On the existence of a fundamental solution for a parabolic differential equation with unbounded coefficients

by P. BESALA (Gdańsk)

Abstract. We prove the existence of a fundamental solution for a linear second order parabolic equation under assumptions which allow the coefficients to grow to infinity in various ways (see Assumption II below).

The fundamental solution for equations with unbounded coefficients was treated in [1] and by the same method the result was improved in [2]. Our result extends those of [1], [2] in two directions. Namely, the assumptions concerning the growth of the coefficients are considerably less restrictive and, moreover, the assumptions on the regularity of the coefficients are weakened by eliminating the Hölder continuity of their derivatives and replacing the classical derivatives by weak ones.

The method used here is based on that applied in papers [1], [2]. The most essential change is introduced in the proof of the boundedness of the sequence of Green functions. Owing to this change we do not make use, contrary to [1], [2], of the existence of the Green function of the adjoint operator.

1. Denote by $x = (x_1, \ldots, x_n)$ points of the Euclidean n-space R^n $(n \geqslant 1)$ and by t points of the interval $\langle 0, T \rangle$, $0 < T < +\infty$. Let $S = (0, T) \times R^n$, $\bar{S} = \langle 0, T \rangle \times R^n$. We consider the equation

$$(1.1) \qquad L(u) \equiv \sum_{i,j=1}^{n} a_{ij}(t, x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t, x) u_{x_j} + c(t, x) u - u_t = 0$$

for $(t, x) \in S$.

A function $\Gamma(t, x; \tau, \xi)$ defined in $D: 0 \leqslant \tau < t \leqslant T$; $x, \xi \in R^n$ is said to be a *fundamental solution* of (1.1) if it has the following two properties:

$1°$ $\Gamma(t, x; \tau, \xi)$ considered as a function of (t, x) for any fixed (τ, ξ) $\in \langle 0, T \rangle \times R^n$ has continuous derivatives $\Gamma_t, \Gamma_{x_i}, \Gamma_{x_i x_j}$ $(i, j = 1, \ldots, n)$ and satisfies (1.1) in $(\tau, T \rangle \times R^n$.

$2°$ for any continuous function $\varphi(x)$ with compact support in R^n we have

$$\lim_{(t,x) \to (\tau+, \bar{x})} \int_{R^n} \Gamma(t, x; \tau, \xi) \varphi(\xi) d\xi = \varphi(\bar{x}).$$

Throughout the paper we make the following assumptions concerning the coefficients of (1.1):

I. a_{ij}, b_j, c are *Hölder continuous with respect to* (t, x) *on every compact subset of* $\bar{S}$, $a_{ij} = a_{ji}$, *and for each* $t \epsilon (0, T\rangle$ *there exist weak derivatives* $(a_{ij})_{x_i}$, $(a_{ij})_{x_i x_j}$, $(b_j)_{x_j}$ *in* R^n (¹) $(i, j = 1, \ldots, n)$. *Moreover, there is a constant* $\varkappa > 0$ *such that*

$$\sum_{i,j} a_{ij}(t, x) \lambda_i \lambda_j \geqslant \varkappa |\lambda|^2$$

for $(t, x) \epsilon S$, $\lambda = (\lambda_1, \ldots, \lambda_n) \epsilon R^n$, $|\lambda|$ *being the Euclidean norm of* λ.

II. *There exists a function* $h(t, x) \epsilon C^2(\bar{S})$ *with Hölder continuous second order* x-*derivatives on every compact subset of* $\bar{S}$ *and such that* $h(t, x) > 0$ *on* $\bar{S}$,

$$(1.2) \qquad L(h) + \eta h \Big[\sum_{i,j} (a_{ij})_{x_i x_j} - \sum_j (\tilde{b}_j)_{x_j} \Big] \leqslant 0$$

for each $t \epsilon (0, T\rangle$, *almost all* $x \epsilon R^n$ *and for* $\eta = 0$, $\eta = 1$, *where*

$$(1.3) \qquad \tilde{b}_j = 2h^{-1} \sum_i a_{ij} h_{x_i} + b_j.$$

Note that if (1.2) holds for $\eta = 0$ and $\eta = 1$, it also holds for each $\eta \epsilon \langle 0, 1 \rangle$. If, in particular, the coefficients satisfy the growth conditions assumed in [1] or [2], Assumption II holds true.

2. THEOREM 1. *If Assumptions* I, II *are satisfied, then there exists a fundamental solution* $\Gamma(t, x; \tau, \xi)$ *of equation* (1.1) *which satisfies the inequalities*

$$(2.1) \qquad 0 \leqslant \Gamma(t, x; \tau, \xi) \leqslant C(t-\tau)^{-n/2} h(t, x)/h(\tau, \xi) \quad in\ D,$$

$$(2.2) \qquad \int_{R^n} \Gamma(t, x; \tau, \xi) h(\tau, \xi) d\xi \leqslant h(t, x) \quad for\ (t, x) \epsilon (\tau, T\rangle \times R^n,$$

$$(2.3) \qquad \int_{R^n} \Gamma(t, x; \tau, \xi)/h(t, x) dx \leqslant 1/h(\tau, \xi) \quad for\ (\tau, \xi) \epsilon \langle 0, t) \times R^n,$$

C *being a positive constant depending only on* n *and* $\varkappa$.

In order to prove Theorem 1 we first prove a similar theorem for a transformed equation. Namely, set $u(t, x) = v(t, x) h(t, x)$ into (1.1). We obtain for v the equation

$$(2.4) \qquad F(v) \equiv \sum_{i,j} a_{ij} v_{x_i x_j} + \sum_j \tilde{b}_j v_{x_j} + \tilde{c} v - v_t = 0,$$

_____

(¹) A function $f_{x_i}(x)$ locally summable in R^n is said to be the *weak derivative of a function* $f(x)$ *locally summable in* R^n if for any function $\psi(x)$ of class C^1 and with compact support in R^n we have

$$\int_{R^n} f \psi_{x_i} dx = - \int_{R^n} f_{x_i} \psi dx.$$

where $\tilde{b}_j$ is given by (1.3) and

$$(2.5) \qquad \tilde{c} = h^{-1}L(h).$$

Evidently, if $\gamma(t, x; \tau, \xi)$ is a fundamental solution of equation (2.4), then

$$(2.6) \qquad \Gamma(t, x; \tau, \xi) = \gamma(t, x; \tau, \xi)h(t, x)/h(\tau, \xi)$$

is a fundamental solution of (1.1). Thus Theorem 1 is an immediate consequence of the following

THEOREM 2. *If Assumptions* I, II *are satisfied, then there exists a fundamental solution* $\gamma(t, x; \tau, \xi)$ *of equation* (2.4) *and satisfies the inequalities*

$$(2.7) \qquad 0 \leqslant \gamma(t, x; \tau, \xi) \leqslant C(t-\tau)^{-n/2} \quad in \ D,$$

$$(2.8) \qquad \int_{R^n} \gamma(t, x; \tau, \xi) d\xi \leqslant 1,$$

$$(2.9) \qquad \int_{R^n} \gamma(t, x; \tau, \xi) dx \leqslant 1,$$

C being the same as in (2.1).

3. **Proof of Theorem 2.** Let $S_m = (0, T) \times (|x| < m) \ (m = 1, 2, \ldots)$. In each S_m the coefficients of equation (2.4) are bounded and Hölder continuous. Hence the Green function $\gamma_m(t, x; \tau, \xi)$ for (2.4) in each S_m exists and has the usual properties [3], [4]. Since $\tilde{c} \leqslant 0$, by the maximum principle we obtain, as in [1],

$$(3.1) \qquad \int_{|\xi|<m} \gamma_m(t, x; \tau, \xi) d\xi \leqslant 1 \quad \text{for } |x| \leqslant m, \ 0 \leqslant \tau < t \leqslant T.$$

Extending the definition of $\gamma_m(t, x; \tau, \xi)$ by setting $\gamma_m = 0$ for $|x| \geqslant m$ or $|\xi| \geqslant m$ one can show, as in [1], that the sequence $\{\gamma_m\}$ $(m = 1, 2, \ldots)$ is non-decreasing in D (and obviously $\gamma_m \geqslant 0$).

We shall prove that the sequence $\{\gamma_m\}$ is convergent and its limit function is a fundamental solution of equation (2.4).

We first show that

$$(3.2) \qquad \int_{|x|<m} \gamma_m(t, x; \tau, \xi) dx \leqslant 1 \quad \text{for } |\xi| < m, \ 0 \leqslant \tau < t \leqslant T,$$

and

$$(3.3) \qquad 0 \leqslant \gamma_m(t, x; \tau, \xi) \leqslant C(t-\tau)^{-n/2}$$

for some constant C which depends on n and $\varkappa$ but which is independent of m.

Under our assumptions on the regularity of the coefficients we cannot assert that $\gamma_m(t, x; \tau, \xi)$, as a function of (τ, ξ) is the Green function of

the adjoint equation. Therefore we prove [(3.2), (3.3) in a somewhat different way from that followed in [1].

For fixed $(\tau, \xi) \epsilon \langle 0, T \rangle \times R^n$ let

(3.4) $g = g_{mp\epsilon} = (\gamma_m^3 + \epsilon^3)^{p/3} - \epsilon^p, \quad m > |\xi|, \quad \epsilon > 0, \quad p \geqslant 1$

and

(3.5) $E_{mp\epsilon}(t) = \int\limits_{|x| < m} g \, dx, \quad t > \tau.$

For $t > \tau$ we have γ_m and its derivatives $(\gamma_m)_{x_i}$, $(\gamma_m)_{x_i x_j}$ are continuous functions of x for $|x| \leqslant m$ and $\gamma_m = 0$ for $|x| = m$ [3], [4]. Extending γ_m by setting $\gamma_m = 0$ for $|x| \geqslant m$ we can show that g, as a function of x for any fixed $t \epsilon (\tau, T)$, is of class $C^2(R^n)$ and $g = g_{x_i} = g_{x_i x_j} = 0$ for $|x| \geqslant m$ $(i, j = 1, \ldots, n)$. Thus g and g_{x_i} are admissible as the test functions ψ appearing in the definition of the weak derivatives. Now, by a direct computation we get from (3.5) and (2.4)

(3.6) $\dfrac{dE_{mp\epsilon}(t)}{dt} = \int\limits_{|x| < m} \sum_{i,j} a_{ij} g_{x_i x_j} dx + \int\limits_{|x| < m} \sum_j \tilde{b}_j g_{x_j} dx +$

$+ \int\limits_{|x| < m} p \tilde{c} (\gamma_m^3 + \epsilon^3)^{\frac{p}{3} - 1} \gamma_m^3 dx - \int\limits_{|x| < m} p (\gamma_m^3 + \epsilon^3)^{\frac{p}{3} - 2} \gamma_m [(p-1)\gamma_m^3 + 2\epsilon^3] \times$

$\times \sum_{i,j} a_{ij} (\gamma_m)_{x_i} (\gamma_m)_{x_j} dx = J_1 + J_2 + J_3 - J_4.$

Assumption I and the properties of function g imply

$J_1 = \int\limits_{R^n} \sum_{i,j} (a_{ij})_{x_i x_j} g \, dx, \quad J_2 = - \int\limits_{R^n} \sum_j (\tilde{b}_j)_{x_j} g \, dx.$

Making use of the inequality

$\int\limits_{|x| < m} \left[p \tilde{c} + \sum_{i,j} (a_{ij})_{x_i x_j} - \sum_j (\tilde{b}_j)_{x_j} \right] g \, dx \leqslant 0$

following Assumption II, we find, by (3.6),

(3.7) $\dfrac{dE_{mp\epsilon}}{dt} \leqslant - \int\limits_{|x| < m} p \tilde{c} \, [\epsilon^3 (\gamma_m^3 + \epsilon^3)^{\frac{p}{3} - 1} - \epsilon^p] \, dx -$

$- \int\limits_{|x| < m} p (\gamma_m^3 + \epsilon^3)^{\frac{p}{3} - 2} \gamma_m [(p-1)\gamma_m^3 + 2\epsilon^3] \sum_{i,j} a_{ij} (\gamma_m)_{x_i} (\gamma_m)_{x_j} dx.$

Consider the case $p = 1$. The second integral on the right-hand side of (3.7) can be omitted since its integrand is non-negative. Next, let $\epsilon \to 0$

in ·(3.7) and then integrate the inequality thus obtained from (3.7) on (τ, t), making use of the relation

$$\lim_{t \to \tau+} \int_{|x|<m} \gamma_m(t_1\,x;\,\tau,\,\xi)\,dx \;=\; 1$$

(cf. [4], p. 83 and 87). Hence we conclude that (3.2) holds true.

Now let $p \geqslant 2$. In this case we first omit in (3.7) the integral

$$-\int_{|x|<m} p\,(\gamma_m^3 + \varepsilon^3)^{\frac{p}{3}-2}\,\gamma_m \cdot 2\varepsilon^3 \sum_{i,j} a_{ij}(\gamma_m)_{x_i}(\gamma_m)_{x_j}\,dx \;\leqslant\; 0,$$

and then take the limit as $\varepsilon \to 0$. In this way we obtain from (3.7) the inequality which, after using parabolicity, can be written in the form

$$(3.8) \qquad \frac{dE_{mp}(t)}{dt} \;\leqslant\; -\frac{4\varkappa(p-1)}{p} \int_{|x|<m} |V_x(\gamma_m^{p/2})|^2\,dx,$$

where we have written E_{mp} instead of E_{mp0}.

On the other hand, since γ_m is an element of the Sobolev space $W^{1,2}$ ($|x|<m$), γ_m is continuous for $|x| \leqslant m$ and $\gamma_m = 0$ for $|x| = m$, we have $\gamma_m \in W_0^{1,2}(|x|<m)$ (see [5], Theorem 2, p. 104). The same is valid on $\gamma_m^{p/2}$ for $p \geqslant 2$. If we set $\gamma_m = 0$ for $|x| \geqslant m$, we have $\gamma_m^{p/2} \in W_0^{1,2}(R^n)$. Applying to the function $\gamma_m^{p/2}$ Nirenberg's form of Sobolev's inequality (see [5], Theorem 4, p. 68), we find

$$(3.9) \qquad \int_{|x|<m} \gamma_m^p\,dx \;\leqslant\; \theta_n \left(\int_{|x|<m} |V_x(\gamma_m^{p/2})|^2\,dx \right)^{\frac{n}{n+2}} \left(\int_{|x|<m} \gamma_m^{p/2}\,dx \right)^{\frac{4}{n+2}},$$

$\theta_n > 0$ being a constant depending only on n. By (3.8), (3.9) we get

$$\frac{dE_{mp}(t)}{dt} \;\leqslant\; -\frac{4\varkappa(p-1)}{p}\,\theta_n^{-\frac{n+2}{n}}\,E^{\frac{n+2}{n}}(t) \left(\int_{|x|<m} \gamma_m^{p/2}\,dx \right)^{-4/n},$$

whence, for $p = 2^k$,

$$(3.10) \qquad \frac{d}{dt}\{E_{m2^k}^{-2/n}\} \;\geqslant\; \varkappa_h\,\frac{2^k-1}{2^k}\,E_{m2^{k-1}}^{-4/n} \qquad (k = 1, 2, \ldots),$$

where $\varkappa_n = (8\varkappa/n)\,\theta_n^{-\frac{n+2}{n}}$.

We shall prove by induction that for every positive integer k the following inequality holds true:

$$(3.11) \qquad E_{m2^k}^{-2/n} \;\geqslant\; \varkappa_n^{2^k-1}\,2^{-2^k s_k}(t-\tau)^{2^k-1},$$

where $s_k = \sum_{l=1}^{k} l \cdot 2^{-l}$.

It follows from (3.10) for $k = 1$ and from (3.2) that

$$\frac{d}{dt}\{E_{m2}^{-2/n}\} \geqslant \varkappa_n/2.$$

Integrating this inequality on (τ, t), we obtain

$$E_{m2}^{-2/n}(t) \geqslant \frac{\varkappa_n}{2}(t-\tau)$$

because (as can easily be shown by using an estimate from below of a fundamental solution [4], p. 83)

(3.12) $\int\limits_{|x|<m} \gamma_m^p\, dx \to \infty$ as $t \to \tau$ for $p > 1$.

Thus (3.11) holds for $k = 1$. We shall show that if (3.11) holds for a certain k, it holds for $k+1$ too. By (3.10) with k replaced by $k+1$ we have

$$\frac{d}{dt}\{E_{m2^{k+1}}^{-2/n}(t)\} \geqslant \varkappa_n\, \frac{2^{k+1}-1}{2^{k+1}}\, E_{m2^k}^{-4/n},$$

whence taking advantage of (3.11) we find

(3.13) $\dfrac{d}{dt}\{E_{m2^{k+1}}^{-2/n}(t)\} \geqslant \varkappa_n^{2^{k+1}-1}\, \dfrac{2^{k+1}-1}{2^{k+1}}\, 2^{-2^{k+1}s_k}(t-\tau)^{2^{k+1}-2}.$

Finally, if we integrate (3.13) on (τ, t) and use (3.12), we obtain (3.11) with k replaced by $k+1$.

Inequality (3.11) yields

(3.14) $\left(E_{m2^k}(t)\right)^{1/2^k} \leqslant \varkappa_n^{-\frac{n}{2}(1-2^{-k})}\, 2^{\frac{n}{2}\sum\limits_{l=1}^{k}l2^{-l}}\,(t-\tau)^{-\frac{n}{2}(1-2^{-k})}.$

Since

$$\left(E_{mp}(t)\right)^{1/p} \to \max_{|x|<m}\gamma_m(t, x; \tau, \xi) \qquad \text{as } p \to \infty$$

and $\sum\limits_{l=1}^{\infty} l2^{-l} = 2$, we deduce from (3.14), after letting $k \to \infty$,

$$\gamma_m(t, x; \tau, \xi) \leqslant \left[\frac{\varkappa_n}{4}(t-\tau)\right]^{-n/2},$$

and thus (3.3) is proved.

Since, moreover, the sequence $\{\gamma_m\}$ is non-decreasing, it is convergent at each point of D. Its limit function $\gamma(t, x; \tau, \xi)$ satisfies — by (3.3), (3.1), (3.2) — inequalities (2.7), (2.8), (2.9).

Now using the interior Schauder estimates one can show, as in [1], that $\gamma(t, x; \tau, \xi)$ satisfies equation (2.4). The proof that γ has the remain-

ing properties required in the definition of a fundamental solution is also similar to that of paper [1] and will be omitted here.

4. Consider now the Cauchy problem:

(4.1)
$$L(u) = f(t, x) \quad \text{for } (t, x) \epsilon S,$$
$$u(0, x) = \varphi(x) \quad \text{for } x \epsilon R^n,$$

where $f(t, x)$, $\varphi(x)$ are given functions defined in S and R^n respectively. Essentially by a similar argument to that given in the proof of the corresponding theorem of paper [1] one can prove the following

THEOREM 3. *Let Assumptions* I, II *be satisfied. We assume that* $\varphi(x)$ *is continuous on* R^n, $f(t, x)$ *is Hölder continuous on every compact subset of* $\bar{S}$ *and there are non-negative constants* K_1, K_2 *such that*

$$|\varphi(x)| \leqslant K_1 h(0, x), \quad x \epsilon R^n, \quad and \quad |f(t, x)| \leqslant K_2 h(t, x), \quad (t, x) \epsilon \bar{S}.$$

Then

$$u(t, x) = \int_{R^n} \Gamma(t, x; 0, \xi)\varphi(\xi)d\xi - \int_0^t d\tau \int_{R^n} \Gamma(t, x; \tau, \xi)f(\tau, \xi)d\xi$$

is a solution of the Cauchy problem (4.1). *Moreover,*

$$|u(t, x)| \leqslant (K_1 + K_2 t)h(t, x) \quad in \quad \bar{S}.$$

References

[1] D. G. Aronson and P. Besala, *Parabolic equations with unbounded coefficients*, J. Diff. Eq. 3 (1967), p. 1–14.

[2] J. Chabrowski, *Sur la construction da la solution fondamentale de l'équation parabolique aux coefficients non borné*, Colloq. Math. 21 (1970), p. 141–148.

[3] A. Friedman, *Partial differential equations of parabolic type*, Prentice–Hall, Englewood–Cliffs, 1964.

[4] A. M. Il'in, A. S. Kalashnikov and O. A. Oleinik, *Second order linear equations of parabolic type*, Russian Math. Surveys (Uspahi Mat. Nauk) 17 (1962), No. 3, p. 1–141.

[5] J. Serrin, *Introduction to differentiation theory*, Lecture Notes, University of Minnesota, 1965.