$. Define $h(x, t) = \varphi_t(x)$.

We assert that h is a Newton–Puiseux trivialization.

Since $d_P(\lambda) \geqslant 1$, (13) implies that for all λ,

$$D_{\tilde{P}}(\lambda) = D_P(\lambda), \quad d_{\tilde{P}}(\lambda) = d_P(\lambda).$$

Hence

$$D_Q(\lambda) \geqslant D_{\tilde{P}}(\lambda) + d_{\tilde{P}}(\lambda).$$

Observe also that $\left| \tilde{P}^{-2} \frac{\partial F}{\partial x_i} \right| \leqslant \tilde{P}^{-1}$, $1 \leqslant i \leqslant n$. Therefore the non-autonomous system

$$\frac{dx_i}{dt} = \tilde{P}^{-2} Q \frac{\partial F}{\partial x_i}, \quad 1 \leqslant i \leqslant n, \tag{15}$$

satisfies the hypothesis of the Main Theorem, for every $\lambda \in \mathscr{P}_0$. Note that the projection of $\varphi_t(x)$ to the hyperplane $t = 0$ is the trajectory of (15). The proof is now complete.

Added in proof. The main ideas in this paper has been developed further in the following papers:

T.-C. Kuo, *Modified trivialization of singularities,* J. Japan Math. Soc. (to appear).

T.-C. Kuo and J. N. Ward, *A theorem on almost analytic equisingularities,* ibidem (to appear).

T.-C. Kuo, *Une classification des singularités réelles,* C. R. Acad. Sci. Paris 288 (1979), p. 809–812.

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Reçu par la Rédaction le 15. 10. 1977