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A generalization of the Malgrange–Zerner theorem

by LUDWIK M. DRUŻKOWSKI (Kraków)

Abstract. The main result of this paper is the following generalization of well-known Malgrange–Zerner theorem on separately holomorphic functions. Let

$$X = \bigcup_{k=1}^{n} \boldsymbol{R}^{k-1} \times P \times \boldsymbol{R}^{n-k} \quad , \quad \text{where } P = R + i[0,1) \subset \boldsymbol{C},$$

let W = convex hull of X and further let $f\colon X \to \boldsymbol{C}$ be a locally bounded function, such that $f|_{\boldsymbol{R}^n}$ is continuous, and for any fixed point $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \boldsymbol{R}^{n-1}$, the function $f(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_n)$ is holomorphic in $\operatorname{int} P$, $k = 1, \ldots, n$. Then the function f may be uniquely continued to a function holomorphic in int W and continuous in. There are also given the two examples which show that the assumptions in the above theorem in some sense are minimal.

Introduction. The classical Hartogs' theorem (see [5]) says that if $f = f(z_1, \ldots, z_n)$ is a function defined in a domain D in the space $\boldsymbol{C}^n$ and f is holomorphic in each variable $z_k \in \boldsymbol{C}$ separately, $k = 1, \ldots, n$, when the other variables have given arbitrary fixed values, then f is holomorphic in D. Hartogs' theorem gives rise to the following natural question: when does a function defined on a non-open lower-dimensional subset of $\boldsymbol{C}^n$, separately holomorphic (in a suitable sense), admit a uniquely extension to a holomorphic function in some open set in the space $\boldsymbol{C}^n$. The Malgrange–Zerner theorem is of theorems of this type.

In this paper we give a version of the Malgrange–Zerner theorem, which is stronger than those given in [1] and [2]. In the classical version given in [2] it is assumed that the function belongs to $\mathscr{C}^\infty(X)$, where as in [1] the function f has to be bounded and belong to $\mathscr{C}^3(R^n)$.

We write shortly: $f \in H(U)$ ($\mathscr{C}(U)$, $\mathrm{Sh}(U)$) if f is a complex or real-valued function holomorphic (continuous, subharmonic) in a subset U of $\boldsymbol{C}^n$. We denote

$$P := \{z \in \boldsymbol{C};\, 0 \leqslant \operatorname{Im} z < 1\} = \boldsymbol{R} + i[0,1),$$

$$X_k := \{z \in \boldsymbol{C}^n;\, z_k \in P,\ z_j \in \boldsymbol{R} \text{ for } j \neq k\},$$

$$X := \bigcup_{k=1}^{n} X_k = \boldsymbol{R}^n + i \operatorname{Im} X, \qquad W := \operatorname{conv} X = \boldsymbol{R}^n + i \operatorname{conv}(\operatorname{Im} X).$$

The main purpose of this note is to prove the following generalization of the Malgrange–Zerner theorem.

THEOREM. *Assume that a function* $f\colon X \to \boldsymbol{C}$ *satisfies the following conditions*:

(i) f *is separately holomorphic in* X, *that is, for each* $k = 1, \ldots, n$ *and for each fixed* $(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \in \boldsymbol{R}^{n-1}$, *the function* $f(z_1, \ldots \ldots, z_{k-1}, \cdot, z_{k+1}, \ldots, z_n) \in H(\operatorname{Int} P) \cap \mathscr{C}(P)$;

(ii) $f|_{\boldsymbol{R}^n} \in \mathscr{C}(\boldsymbol{R}^n)$;

(iii) f *is locally bounded in* X.

Then there exists exactly function $\tilde{f} \in H(\operatorname{Int} W) \cap \mathscr{C}(W)$ *such that* $\tilde{f}|_X = f$.

Assumptions (ii) and (iii) are independent, as will be shown in Section 2. In view of the results of Section 2, this generalization of the Malgrange–Zerner Theorem is sharp.

1. Proof of the generalization of the Malgrange-Zerner Theorem.

LEMMA A. *If a function* $f\colon X \to \boldsymbol{C}$ *satisfies the conditions*:

(i) f *is reparately holomorphic in* X,

(ii) $f|_{\boldsymbol{R}^n} \in \mathscr{C}(\boldsymbol{R}^n)$,

(iii) f *is bounded on* X,

then $f \in \mathscr{C}(X)$.

Proof of Lemma A. Consider the function $g(z) := f(z)\cdot \exp(-z^2)$, where $z^2 := z_1^2 + \ldots + z_n^2$ for $z \in X$. The function g has all the properties imposed on f and, moreover, g is uniformly continuous in $\operatorname{Re} X = \boldsymbol{R}^n$. We will prove that $g \in \mathscr{C}(X)$ which implies $f \in \mathscr{C}(X)$. We take a fixed point $\underset{0}{w} \in X$ and a number $\varepsilon > 0$. Let, for example, $\underset{0}{w} = (\underset{0}{x}, \underset{0}{z}) \in X_n = \boldsymbol{R}^{n-1} \times P \subset X$ and $w = (x, z) \in X_n \subset X$. We have

$$|g(\underset{0}{x}, \underset{0}{z}) - g(x, z)| \leqslant |g(\underset{0}{x}, \underset{0}{z}) - g(\underset{0}{x}, z)| + \exp U_x(z), \tag{A1}$$

where $U_x(z) := \ln|g(x, z) - g(\underset{0}{x}, z)|$, $z = t + iy \in P$. In view of (i) and (iii) we get

$$U_x \in \operatorname{Sh}(P) \quad \text{and} \quad U_x(z) \leqslant M \quad \text{for } z \in P. \tag{A2}$$

Since $g(\underset{0}{x}, \cdot) \in H(\operatorname{Int} P) \cap \mathscr{C}(P)$, we infer that

$$|g(\underset{0}{x}, \underset{0}{z}) - g(\underset{0}{x}, z)| < \varepsilon/2, \quad \text{where } |z - \underset{0}{z}| < r,\ p := 1 - r - \operatorname{Im} \underset{0}{z} > 0. \tag{A3}$$

Now we quote following versions of the lemma of Two Constants for Subharmonic Functions.

LEMMA. *If a function* $U \in \mathrm{Sh}(P)$ *is upper bounded,* $U(z) \leqslant m$ *for* $\mathrm{Im}\, z = 0$ *and* $U(z) \leqslant M$ *for* $\mathrm{Im}\, z = 1$, *then*

$$U(t+iy) \leqslant m+(M-m)y \quad \text{for } t+iy \in P.$$

(The proof of the lemma is analogous to the proof of the Maximum Principle for bounded holomorphic functions in a strip, see [3], p. 244.)

Since g is a uniformly continuous functions in $\boldsymbol{R}^n$, we have

$$|g(\underset{0}{x}, t)-g(x, t)| < m \quad \text{for } \|\underset{0}{x}-x\| < r',$$

where

$$m := \exp\left(p^{-1}[\ln \varepsilon/2 - M(1-p)]\right) < 1.$$

Hence

$$\text{(A4)} \qquad U_x(z) < \ln m \quad \text{for } \mathrm{Im}\, z = 0, \ \|\underset{0}{x}-x\| < r'.$$

Applying the above lemma to (A2) and (A4), we obtain

$$U_x(z) < p\cdot \ln m + M(1-p) \quad \text{for } \|\underset{0}{x}-x\| < r', \ 0 \leqslant \mathrm{Im}\, z < 1-p.$$

From the definitions of the numbers m and p we have

$$\text{(A5)} \qquad \exp U_x(z) < \varepsilon/2 \quad \text{when } \|\underset{0}{x}-x\| < r', \ 0 \leqslant \mathrm{Im}\, z < \mathrm{Im}\, \underset{0}{z}+r < 1.$$

(A1), (A3) and (A5) show that the function g is continuous at the point $\underset{0}{w}$.

LEMMA B (see for example [4]). *If a function* $f\colon \bar{X} \to \boldsymbol{C}$ *satisfies the conditions*

(i) f *is separately holomorphic in* X,

(ii) $f \in \mathscr{C}(\bar{X})$,

(iii) f *is bounded in* X,

then there exists exactly one function $\tilde{f} \in H(\mathrm{Int}\, W) \cap \mathscr{C}(\bar{W})$ *such that* $\tilde{f}|_{\bar{X}} = f$.

Proof of the theorem. We introduce the following notation:

$$P(h) := \{z \in \boldsymbol{C};\ 0 \leqslant \mathrm{Im}\, z < h < 1\},$$

$$X_m(h) := \{z \in \boldsymbol{C}^n;\ z_m \in P(h),\ z_j \in \boldsymbol{R},\ j \neq m\},$$

$$X(h) := \bigcup_{m=1}^{n} X_m(h), \qquad B(\underset{0}{z}, r) := \{z \in \boldsymbol{C};\ |z-\underset{0}{z}| < r\},$$

$$E(k, a) := \left\{z \in \boldsymbol{C};\ 0 < \mathrm{Arg}\frac{k+z}{k-z} < a < \pi\right\}$$

$$= B\left(-ik\,\mathrm{cotan}\, a,\ k\frac{1}{\sin a}\right) \cap \{z \in \boldsymbol{C};\ \mathrm{Im}\, z > 0\},$$

where $k = 1, 2, \ldots$

Fix $h \in (0,1)$ and put

$$E(k,h) := E(k, a_k), \quad \text{where } a_k = 2/h \tan^{-1} h/k,$$
$$E'(k,h) := \overline{E(k,h)} \setminus \{-k; k\}.$$

The following relations are evident:

$$(1) \qquad E(k,h) \subset E(k+1,h) \subset P(h) \subset \bigcup_{k+1}^{\infty} E'(k,h).$$

We denote

$$g_k(z) := k\,th\frac{za_k}{2} \quad \text{for } z \in \overline{P(h)}.$$

It is easy to see that g_k is a conformal mapping from $\mathrm{Int}P(h)$ onto $E(k,h)$ and a homeomorphism of $P(h)$ onto $E'(k,h)$. We write

$$G_k(z) := \big(g_k(z_1), \ldots, g_k(z_n)\big), \quad \text{where } z_j \in \overline{P(h)}.$$

Since $\lim\limits_{k\to\infty} g_k(z) = z$ for $z \in P(h)$, we obtain by (1)

$$(2) \quad G_k[\mathrm{conv}\,X(h)] \subset G_{k+1}[\mathrm{conv}\,X(h)] \subset \mathrm{conv}\,X(h) \subset \bigcup_{k=1}^{\infty} G_k[\mathrm{conv}\,X(h)].$$

Let $F_k(z) := (f \circ G_k)(z)$ for $z \in \overline{X(h)}$. By Lemma A and Lemma B, there exists exactly one function

$$\tilde{F}_k \in H\big(\mathrm{Int\,conv}\,X(h)\big) \cap \mathscr{C}\big(\mathrm{conv}\,\overline{X(h)}\big) \quad \text{such that } \tilde{F}_k|_{\overline{X(h)}} = F_k.$$

Putting $f_k(z) := (\tilde{F}_k \circ G_k^{-1})(z)$ for $z \in G_k\big(\mathrm{conv}\,X(h)\big)$, we see that

$$(3) \quad f_k \in H\big[\mathrm{Int}G_k\big(\mathrm{conv}\,X(h)\big)\big] \cap \mathscr{C}\big[G_k\big(\mathrm{conv}\,X(h)\big)\big] \quad \text{and} \quad f_k = f \text{ in } G_k\big(X(h)\big).$$

Therefore

$$f_k = f_m \quad \text{in } G_k\big(\mathrm{conv}\,X(h)\big) \quad \text{for } k \leqslant m$$

and we can define

$$f^h(z) := \bigcup_{k=1}^{\infty} f_k(z) \quad \text{for } z \in \mathrm{conv}\,X(h).$$

By (2) and (3), it is obvious that

$$f^h \in H\big(\mathrm{Int\,conv}\,X(h)\big) \cap \mathscr{C}\big(\mathrm{conv}\,X(h)\big) \quad \text{and} \quad f^h = f \quad \text{in } X(h).$$

Writing $\tilde{f} := \bigcup\limits_{0<h<1} f^h$, we hence obtain

$$\tilde{f} \in H(\mathrm{Int\,conv}\,X) \cap \mathscr{C}(\mathrm{conv}\,X) \quad \text{and} \quad \tilde{f}|_X = f.$$

Remark. Let D_k be a Jordan region in the complex plane C and let $E_k \subsetneq \partial D_k$ be a connected open subset of ∂D_k (in the sense of the topology of ∂D_k), $k = 1, \ldots, n$. Let us denote $Y_k := \{z \in C^n\colon z_k \in E_k \cup D_k,$

$z_j \in E_j, j \neq k\}$ and $Y := \bigcup_{k=1}^{n} Y_k$. If a function $f\colon Y \to \boldsymbol{C}$ is separately holomorphic in Y (in the sense of assumption (i)), continuous in the set $E_1 \times \ldots \times E_n$ and locally bounded in Y, then there exists exactly one function $\tilde{f} \in H(\operatorname{Int} V) \cap \mathscr{C}(V)$ such that $\tilde{f}|_Y = f$, where

$$V := \{z \in \bar{D}_1 \times \ldots \times \bar{D}_n;\ \operatorname{Im}[g_1(z_1) + \ldots + g_n(z_n)] < 1\}$$

and g_k is a conformal mapping from D_k onto the strip $\operatorname{Int} P$, such that $g_k(E_k) = \boldsymbol{R}$, $k = 1, \ldots, n$.

2. The independence of assumptions (ii) and (iii) in the Malgrange–Zerner Theorem.

EXAMPLE 1. Consider the sets

$$B := \{z \in \boldsymbol{C};\ |z| < 1\}, \qquad E := \{z \in \boldsymbol{C};\ |z| = 1, \operatorname{Re} z > 0\},$$
$$Y := E \times (B \cup E) \cup (B \cup E) \times E \subset \boldsymbol{C}^2.$$

We define a function $f\colon Y \to \boldsymbol{C}$ by the formula

$$f(z_1, z_2) := \begin{cases} \exp\left[-\operatorname{Log}(1-z_1)(1-z_2)\cdot \operatorname{Log}\dfrac{2+z_1 z_2}{3}\right] & \text{for } z_1 \neq 1,\ z_2 \neq 1, \\ 0 & \text{for } z_1 = 1 \text{ or } z_2 = 1, \end{cases}$$

where $(z_1, z_2) \in Y$, $-\frac{1}{2}\pi < \operatorname{Arg} z < \frac{1}{2}\pi$. It is not defficult to check that

$$f(\cdot, z_2) \in H(B) \cap \mathscr{C}(B \cup E) \quad \text{for } z_2 \in E,$$
$$f(z_1, \cdot) \in H(B) \cap \mathscr{C}(B \cup E) \quad \text{for } z_1 \in E,$$

f is bounded in Y, but $f \notin \mathscr{C}(E \times E)$ because $\lim_{t \to 0} f(e^{it}, e^{-it}) = 1 \neq f(1, 1) = 0$.

EXAMPLE 2. Preserving notation of the preceding example we define

$$g(z_1, z_2) := \begin{cases} \exp\left[-(z_1 + a)\operatorname{Log}^2 \dfrac{3+z_2}{1-z_2}\right] & \text{for } z_2 \neq 1, \\ 0 & \text{for } z_2 = 1, \end{cases}$$

where $(z_1, z_2) \in Y$, $0 < a < \frac{1}{2}$.

Obviously, g is separately holomorphic in Y, $g \in \mathscr{C}(E \times E)$, but g is not locally bounded in Y.

Namely, taking

$$z_2(t) = e^{it} \in E \quad \text{and}$$
$$z_1(t) = -a + \left(\operatorname{Re}\operatorname{Log}^2 \frac{3+z_2(t)}{1-z_2(t)}\right)^{-1} + a\, i \in B \quad \text{for } 0 < t < 1,$$

we have $|g(z_1(t), z_2(t))| \to \infty$ when $t \to 0^+$.

Thus g is an unbounded function in each neighbourhood of the point $(-a + ia, 1) \in B \times E$.

References

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