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Approximate determination of eigenvalues and eigenvectors of self-adjoint operators

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Abstract. A method (1), (2) for calculation of eigenvalues and eigenvectors of self-adjoint operators is investigated. It is shown that the sequence (μ_n) defined by (1), (2) converges to λ_1, where the least upper bound of the spectrum $\sigma(A)$ of A is not necessarily an eigenvalue of A. If λ_1 is an eigenvalue of A (necessarily isolated), then (u_n) converges to the corresponding eigenvector of A.

Let X be a real Hilbert space with the scalar product (,), and let $A\colon X \to X$ be a linear self-adjoint positive operator defined on X. By saying that A is positive we mean that $(Au, u) > 0$ for all $u \in X$, $u \neq 0$, and that $(Au, u) = 0$ implies $u = 0$. Since A is self-adjoint and is defined on all of X, A is bounded by the closed graph theorem. The spectrum $\sigma(A)$ of A lies in the segment $[m, \lambda_1]$, where $m = \inf\{(Au, u)\colon \|u\| = 1\}$, $m \geqslant 0$, and $\lambda_1 = \sup\{(Au, u)\colon \|u\| = 1\}$. We introduce an iterative method of calculation of eigenvalues and eigenvectors of A as follows. Let R denote the set of all real numbers, and let $u_0 \in X$ be an arbitrary element such that $u_0 \neq 0$. Define a function $f\colon R \times X \to R$ by $f(\tau, u) = \|Au - \tau u\|^2$, $\tau \in R$, $u \in X$. Let μ_1 denote that value τ at which the function $\tau \to f(\tau, u_0)$, $\tau \in R$, assumes its minimum on R, i.e., $f(\mu_1, u_0) = \min\{\|Au_0 - \tau u_0\|^2\colon \tau \in R\}$. The condition $f'_\tau(\mu_1, u_0) = 0$ implies $\mu_1 = (Au_0, u_0)\|u_0\|^{-2}$. Put $u_1 = \mu_1^{-1}Au_0$. Since A is positive, $\mu_1 > 0$ and $u_1 \neq 0$. Having μ_1 and u_1, the condition $f'_\tau(\mu_2, u_1) = 0$ gives $\mu_2 = (Au_1, u_1)\|u_1\|^{-2}$. Put $u_2 = \mu_2^{-1}Au_1$. By repeating this procedure, we obtain the following iterative process

$$u_{n+1} = \mu_{n+1}^{-1}Au_n, \tag{1}$$

$$\mu_{n+1} = (Au_m, u_n)\|u_n\|^{-2}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $\mu_n > 0$, $u_n \neq 0$ for all n. The procedure (1), (2) is similar to that of Birger; the latter runs as follows:

$$y_{n+1} = a_{n+1}Ay_n, \quad a_{n+1} = (Ay_n, y_n)\|Ay_n\|^{-2}. \tag{3}$$

I. A. Birger [2] suggested this method without any motivation of convergence and estimates. He has found (applying it to engineering problems) some advantages of this method in comparison with the other ones; see also G. I. Marchuk [11], where a similar observation has been done on the ground of physical ideas.

The proofs of convergence in this case of the processes under discussion for compact and symmetrizable operators were given in [5], [6], while the estimates for those methods were derived in [7]. I. Marek [9], [10] proved the convergence of those procedures in Banach spaces for linear bounded operators having a dominated eigenvalue. W. V. Petryshyn [13] deduced from his very general theorems the convergence of the method (1), (2) for unbounded linear operators having the dominated eigenvalue. Results related to that of [13] have been obtained by R. I. Andrushkiw [1].

The purpose of this paper is to show that the sequence (μ_n) also converges in the case when the least upper bound λ_1 of the spectrum $\sigma(A)$ of the self-adjoint operator A is not an eigenvalue of A and to get rid of the condition that λ_1 is an isolated eigenvalue of A.

We shall use in the sequel two following lemmas.

Lemma 1. *Let X be a real Hilbert space and $A: X \to X$ a linear positive self-adjoint operator on X. Then the sequence (μ_n) defined by* (1), (2), *where $\mu_0 \neq 0$, is monotone increasing and convergent.*

Proof. First of all, note that $0 < \mu_n \leqslant \|A\|$ and $\|u_n\| \leqslant \|u_{n+1}\|$ for each n. The last relation together with (1), (2) show that $(Au_n, u_n) \leqslant (Au_n, u_{n+1})$, while the Schwarz inequality and the positiveness of A imply

$$(Au_n, u_n) \leqslant (Au_{n+1}, u_{n+1}), \quad n = 0, 1, 2, \ldots \tag{4}$$

We have $\mu_n(Au_n, u_n) = \mu_{n+1}(Au_{n-1}, u_n)$. Indeed,

$$\begin{aligned}\mu_{n+1}(Au_{n-1}, u_n) &= (Au_n, u_n)\|u_n\|^{-2}(Au_{n-1}, u_n)\\ &= (Au_n, u_n)\|u_n\|^{-2}\|u_n\|^2\mu_n.\end{aligned}$$

Using the last equality, the Schwarz inequality and (4), we obtain $\mu_n \leqslant \mu_{n+1}$ for each n. Hence there exists $\lim_{n\to\infty}\mu_n = \mu$ and $\mu_1 \leqslant \mu \leqslant \lambda_1$.

Lemma 2. *Under the assumptions of Lemma* 1 *the sequence* $(\|u_n\|)$, *where the u_n are defined by* (1), (2), *is bounded.*

Proof. Put $v_n = u_n/\|u_n\|$; then $v_{n+1} = a_{n+1}Av_n$, where $a_{n+1} = \|u_n\|^2\|u_{n+1}\|^{-1}(Au_n, u_n)^{-1}$. Then $(v_n, v_{n+1}) = \|u_n\|\cdot\|u_{n+1}\|^{-1}$, because $(u_n, u_{n+1}) = \|u_n\|^2$. In view of the last relation, it is sufficient to prove that the product $\prod_{n=0}^{\infty}(v_n, v_{n+1})^{-1}$ converges. Now, we have

$$\begin{aligned}Av_n = Au_n\cdot\|u_n\|^{-1} &= \mu_{n+1}\|u_{n+1}\|\|u_n\|^{-1}v_{n+1}\\ &= \mu_{n+1}(v_n, v_{n+1})^{-1}v_{n+1}.\end{aligned} \tag{5}$$

Taking (5) into account we get

$$(6)\qquad \begin{aligned}(v_{n-1}, v_{n+1}) &= \mu_{n+1}^{-1}(v_n, v_{n+1})(v_{n-1}, Av_n)\\ &= \mu_{n+1}^{-1}(v_n, v_{n+1})(Av_{n-1}, v_n)\\ &= \mu_n\mu_{n+1}^{-1}(v_n, v_{n+1})(v_{n-1}, v_n)^{-1}.\end{aligned}$$

Again by (5)

$$\begin{aligned}0 &\leqslant \big(v_n - v_{n-1}, A(v_n - v_{n-1})\big)\\ &= \big(v_n - v_{n-1}, \mu_{n+1}(v_n, v_{n+1})^{-1}v_{n+1} - \mu_n(v_{n-1}, v_n)^{-1}v_n\big)\\ &= \mu_{n+1} - \mu_{n+1}(v_n, v_{n+1})^{-1}(v_{n-1}, v_{n+1}) - \mu_n(v_{n-1}, v_n)^{-1} + \mu_n.\end{aligned}$$

These relations and (6) imply

$$\mu_{n+1} + \mu_n - 2\mu_n(v_{n-1}, v_n)^{-1} \geqslant 0.$$

Hence

$$\begin{aligned}1-(v_{n-1}, v_n) &\leqslant 1 - 2\mu_n(\mu_n + \mu_{n+1})^{-1}\\ &= (\mu_{n+1} - \mu_n)(\mu_n + \mu_{n+1})^{-1} \leqslant \frac{1}{2\mu_1}(\mu_{n+1} - \mu_n).\end{aligned}$$

Therefore

$$(8)\qquad \sum_{n=1}^{\infty}[1-(v_{n-1}, v_n)] \leqslant \sum_{n=1}^{\infty}\frac{1}{2\mu_1}(\mu_{n+1} - \mu_n).$$

The sequence (μ_n) is convergent by Lemma 1; thus the series on the left of (8) converges and this proves our lemma.

The proofs of theorems which follow depend on Lemmas 1, 2 and the spectral analysis of self-adjoint operators. Let $\{E_\lambda\}$ be the spectral resolution of the self-adjoint linear operator $A\colon X \to X$. We show that the sequence (μ_n) converges to λ_1 for "almost all" starting approximations u_0.

THEOREM 1. *Let X be a real Hilbert space and $A\colon X \to X$ a linear self-adjoint positive operator on X. Suppose that $u_0 \in X$ is a vector such that $E_\lambda u_0 \neq u_0$ for each $\lambda < \lambda_1$. Then $\mu_n \nearrow \lambda_1$, where (μ_n) is defined by* (1) *and* (2).

Proof. By Lemma 1, $\mu_n \nearrow \mu$ and $\mu \leqslant \lambda_1$. We have to prove $\mu = \lambda_1$. Suppose $\mu < \lambda_1$ and put $a = \frac{1}{2}(\mu + \lambda_1)$; then $\mu_n \leqslant \mu < a < \lambda_1$ for $n \geqslant 1$. Put $\beta = [a, \lambda_1]$, $\alpha = a\mu^{-1}$. Then for $n \geqslant 0$ we have

$$\begin{aligned}\|E(\beta)u_{n+1}\|^2 &= \big(E(\beta)u_{n+1}, u_{n+1}\big) = \mu_{n+1}^{-2}\int_a^{\lambda_1}\lambda^2 d(E_\lambda u_n, u_n)\\ &\geqslant \alpha^2\int_a^{\lambda_1} d(E_\lambda u_n, u_n) = \alpha^2\|E(\beta)u_n\|^2.\end{aligned}$$

Continuing this process, we get

$$\|E(\beta)u_n\|^2 \geqslant a^{2n}\|E(\beta)u_0\|^2$$

for all $n \geqslant 1$. Since $E(\beta)u_0 \neq 0$ by our assumption and $a > 1$, we obtain $\|E(\beta)u_n\| \to +\infty$ as $n \to \infty$, which contradicts the boundedness of $\|u_n\|$. The theorem is proved.

Remark 1. Let $A: X \to X$ be a linear self-adjoint operator on a Hilbert space X and assume that the starting approximation u_0 of the procedure (1), (2) has the form $u_0 = a_0x_0 + Z_0$, where $x_0 \in \ker(A-\lambda_1)$, $\|x_0\| = 1$, $Z_0 \in \ker(A-\lambda_1)^{\perp}$ and $a_0 > 0$. Then the sequence (u_n) defined by (1), (2) can be expressed as $u_n = a_nx_0 + Z_n$, where $Z_n \in \ker(A-\lambda_1)^{\perp}$, $a_n > 0$, $n \geqslant 1$.

Indeed, let us assume that the representation of (u_n) is valid for $n = i$, i.e., let $u_i = a_ix_0 + Z_i$, where $Z_i \in \ker(A-\lambda_1)^{\perp}$ and $a_i > 0$. Then $u_{i+1} = \mu_{i+1}^{-1}Au_i = a_{i+1}x_0 + Z_{i+1}$, where $a_{i+1} = \mu_{i+1}^{-1}a_i\lambda_1$, $Z_{i+1} = \mu_{i+1}^{-1}AZ_i$. We have $(x, Z_{i+1}) = 0$ for each $x \in \ker(A-\lambda_1)$, because $Z_i \in \ker(A-\lambda_1)^{\perp}$ and this subspace is invariant with respect to A. Thus the assertion of Remark 1 is valid for $n = i+1$ and hence for all n.

Theorem 2. *Let A be a positive linear self-adjoint operator from a real Hilbert space X into X. Suppose that λ_1 is necessarily an isolated point of $\sigma(A)$ of A and that $E_\lambda u_0 \neq u_0$ for each $\lambda < \lambda_1$ and any starting approximation $u_0 \in X$. If u_0 is of the form $u_0 = a_0x_0 + Z_0$, where $\|x_0\| = 1$, $x_0 \in \ker(A-\lambda_1)$, $Z_0 \in \ker(A-\lambda_1)^{\perp}$ and $a_0 > 0$, then $\|u_n - Nx_0\| \to 0$ as $n \to \infty$, where $N = \sup_{n=1,2,\ldots}\|u_n\| < \infty$.*

Proof. Suppose $u_0 = a_0x_0 + Z_0$, where

$$x_0 \in \ker(A-\lambda_0), \quad \|x_0\| = 1, \quad Z_0 \in \ker(A-\lambda_1)^{\perp}, \quad a_0 > 0.$$

Assume $m < \lambda_1$ (if not, then, by the spectral theorem, $A = mI$, where I denotes the identity mapping in X). Taking λ such that $m < \lambda < \lambda_1$, we have

$$\lambda_1\|u_n\|^2 - (Au_n, u_n) \geqslant (\lambda_1 - \lambda)\|E_\lambda u_n\|^2, \quad n = 0, 1, 2, \ldots$$

Indeed,

$$\begin{aligned}\lambda_1\|u_n\|^2 - (Au_n, u_n) &= \int_m^{\lambda_1} (\lambda_1 - t)\,d(E_tu_n, u_n)\\ &\geqslant \int_m^{\lambda} (\lambda_1 - t)\,d(E_tu_n, u_n) \geqslant (\lambda_1 - \lambda)\int_m^{\lambda} d(E_tu_n, u_n)\\ &= (\lambda_1 - \lambda)\|E_\lambda u_n\|^2.\end{aligned}$$

Hence $\|E_\lambda u_n\| \to 0$ if $n \to \infty$, by Theorem 1. Thus $\|(I - E_\lambda)u_n\| \to N$ as $n \to \infty$, because $\|u_n\| \nearrow N$. Put $P_0 = I - E_{\lambda_1 - 0}$. We have $E_\lambda \to E_{\lambda_1 - 0}$ if $\lambda \to \lambda_1$

in the strong point operator topology. Therefore $I - E_\lambda \to P_0$ strongly on X as $\lambda \to \lambda_1$. Moreover, P_0 is the projection of X onto $\ker(A - \lambda_1)$. By Remark 1, the sequence (u_n) determined by (1), (2) is of the form $u_n = a_n x_0 + Z_n$, where $Z_n \in \ker(A - \lambda_1)^\perp$, $a_n > 0$. Hence $P_0 u_n = a_n x_0$ and $a_n^2 = \|P_0 u_n\|^2$. Now, λ_1 is isolated, i.e. $\sigma(A) - \{\lambda_1\} \subset [m, M]$ for some $M > 0$. If $M < \lambda < \lambda_1$, then

$$|a_n - N| = \big|\|P_0 u_n\| - N\big| \leqslant \big|\|P_0 u_n\| - \|(I - E_\lambda) u_n\|\big| + \big|\|(I - E_\lambda) u_n\| - N\big|$$
$$= \big|\|(I - E_\lambda) u_n\| - N\big|$$

and hence $a_n \to N$. It follows from the equality $\|u_n\|^2 = a_n^2 + Z_n^2$ that $\|Z_n\| \to 0$ if $n \to \infty$. The equality $\|u_n - N x_0\|^2 = (a_n - N)^2 + \|Z_n\|^2$ and the above conclusions finish the proof.

Similar results to these obtained in this note are valid also for the Birger method and Kellogg method. Let us remark that the method (3) for non-linear operators has been investigated in [3] and that the processes (1), (2) and (3) are related to the Schwarz constants method [4]. An extensive bibliography concerning the various methods of calculation of eigenvalues and eigenvectors of linear continuous operators is given in [8], [13], [14].

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