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Global asymptotic behavior of solutions of positively damped Liénard equations

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Abstract. If f and g are continuous on R to R, and satisfy $f(x) > 0$ for all x, $xg(x) > 0$ for all $x \neq 0$, we show that the equation $x''+f(x)x'+g(x) = 0$ may have solutions $x(t)$ such that $(x(t), x'(t))$ does not tend to $(0, 0)$ as $t \to \infty$. If the solutions are uniquely determined by their initial values, a general classification of the global asymptotic behavior of all solutions of such equations is given.

We consider the scalar second order differential equation

$$x''+f(x)x'+g(x) = 0 \tag{1}$$

where f and g are continuous on the set of real numbers R and satisfy $f(x) > 0$ for all x and $xg(x) > 0$ for $x \neq 0$.

It is well known that if either of the following conditions hold:

$$\text{(i)} \qquad \int_0^x g(s)ds \to +\infty \quad \text{as } x \to \infty \text{ and as } x \to -\infty,$$

$$\text{(ii)} \qquad \int_0^z f(s)ds \to +\infty \quad \text{as } x \to \infty, \qquad \to -\infty \quad \text{as } x \to -\infty,$$

then all solutions $x(t)$ satisfy $\big(x(t), x'(t)\big) \to (0, 0)$ as $t \to \infty$; cf. for example, [1]; pp. 224–227.

It also is easy to show that if $x^2(0)+x'^2(0)$ is sufficiently small, then $\big(x(t), x'(t)\big) \to (0, 0)$ as $t \to \infty$; in fact, the critical point $(0, 0)$ of the equivalent system

$$x' = y, \qquad y' = -f(x)y-g(x) \tag{2}$$

is uniformly asymptotically stable, and if either (i) or (ii) hold it is globally uniformly asymptotically stable; cf. [1] or any nonelementary textbook on ordinary differential equations.

This note concerns itself with the question: are there systems (2) for which there exist solutions which do not approach (0, 0) as $t \to \infty$? Our first result answers this question in the affirmative and shows that for a fairly large class of systems (2), there exist such solutions. We then give an analysis of the asymptotic behavior of all solutions of (2) where only the basic conditions on f and g as stated after (1) are assumed with the additional requirement that solutions of (2) be uniquely determined by their initial values. Our methods are fairly standard and primarily geometric.

Before stating our first result, we note that if $x(t)$ repesents the state of a physical system such as the displacement from equilibrium of a damped spring, the corresponding total energy $E = (x'(t))^2/2 + G(x(t))$, where

$$G(x) = \int_0^x g(v)dv$$

satisfies $dE/dt = -(x'(t))^2 f(x(t))$, from which we see that if $x(t)$ is not trivial solution of (1), E is strictly decreasing as t increases. One may thus refer to such a system as dissipative.

THEOREM 1. *Suppose there exists a constant $\alpha > 0$ such that one of the following holds*:

(a) $\sup\{f(x)e^{\alpha x}: x \geqslant 0\} < \infty$ *and* $\sup\{g(x)e^{2\alpha x}: x \geqslant 0\} < \infty$,

(b) $\sup\{f(x)e^{-\alpha x}: x \leqslant 0\} < \infty$ *and* $\sup\{-g(x)e^{-2\alpha x}: x \leqslant 0\} < \infty$.

Then there exist solutions $(x(t), y(t))$ of (2) which do not approach (0, 0) as $t \to \infty$. More precisely, if (a) holds there exists an $a_0^+ > 0$ such that any solution $(x(t), y(t))$ with $x(0) = 0$, $y(0) > a_0^+$ satisfies $|x(t)| \to \infty$ as $t \to \infty$. If (b) holds, an analogous constant $a_0^- < 0$ exists.

Proof. Fixed $a > 0$, $b > 0$, we denote by $C(a, b)$ the graph of $y = ae^{-\alpha x} + b$, $x \geqslant 0$, and by Γ^+ the positive semi-orbit of a solution $(x(t), y(t))$ of (2) with $x(0) = 0$, $y(0) > a + b$. As long as $y(t) > 0$, $x(t)$ increases with t; so either Γ^+ intersects $C(a, b)$ at a first point (x_0, y_0), $x_0 > 0$, $y_0 > 0$, or it does not. If not, the conclusion of our theorem follows since the only point any solution of (2) can approach as $t \to \infty$ is (0,0).

So suppose $(x_0, y_0) \in C(a, b) \cap \Gamma^+$ where $(x(t_0), y(t_0)) = (x_0, y_0)$ and $y(t) > ae^{-x(t)} + b$ for $0 \leqslant t < t_0$. Clearly the slope of $C(a, b)$ must not be less than the slope of Γ^+ at (x_0, y_0); i.e.,

$$-\alpha a e^{-\alpha x_0} \geqslant -f(x_0) - g(x_0)/(ae^{-\alpha x_0} + b) \geqslant -f(x_0) - g(x_0)e^{\alpha x_0}/a;$$

$$\alpha a^2 \leqslant af(x_0)e^{\alpha x_0} + g(x_0)e^{2\alpha x_0} \leqslant af_0^+ + g_0^+,$$

where $f_0^+ = \sup\{f(x)e^{\alpha x}: x \geqslant 0\}$, $g_0^+ = \sup\{g(x)e^{2\alpha x}: x \geqslant 0\}$. Clearly, for $a > a_0^+ \equiv [f_0^+ + (f_0^{+2} + 4g_0^+\alpha)^{1/2}]/2\alpha$ this is impossible, and the conclusion of our theorem follows.

Using the graph of $y = -(ae^{\alpha x}+b)$, $x \leqslant 0$ and a completely similar argument, we obtain the existence of $a_0^- < 0$ such that for any solution $(x(t), y(t))$ with $x(0) = 0$, $y(0) < a_0^-$ it follows that $x(t) \to -\infty$ as $t \to \infty$.

COROLLARY. *Let hypotheses* (a) *of Theorem* 1 *hold and let* a_0^+ *be as in the conclusion of this theorem. Then for any* $b \geqslant 0$ *and any solution* $(x(t), y(t))$ *with* $x(0) = 0$, $y(0) > a_0^+ + b$, *it follows that*

$$b \leqslant \lim_{t \to \infty} y(t) \leqslant a_0 + b.$$

An analogous result holds if (b) *is satisfied.*

A proof of this corollary follows easily from the details of the proof of the theorem.

We now discuss the asymptotic behavior of all solutions of (2) without conditions such as in Theorem 1. We first give a result for a certain large class of solutions; i.e., solutions $(x(t), y(t))$ such that $x(0) = 0$, $y(0) \geqslant 0$ and indicate how extensions to other solutions can be made.

THEOREM 2. *Suppose the solutions of* (2) *are uniquely determined by their initial values. Then there exist* a_0 *and* a_1, $0 < a_0 \leqslant a_1 \leqslant \infty$, *such that if* $(x(t), y(t))$ *solves* (2) *with* $(x(0), y(0)) = (0, a)$ *then*

(i) $a \geqslant a_1$ *implies* $y(t) > 0$, $t \geqslant 0$ *and*

$$\lim_{t \to \infty} (x(t), y(t)) = (+\infty, L(a)).$$

(ii) $a_0 \leqslant a < a_1$ *implies there exists* $t_1(a) > 0$, $L(a) \leqslant 0$ *such that* $y(t) > 0$, $0 \leqslant t < t_1(a)$, $x(t_1(a)) > 0$, $y(t_1(a)) = 0$, $y(t) < 0$ *for* $t > t_1(a)$, *and*

$$\lim_{t \to \infty} (x(t), y(t)) = (-\infty, L(a)).$$

(iii) $0 \leqslant a < a_0$ *implies* $\lim_{t \to \infty} (x(t), y(t)) = (0, 0)$.

Note.
If $a_1 = \infty$, only cases (ii) and (iii) can arise;
if $a_0 = a_1$, only cases (i) and (iii) can arise;
if $a_0 = a_1 = \infty$, only case (iii) can arise; in this case $(0, 0)$ in fact is globally asymptotically stable.

Proof. If for every solution $(x(t), y(t))$ of (2) as in our hypotheses, $\lim_{t \to \infty} (x(t), y(t)) = (0, 0)$, we take $a_0 = a_1 = \infty$, and we are done; the fact that in this case all solutions are attracted to $(0, 0)$ as $t \to \infty$ is easy to verify.
Suppose this is not the case. Then define

$$a_1 = \inf\{a > 0\colon\ y(t) > 0 \text{ for } t \geqslant 0\}.$$

If $a_1 < \infty$ and $a > a_1$, $x(t)$ is increasing for $t \geqslant 0$ and $x(t) \to +\infty$ as $t \to \infty$; if not, $(x(t), y(t))$ would approach a critical point of (2) distinct from (0, 0), and none such exist. If $a_1 = \infty$ we go, as noted above, to cases (ii) and (iii). Since (0, 0) is asymptotically stable, $a_1 > 0$. Suppose $0 < a < a_1$. Then there exists $t_1 = t_1(a) > 0$ such that $x(t_1) > 0$ $y(t_1) = 0$, $y(t) > 0$, $0 \leqslant t < t_1$. By the continuity of solutions of (2) in terms of their initial values, it follows that if $a_1 < \infty$, the solution $(x(t), y(t))$ with $(x(0), y(0)) = (0, a_1)$ satisfies $y(t) > 0$ for $t \geqslant 0$, $x(t) \to \infty$ as $t \to \infty$.

If for all a, $0 < a < a_1$, $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$, we take $a_0 = a_1$; (iii) then holds and we are done.

Suppose there exists a, $0 < a < a_1$, so that this last last condition does not hold. We claim that in this case, $x(t) \to -\infty$ and $y(t) < 0$ for $t > t_1$. Suppose not; then there exists a $t_2 = t_2(a) > t_1$ such that $y(t_2) = 0$, $x(t_2) < 0$, and $y(t) < 0$ for $t_1 < t < t_2$. It follows that in this case there exists $t_3 > t_2$ such that $y(t_3) > 0$, $x(t_3) = 0$, and $y(t) > 0$, $t_2 < t < t_3$. But it must then follow that $y(t_3) < a$; to see this consider $V(x, y) = y^2/2 + G(x)$ where $G(x) = \int_0^x g(s)ds$; then we easily get

$$\frac{dV}{dt}(x(t), y(t)) = -f(x(t))y^2(t),$$

and integrating this from 0 to t_3 yields

$$y^2(t_3) < y^2(0).$$

Hence using the Poincaré–Bendixson theorem, it follows that $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$. Thus we have $a_0 = a_1$ again and we are done.

So as claimed above, there exists a, $0 < a < a_1$ such that $x(t) \to -\infty$, as $t \to \infty$, and $y(t) < 0$ for $t > t_1$. Define

$$a_0 = \inf\{a > 0\colon\ y(t) < 0 \text{ for } t > t_1\}.$$

Clearly $a_0 < a_1$, and our proof is complete. The cases $a_0 < a_1 < \infty$ and $a_0 = a_1 < \infty$ are illustrated respectively in Fig. 1 and Fig. 2. In the following

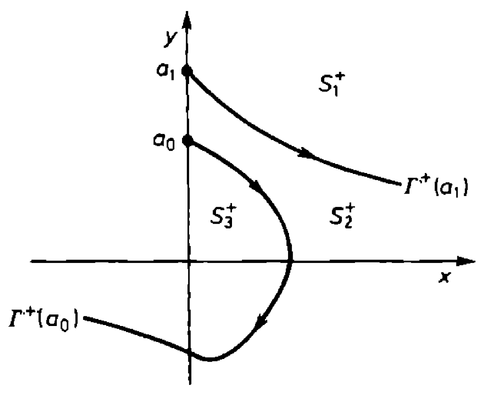

Fig. 1

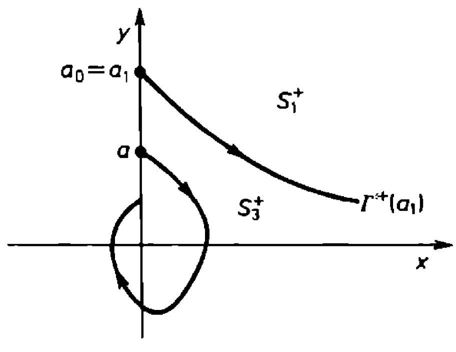

Fig. 2

discussion we always assume as in Theorem 2 that solutions of (2) are uniquely determined by their initial values.

Let $\Gamma^+(a)$ be the positive semi-orbit of the solution $(x(t), y(t))$ of (2) such that $(x(0), y(0)) = (0, a)$, $a > 0$. Define

$$S_2^+ = \bigcup\{\Gamma^+(a)\colon a \geqslant a_1\}, \quad S_1^+ = \bigcup\{\Gamma^+(a)\colon a_0 \leqslant a < a_1\},$$
$$S_0^+ = \bigcup\{\Gamma^+(a)\colon 0 < a < a_0\}.$$

These are mutually disjoint regions of the (x, y) plane; cf. Fig. 3. Note that S_1^+ and S_2^+ may be empty, but S_0^+ is always nonempty. If $S^+ = S_0^+ \cup S_1^+ \cup S_2^+$, it follows easily that if $(x(t_0), y(t_0)) \in S^+$, then one and only one of 3 possibilities exist for the solution $(x(t), y(t))$:

(a) $(x(t), y(t)) \to (+\infty, L)$ as $t \to \infty$, $L \geqslant 0$;

(b) $x(t_1) > 0$, $y(t_1) = 0$ for some $t_1 > t_0, y(t) < 0$ for $t > t_1$ and $(x(t), y(t)) \to (-\infty, -L)$, $L \geqslant 0$;

(c) $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$.

If we consider the negative semi-orbits $\Gamma^-(a)$ of solutions $(x(t), y(t))$ such that $(x(0), y(0)) = (0, a)$; i.e.,

$$\Gamma^-(a) = \{(x(t), y(t))\colon t \leqslant 0\}$$

we can extend our considerations to parts of the (x, y) plane outside of S^+. Consider first $\Gamma^-(a_1)$ with a_1 as in Theorem 2; either it intersects the negative x-axis or not. If it does, it must also intersect the negative y-axis and stay below the x-axis as $t \to -\infty$, for if this were not the case, it would intersect the positive x-axis and subsequently the positive y-axis at a point $(0, a)$ with $a < a_1$; this would again lead to a contradiction using $V(x, y)$ as above.

We thus have the two remaining possibilities: either $\Gamma^-(a_1)$ is entirely above the x-axis or it intersects the negative x-axis and remains below it for decreasing t. In the latter case, using the definition of a_1 and a_0 we see easily

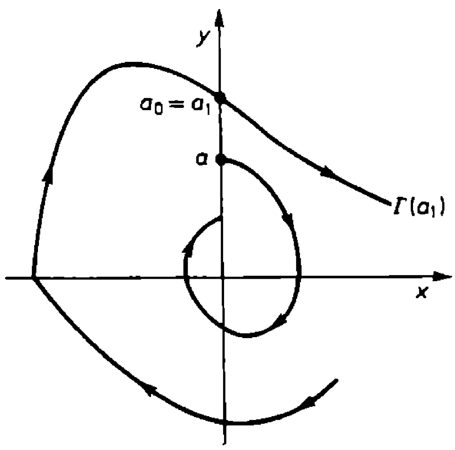

Fig. 3

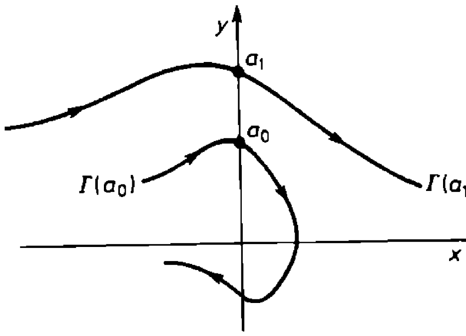

Fig. 4

that we must have $a_0 = a_1$, i.e., all orbits $\Gamma^+(a)$, $a < a_1$, are attracted to (0, 0) as $t \to \infty$. In the former case, we may have $a_0 < a_1$. (cf. Figs. 3, 4.) We also note that if $a_0 < a_1$, the first of these two possibilities must hold.

Thus in any case, $\Gamma(a_1) = \Gamma^+(a_1) \cup \Gamma^-(a_1)$ bounds a portion of the (x, y) plane, namely that portion not containing (0, 0), which consists of orbits which do not approach (0, 0) as $t \to \infty$; in fact, in case $a_0 = a_1$, it consists of all such orbits while the remaining orbits do approach (0, 0) as $t \to \infty$. In case $a_0 < a_1$, this maximal region of nonattraction is the portion of the plane bounded by $\Gamma(a_0)$ which does not contain (0, 0).

If we now define $\Gamma^-(a)$ to be the negative semi-orbit of the solution $(x(t), y(t))$ with $(x(0), y(0)) = (0, a)$, $a > 0$, we can consider regions in the (x, y) plane outside S^+. Again various cases arise.

We note that any solution $x(t)$ of (1) which does not approach 0 as $t \to \infty$ can have at most two zeros for $t \in R$.

We conclude with some observations about solutions with orbits $\Gamma(a)$, $a \geqslant a_1$. From Theorem 2, we see that for each corresponding solution $(x(t), y(t))$, we have $\lim_{t \to \infty} y(t) = L(a) \geqslant 0$.

It follows easily that $L(a)$ is nondecreasing for $a \geqslant a_1$. It also follows, using a standard argument involving continuous dependence of solutions on initial values, that the set $\{L(a)\colon a \geqslant a_1\}$ must be dense in $[L(a_1), \infty)$. It follows then that $L(a)$ is in fact continuous for $a \geqslant a_1$.

Some interesting and apparently open questions arise regarding this function $L(a)$;

(i) Can $L(a_1) > 0$? If so, under what conditions will $L(a_1) = 0$?

(ii) Is $L(a)$ strictly increasing for $a \geqslant a_1$? Again, if not, are there conditions under which it is?

Reference

[1] F. Brauer and J. A. Nohel, *Qualitative Theory of Ordinary Differential Equations*, W. A. Benjamin, Inc., New York, N.Y., 1969.

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