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On the extension of holomorphic functions on a locally convex space

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Abstract. The map between the spaces of holomorphic functions on locally convex spaces induced by a continuous linear map is investigated. The obtained results allows us to get the s-nuclearity (resp. the nuclearity) of the space $\mathcal{O}(U)$ of holomorphic functions on an open subset U of a quasi-complete dual s-nuclear space (resp. of a quasi-complete dual nuclear space). Certain extension and lifting theorems for holomorphic maps between locally convex spaces are also established.

The space of holomorphic functions on an open set in a locally convex space was investigated by several authors [2], [3], [4]. In Section 1 of this paper we find conditions for a map between spaces of holomorphic functions on open sets in locally convex spaces induced by a continuous linear map to be s-nuclear. Section 2 is devoted to an application of obtained results to the extension and lifting problem for holomorphic maps between Frechet and dual Frechet spaces.

Notations and definitions. Let L be a complex locally convex space and let L' denote the strong dual of L. By $\mathcal{U}(L)$ we denote the set of all absolutely convex neighbourhoods of zero in L. For every $U \in \mathcal{U}(L)$, let $L(U)$ denote the completion of $L/p(U)^{-1}(0)$ equipped with the norm $p(U)$, where $p(U)$ is the Minkowski functional of U, and let $\pi(U)$ denote the canonical map from L into $L(U)$. If $U, V \in \mathcal{U}(L)$ and $V \subset U$, then $\omega(V, U)$ denotes the canonical map from $L(V)$ into $L(U)$.

A continuous linear map T from a locally convex space L into a locally convex space F is called *s-nuclear* [11] iff for every $V \in \mathcal{U}(F)$ there exists $U \in \mathcal{U}(L)$ such that $TU \subset V$ and the map $T(U, V)$: $L(U) \to F(V)$ induced by T is s-nuclear, i.e., there exist sequences $\{\Lambda_j\} \subset \boldsymbol{R}$, $\{u_j'\} \subset L'(U)$ and $\{v_j\} \subset L(V)$ such that

$$\sup_{\Lambda_1 \geqslant \Lambda_2 \geqslant \ldots > 0} \{\|u_j'\|, \|v_j\|: j = 1, 2, \ldots\} < \infty, \qquad \sum_{j=1}^{\infty} \Lambda_j^p < \infty \quad \text{for every } p > 0$$

and

$$T(U, V)u = \sum_{j=1}^{n} \Lambda_j u'_j(u) v_j \quad \text{for } u \in L(U).$$

A map T is said to be *quasi-s-nuclear* iff for every $V \in \mathscr{U}(F)$ there exists $U \in \mathscr{U}(L)$ such that $TU \subset V$ and $\alpha T(U, V)$ is s-nuclear for some embedding α of $L(V)$ into a Banach space B.

Given a locally convex space L, denote by $L(K)$, for any bounded set K in L, the normed space span K equipped with the norm generated by the absolutely convex envelope $\omega(K)$ of K. Let $\mathscr{B}(L)$ denote the set of all absolutely convex bouned sets in L.

Let G be an open subset of L. A subset K of G is called *G-precompact* iff K is precompact and $K \Subset G$, where we write $K \Subset G$ iff there exists $U \in \mathscr{U}(L)$ such that $K + U \subset G$.

A complex valued function f on G is said to be *bc-holomorphic* iff

(a) $f | G \cap L_0$ is holomorphic for every finite dimensional subspace L_0 of L,

(b) f is bounded and continuous on all G-precompact subsets of G.

Obviously, if L is quasi-complete, then f is bc-holomorphic iff f is hypoholomorphic.

By $\mathcal{O}_{bc}(G)$ we denote the space of bc-holomorphic functions on G equipped with the topology of uniform convergence on all G-precompact subsets of G.

Put

$$\mathcal{O}_b(G) = \{f \in \mathcal{O}(G) \colon f \text{ is bounded on every bounded set } K \Subset G\},$$

$$\mathcal{O}_{HY}(G) = \{f \colon G \to C \colon f \text{ is hypoholomorphic}\},$$

where $\mathcal{O}(G)$ denotes the space of holomorphic functions on G. This space is endowed with the compact-open topology. The space $\mathcal{O}_b(G)$ (resp. $\mathcal{O}_{HY}(G)$) is equipped with the topology of uniform convergence on all G-bounded sets (resp. on all compact sets) in G.

1. The spaces $\mathcal{O}(G)$, $\mathcal{O}_b(G)$, $\mathcal{O}_{HY}(G)$ and $\mathcal{O}_{bc}(G)$. Let T be a continuous linear map from a locally convex space E into a locally convex space F. Then T induces naturally the continuous linear maps

$$\hat{T} \colon \mathcal{O}(F) \to \mathcal{O}(E), \qquad \hat{T}_b \colon \mathcal{O}_b(F) \to \mathcal{O}_b(E)$$

and

$$\hat{T}_{HY} \colon \mathcal{O}_{HY}(F) \to \mathcal{O}_{HY}(E), \qquad \hat{T}_{bc} \colon \mathcal{O}_{bc}(F) \to \mathcal{O}_{bc}(E).$$

We now prove the following

THEOREM 1.1. *Let T be a continuous linear map from a normed space E into a normed space F. Then the following conditions are equivalent:*

(i) *The dual map $T'\colon F' \to E'$ is s-nuclear.*

(ii) *If G and $\tilde{G}$ are open sets in E and F, respectively, such that $T\omega G \Subset \tilde{G}$, then the map $\hat{T}_b\colon \mathcal{O}_b(\tilde{G}) \to \mathcal{O}_b(G)$ induced by $T|G$ is s-nuclear.*

(iii) *$\hat{T}_b\colon \mathcal{O}_b(F) \to \mathcal{O}_b(E)$ is s-nuclear.*

Moreover, if E and F are locally convex spaces and if T satisfies the condition:

For every $K \in \mathscr{B}(E)$ there exists $\tilde{K} \in \mathscr{B}(F)$ such that $TK \subset \tilde{K}$ and the map $T_K\colon E(K) \to F(\tilde{K})$ induced by T is s-nucelar,

then,

(iv) *$\hat{T}_b\colon \mathcal{O}_b(\tilde{G}) \to \mathcal{O}_b(G)$ and $\hat{T}_{bc}\colon \mathcal{O}_{bc}(\tilde{G}) \to \mathcal{O}_{bc}(G)$ are quasi-s-nucelar, where G and $\tilde{G}$ are open sets in E and F, respectively, such that $T\omega G \Subset \tilde{G}$.*

(v) *$\hat{T}\colon \mathcal{O}_{bc}(\tilde{G}) \to \mathcal{O}_{HY}(G)$ is quasi-s-nuclear for all open sets G and $\tilde{G}$ in E and F, respectively, such that $TG \subset \tilde{G}$.*

Proof. (i)$\Rightarrow$(ii). Since $T\colon E \to F''$ is s-nuclear, there exist sequences $\{\Lambda_j\} \subset E'$ and $\{e_j\} \subset F''$ such that

$$\sum_{j=1}^{\infty} \|\Lambda_j\|^p < \infty \qquad \text{for every } p > 0,$$

$$\sum_{j=1}^{\infty} \|e_j\| < \infty, \tag{1.1}$$

$$T_u = \sum_{j=1}^{\infty} \lambda_j(u) e_j \qquad \text{for } u \in E.$$

Put $\tilde{G}_1 = \omega TG + \hat{B}(0, \varepsilon/3e)$, where $\varepsilon = d(\omega TG, \partial\tilde{G}) > 0$ and $\hat{B}(x, r) = \{y \in F''\colon \|x-y\| < r\}$. Obviously,

$$\omega TG \Subset \tilde{G}_1 \subset W = U\{\hat{B}(y, d(y, \partial\tilde{G})/3e)\colon y \in \tilde{G}\}.$$

Since $r_b(x,f) = d(x, \partial\tilde{G}) = d(x)$ for $f \in \mathcal{O}_b(\tilde{G})$, where $r_b(x,f)$ denotes the radius of boundedness of f at x, by a lemma of Aron and Berner (see [1], p. 10) there exists a linear extension map $\theta\colon \mathcal{O}_b(\tilde{G}) \to \mathcal{O}(W)$. This map is given by the formula

$$(\theta f)x = \sum_{k=0}^{\infty} T_k P_k f(y)(x-y) \qquad \text{for } x \in \hat{B}(y, d(y)/e),\ y \in \tilde{G},$$

where $P_k f(y)$ denotes the k-th Taylor coefficient of f at y and T_k is the natural extension map from the space of continuous k-homogeneous polynomials on F into the space of continuous k-homogeneous polynomials on F''. Observe that $\theta\colon \mathcal{O}_b(\tilde{G}) \to \mathcal{O}(\tilde{G}_1)$ is continuous. Indeed, let $\mathscr{U} = \{V_y\colon y \in G\}$ be a locally finite open cover of W such that $\bar{V}_y \subset \hat{B}(y, d(y)/3e)$ for $y \in \tilde{G}$ and let K be a compact set in $\tilde{G}_1$. By the local finiteness of $\mathscr{U}$ without loss of generality we may assume that $K \subset V_{y_0}$ for some y_0. Hence, for $f \in \mathcal{O}_b(\tilde{G})$ and $x \in K$, we have

$$|(\theta f)x| \leqslant \sum_{k=0}^{\infty} \|T_k\| \, \|P_k f(y_0)\| \, \|x - \dot{y}_0\|^k$$

$$\leqslant \sum_{k=0}^{\infty} k^k/k!\,(e/d(y_0))^k (d(y_0)/3e)^k p_{B(y_0, d(y_0)/e)}(f),$$

where $B(y, r) = \hat{B}(y, r) \cap F$ and p_K denotes the seminorm on $\mathcal{O}_b(\tilde{G})$ generated by a $\tilde{G}$-bounded set K in $\tilde{G}$. Put $C = \sum_{k=0}^{\infty} (k/3)^k 1/k! < \infty$.

Then

$$p_K(\theta f) \leqslant C p_{B(y_0, d(y_0)/e)}(f) \quad \text{for } f \in \mathcal{O}_b(\tilde{G}).$$

Since $\overline{T(K)}$ is compact in $\tilde{G}_1$ for all G-bounded sets K in G, it follows that $\hat{T}(\mathcal{O}(\tilde{G}_1)) \subset \mathcal{O}_b(G)$. Thus, by the continuity of θ, it suffices to show that the map $\hat{T}$: $\mathcal{O}(\tilde{G}_1) \to \mathcal{O}_b(G)$ is s-nuclear.

Without loss of generality we may assume that $\{e_j\} \subset \hat{B}(0, \varepsilon/3e)$. Let K be a G-bounded set in G. Put $\tilde{K} = \overline{\omega\{e_j\} + \omega T(K)}$. Observe that $\tilde{K}$ is a compact set in $\tilde{G}$. We will show that the canonical map

$$\hat{T}_K\colon\ \mathcal{O}(\tilde{G}_1)/p_{\tilde{K}}^{-1}(0) \to \mathcal{O}_b(G)/p_K^{-1}(0)$$

induced by T is s-nuclear.

Let $p > 0$ and let

$$q = \max\Big\{\sum_{j=1}^{\infty} p_K^p(\lambda_j),\ \sum_{j=1}^{\infty} \|e_j\|\Big\} < \infty.$$

Given an $f \in \mathcal{O}(\tilde{G}_1)$, consider the Taylor expansion of f at zero

$$f(u) = \sum_{n=0}^{\infty} (P_n f)u \quad \text{for } u \in \tilde{G}_1.$$

Since

$$(P_n f)u = \tfrac{1}{2}\pi i \int_{|\lambda|=2} f(\lambda u)\lambda^{-n-1} d\lambda$$

for $u \in \tilde{K}_1 = \tilde{K}/2$, we have

$$p_{\tilde{K}_1}(P_n f) \leqslant 2^{-n} p_{\tilde{K}}(f) \quad \text{for } n \geqslant 0. \tag{1.2}$$

By (1.2) we have

$$(1.3)\qquad |P_nf(e_{j_1},\ldots,e_{j_n})| \leqslant (1/n!) \sum_{(i_1<\ldots<i_p)\subset(j_1,\ldots,j_n)} (2a)^n|P_nf(e_{i_1}+\ldots+e_{i_p}/2a)|$$
$$\leqslant (2a)^n|n!\, p_{\tilde K}(f) \quad \text{for all } n, j_1,\ldots,j_n \geqslant 1.$$

Setting

$$\beta_{(n,j_1,\ldots,j_n)}(u) = \lambda_{j_1}(u)\ldots\lambda_{j_n}(u) \qquad \text{for } u\in G$$

and

$$u'_{(n,j_1,\ldots,j_n)}(f) = (P_nf)(e_{j_1},\ldots,e_{j_n}) \quad \text{for } f\in\mathcal{O}(\tilde G_1),$$

$n, j_1,\ldots,j_n \geqslant 1$, we get the elements belonging to $\mathcal{O}_b(G)/p_K^{-1}(0)$ and $[\mathcal{O}(\tilde G_1)/p_{\tilde K}^{-1}(0)]$, respectively, such that

$$(Tf)u = f(0)+\sum_{n=1}^{\infty}(P_nf)\,Tu = f(0)+\sum_{n=1}^{\infty}(P_nf)\Big(\sum_{j=1}^{\infty}\lambda_j(u)e_j\Big)$$
$$= f(0)+\sum_{n,j_1,\ldots,j_n\geqslant 1}\lambda_{j_1}(u)\ldots\lambda_{j_n}(u)\,P_nf(e_{j_1},\ldots,e_{j_n})$$
$$= f(0)+\sum_{n,j_1,\ldots,j_n\geqslant 1}\beta_{(n,j_1,\ldots,j_n)}(u)\,u'_{(n,j_1,\ldots,j_n)}(f).$$

On the other hand, by (1.3) we have

$$\sum \|\beta_{(n,j_1,\ldots,j_n)}\|^p\|u'_{(n,j_1,\ldots,j_n)}\|^p \leqslant \sum p_K^p(\lambda_{j_1})\ldots p_K^p(\lambda_{j_n})\big((2a)^n/n!\big)^p$$
$$= \sum_{n=1}^{\infty}\Big(\sum_{j=1}^{\infty}p_K^p(\lambda_j)\Big)^n\big((2a)^n/n!\big)^p = \sum_{n=1}^{\infty}(2^na^{2n}/n!)^p < \infty.$$

Hence we infer that $\hat T_K$ is s-nuclear.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) follows from the relation $T' = P_1\hat T|F$.

Proof of (iv). (a) First we show that $\hat T\colon \mathcal{O}_b(\tilde G)\to\mathcal{O}_b(G)$ is quasi-s-nuclear. Take $V\in\mathcal{U}(F)$ such that

$$(1.4)\qquad \omega TG \subset \tilde G_1 = \omega TG + V \subset \tilde G_1 + V \subset \tilde G.$$

Obviously $\tilde G_1\in\mathcal{U}(F)$. Let K be a G-bounded set in G and let $\tilde K\in\mathcal{B}(F)$ such that $T(K)\subset\tilde K$ and the map $T_K\colon E(K)\to F(\tilde K)$ is s-nuclear. Observe that $K\Subset E(K)\cap G$ and K is bounded in $E(K)$. By (1.4) we have $f|\tilde G_1\cap F(\tilde K)\in\mathcal{O}_b(\tilde G_1\cap F(\tilde K))$ for $f\in\mathcal{O}_b(\tilde G)$ and

$$\omega T(E(K)\cap G)\Subset \tilde G_1\cap F(\tilde K).$$

Thus applying implication (i) $\Rightarrow$ (ii) to T_K we get the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_b(\tilde{G}) \arrow[rr, "r"] \arrow[d, "\tilde{\omega}_K \hat{T}"'] & & \mathcal{O}_b(\tilde{G}_1 \cap F(\tilde{K})) \arrow[d, "\tilde{\omega}_K \hat{T}_K"] \\
\mathcal{O}_b(G)/p_K^{-1}(0) \arrow[dr, "i"'] & & \mathcal{O}_b(G \cap E(K))/p_K^{-1}(0) \arrow[dl, "i_0"] \\
& \mathcal{O}_0(G \cap E(K))/p_K^{-1}(0) &
\end{tikzcd}$$

in which $\hat{T}_K$ is s-nuclear, where

$$\mathcal{O}_0(G \cap E(K)) = \{f \in \mathcal{O}(G \cap E(K)) \colon f \text{ is bounded on } K\},$$

r is the restriction map, i, i_0, $\tilde{\omega}_K$ and ω_K are canonical maps. Hence $\hat{T}$ is quasi-s-nuclear.

(b) In notations of (a), since T_K is s-nuclear, it can be represented by formula (1.1) with $\{e_j\} \subset F(\tilde{K})$ ([11], Theorem 8.5.6). Hence, by the proof of (i) $\Rightarrow$ (ii), it follows that for each G-precompact set K in G there exists an absolutely convex set K'' in $F(\tilde{K})$ such that K'' is precompact in $F(\tilde{K})$, $rK'' \Subset \tilde{G} \cap F(\tilde{K})$ for some $r > 1$ and

$$\hat{T}_K \colon \mathcal{O}_b(G \cap F(\tilde{K}))/p_{K''}^{-1}(0) \to \mathcal{O}_b(G \cap E(K))/p_K^{-1}(0)$$

is s-nuclear. By the Taylor expansion at zero of each element $f \in \mathcal{O}_{bc}(\tilde{G})$ we infer that $f|\tilde{G}_1 \cap F(\tilde{K}) \in \widehat{\mathcal{O}_b(\tilde{G}_1 \cap F(\tilde{K}))/p_{K''}^{-1}(0)}$ for $f \in \mathcal{O}_{bc}(\tilde{G})$. Thus we get the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{bc}(\tilde{G}) \arrow[rr, "r"] \arrow[d, "\omega_K \hat{T}"] & & \widehat{\mathcal{O}_b(\tilde{G}_1 \cap F(\tilde{K}))/p_{K''}^{-1}(0)} \arrow[d, "\hat{T}_K"] \\
\mathcal{O}_{bc}(G)/p_K^{-1}(0) \arrow[dr, "i"'] & & \mathcal{O}_b(G \cap E(K))/p_K^{-1}(0) \arrow[dl, "i_0"] \\
& \mathcal{O}_0(G \cap E(K))/p_K^{-1}(0) &
\end{tikzcd}$$

in which $\hat{T}_K$ is s-nuclear. Consequently $\hat{T} \colon \mathcal{O}_{bc}(\tilde{G}) \to \mathcal{O}_{bc}(G)$ is quasi-s-nuclear.

Proof of (v). Let K be a compact set in G. Take $V \in \mathcal{U}(F)$ such that

$$T(K) \subset \bigcup_{j=1}^{m} (v_j + V) \subset \bigcup_{j=1}^{m} (v_j + 2V) \subset \bigcup_{j=1}^{m} (v_j + 3V) \subset \tilde{G}$$

for some m, where $v_j = Tu_j$, $u_j \in G$, $j = 1, 2, \ldots, m$. Put

$$G_j = G \cap T^{-1}(v_j + 2V), \qquad \tilde{G}_j = v_j + 3V \quad \text{and} \quad K_j = K \cap T^{-1}(v_j + \bar{V}).$$

Since $G_j \simeq G_j - u_j$, $\tilde{G}_j \simeq \tilde{G}_j - v_j = 3V$ and $T(G_j - u_j) \subset 2V \Subset 3V$, applying (iv) to T_j: $T|G_j$: $G_j \to \tilde{G}_j$ we get a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{bc}(\tilde{G}) \arrow[rr] \arrow[d, "\tilde{\omega}_K \hat{T}"] & & \oplus \mathcal{O}_{bc}(\tilde{G}_j) \arrow[d, "\oplus \omega_{K_j} \hat{T}_{jK_j}"] \\
\mathcal{O}_{HY}(G)/p_K^{-1}(0) \arrow[dr, "i"'] & & \oplus \mathcal{O}_{bc}(G_j)/p_{K_j}^{-1}(0) \arrow[dl, "i_0"] \\
& \oplus \mathcal{O}_{HY}(G_j)/p_{K_j}^{-1}(0) &
\end{tikzcd}$$

in which $\omega_{K_j} \hat{T}_{jK_j}$ are quasi-s-nucelar. Hence $\hat{T}$ is quasi-s-nuclear. The theorem is proved.

A locally convex space L is called *dual s-nuclear* (resp. *dual nuclear*) if L' is s-nuclear (resp. nuclear).

Since each quasi-s-nuclear map is absolutely summing, it follows that the product of two quasi-s-nuclear maps is nuclear. On the other hand, since each nuclear map from a subspace of a Banach space into a Banach space can be extended to a nuclear map, we infer that the product of three quasi-s-nuclear maps is s-nuclear. Combining this with Theorem 1.1 ((iv) and (v)) we get the following

COROLLARY 1.1. *If E is dual s-nuclear, then $\mathcal{O}_b(E)$ and $\mathcal{O}_{bc}(E)$ are s-nuclear.*

COROLLARY 1.2 [6]. *If G is an open set in a quasi-complete s-nuclear space, then $\mathcal{O}(G)$ and $\mathcal{O}_{HY}(G)$ are s-nuclear.*

Now assume that E is dual nuclear. Let $K \in \mathscr{B}(E)$. Since E' is nuclear, there exists $\tilde{K} \in \mathscr{B}(E)$ such that $K \subset \tilde{K}$ and the canonical map id_K: $E(K) \to E(\tilde{K})$ belongs to class $l^{1/2}$ ([11], p. 142). Thus id_K is represented by the formula

$$\mathrm{id}_K(u) = \sum_{j=1}^{\infty} \lambda_j(u) e_j \quad \text{for } u \in E(K),$$

where $\{\lambda_j\} \subset E'(K)$, $\{e_j\} \subset E(\tilde{K})$ and $\sum_{j=1}^{\infty} (\|\lambda_j\| + \|e_j\|) < \infty$ ([11], p. 136).

Hence, by an argument as in Theorem 1.1, we get the following

COROLLARY 1.3. *If E is dual nuclear, then $\mathcal{O}_b(E)$ and $\mathcal{O}_{bc}(E)$ are nuclear.*

COROLLARY 1.4. *If G is an open set in a quasi-complete dual nuclear space, then $\mathcal{O}(G)$ and $\mathcal{O}_{HY}(G)$ are nuclear.*

Remark 1.1. The nuclearity of $\mathcal{O}(G)$ has been established by Boland [3].

THEOREM 1.2. *Let T be a continuous linear map from a normed space E*

into a normed space F. Then $T'\colon F'\to E'$ is nuclear if and only if $\hat{T}\colon \mathcal{O}_b(F)\to\mathcal{O}_b(G)$ is nuclear for every open set G in E.

Proof. The necessity of Theorem 1.2 is trivial.

Let $T'\colon F'\to E'$ be nuclear and let G be an open set in E. Since $T\colon E\to F''$ is nuclear, we have

$$Tu=\sum_{j=1}^{\infty}\lambda_j(u)e_j \quad \text{for } u\in E,$$

where $\{\lambda_j\}$ and $\{e_j\}$ are sequences in E' and F'', respectively, such that

$$\sum_{j=1}^{\infty}\|\lambda_j\|<\infty \quad \text{and} \quad \sup\{\|e_j\|\colon j=1,2,\ldots\}\leqslant 1.$$

Since there exists a continuous linear extension map $\mathcal{O}_b(F)\to\mathcal{O}_b(F'')$ [1], it suffices to show that $\hat{T}\colon \mathcal{O}_b(F'')\to\mathcal{O}_b(G)$ is nuclear.

Let K be a G-bounded set in G and let $a=\sum_{j=1}^{\infty}p_K(\lambda_j)$. Put

$$\tilde{K}=\omega(\{6ae_j\}\,U\,TK), \qquad \tilde{K}_1=\tilde{K}/2.$$

Let $f\in\mathcal{O}_b(F'')$. In notations of Theorem 1.1 we have

$$p_{\tilde{K}_1}(P_nf)\leqslant 2^{-n}p_{\tilde{K}}(f) \quad \text{for } n\geqslant 0$$

and

$$|P_nf(e_{j_1},\ldots,e_{j_n})|\leqslant(1/n!)\sum_{(i_1<\ldots<i_p)\subset(j_1,\ldots,j_n)}(n/3a)^n|P_nf(e_{i_1}+\ldots+e_{i_p})3a/n|$$

$$\leqslant(1/n!)(n/3a)^n\sum_{p=1}^{n}\binom{p}{n}2^{-n}p_{\tilde{K}}(f)\leqslant(1/n!)(n/3a)^np_{\tilde{K}}(f)$$

for $n\geqslant 1$. Since $\sum_{n=1}^{\infty}(n/3)^n/n!<\infty$, by an argument as in Theorem 1.1 it follows that $\hat{T}_K\colon \mathcal{O}_b(F'')/p_{\tilde{K}}^{-1}(0)\to\mathcal{O}_b(G'')/p_K^{-1}(0)$ is nuclear.

The theorem is proved.

THEOREM 1.3. *Let T be a continuous linear map from a locally convex space E into a locally convex space F. Then the following conditions are equivalent:*

(i) *$T'\colon F'\to E'$ is precompact, i.e., for every $V\in\mathcal{U}(E')$ there exists $U\in\mathcal{U}(F')$ such that $T'(U,V)\colon F'(U)\to E'(V)$ is precompact.*

(ii) *If G and $\tilde{G}$ are open sets in E and F, respectively, such that $TG\Subset\tilde{G}$, then $\hat{T}\colon\mathcal{O}_b(\tilde{G})\to\mathcal{O}_b(G)$ is precompact.*

(iii) *$\hat{T}\colon\mathcal{O}_b(F)\to\mathcal{O}_b(E)$ is precompact.*

Moreover, $\hat{T}\colon\mathcal{O}_b(\tilde{G})\to\mathcal{O}_{HY}(G)$ is precompact for all open sets G and $\tilde{G}$ in E and F, respectively, such that $TG\subset\tilde{G}$.

Proof. (ii)$\Rightarrow$(iii)$\Rightarrow$(i) are trivial.

(i) $\Rightarrow$ (ii). Since $L(K)$ is a subspace of $[L'/p_K^{-1}(0)]'$, where L is a locally convex space and $K \in \mathscr{B}(L)$, it follows that for each $K \in \mathscr{B}(E)$ there exists $\tilde{K} \in \mathscr{B}(F)$ such that $T(K) \subset \tilde{K}$ and T_K: $E(K) \to F(\tilde{K})$ is precompact. Hence, by an argument as in Theorem 1.1 (iv), we may assume that E and F are normed spaces. Let K be a G-bounded set. Since $d(TK, \partial\tilde{G}) > 0$, there exists a bounded open neighbourhood $\tilde{G}_1$ of TK such that $\tilde{G}_1 \Subset G$. We now show that the canonical map θ: $\mathcal{O}_b(\tilde{G})/p_{\tilde{G}_1}^{-1}(0) \to \mathcal{O}_b(G)/p_K^{-1}(0)$ is precompact. Assume that $\{f_k\}$ is a bounded sequence in $\mathcal{O}_b(\tilde{G})/p_{\tilde{G}_1}^{-1}$. Since $TK \Subset \tilde{G}_1$, it is easy to see that $\{f_k| TK\}$ is equicontinuous. Hence, by the precompactness of TK, it follows that $\{f_k T|_K\}$ is precompact in $\mathcal{O}_b(G)/p_K^{-1}(0)$. Using the proof of (i) $\Rightarrow$ (ii) and examining the proof of Theorem 1.1 ((iv), (v)) we get the precompactness of $\hat{T}$: $\mathcal{O}_b(\tilde{G}) \to \mathcal{O}_{HY}(G)$. A locally convex space L is called *dual Schwartz* if L' is a Schwartz space.

The following are an immediate consequence of Theorem 1.3.

COROLLARY 1.5. *If E is a dual Schwartz space, then $\mathcal{O}_b(E)$ is a Schwartz space.*

COROLLARY 1.6. *If G is an open subset of a quasi-complete dual Schwartz space, then $\mathcal{O}(G)$ and $\mathcal{O}_{HY}(G)$ are Schwartz spaces.*

Let Λ be an index set. Put $C^\Lambda = \prod_{i \in \Lambda} C_i$, where $C_i = C$ for $i \in \Lambda$. We will prove the following

THEOREM 1.4. *Let Λ be infinite and let*

$$K = \prod_{i \in \Lambda} \Delta(r_i), \qquad \textit{where } \Delta(r_i) = \{z \in C\colon |z| \leqslant r_i\}.$$

Then the canonical map

$$\pi(\tilde{K}, K)\colon \mathcal{O}_{HY}(C^\Lambda)/p_{\tilde{K}}^{-1}(0) \to \mathcal{O}_{HY}(C^\Lambda)/p_K^{-1}(0)$$

is s-nuclear for some compact set $\tilde{K}$ in C^Λ containing K if and only if the set $\Lambda(K) = \{i \in \Lambda\colon r_i > 0\}$ is finite or countable.

Moreover, if $\Lambda(K)$ is countable, then $\pi(\tilde{K}, K)$ is s-nuclear for every compact set

$$\tilde{K} = \prod_{i \in \Lambda} \Delta(r_i + \varepsilon_i), \qquad \sum_{i \in \Lambda(K)} r_i/\varepsilon_i < \infty.$$

Proof. Assume that $\pi(\tilde{K}, K)$ is s-nuclear for some compact set $\tilde{K}$ containing K. Since $\pi(\tilde{K}, K)$ is surjective, $\mathcal{O}_{HY}(C^\Lambda)/p_K^{-1}(0)$ is separable. For $i \in \Lambda(K)$ and $\xi \in C^\Lambda$ put $e_i'(\xi) = \xi_i/r_i$. Obviously, $e_i' \in \mathcal{O}(C^\Lambda)$ and $p_K(e_i' - e_j') = l$ for $i, j \in \Lambda(K)$, $i \neq j$. Hence, by the separability of $\mathcal{O}_{HY}(C^\Lambda)/p_K^{-1}(0)$, we infer that $\Lambda(K)$ is finite or countable.

Now assume that $\Lambda(K) = \{i_1, i_2, \ldots\}$.

(a) Put

$$K_0 = \prod_{j=1}^{\infty} \Delta(r'_j), \qquad \tilde{K}_0 = \prod_{j=1}^{\infty} \Delta(r'_j + \varepsilon'_j),$$

where $r'_j = r_{i_j}$, $\varepsilon'_j = i_j$.

By the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{O}_{HY}(C^A)/p_{\tilde{K}}^{-1}(0) \arrow[r, "\pi(\tilde{K},K)"] \arrow[d, "\tilde{i}"] & \mathcal{O}_{HY}(C^A)/p_K^{-1}(0) \arrow[d, "i"] \\
\mathcal{O}(C^{A(\tilde{K}_0)})/p_{\tilde{K}_0}^{-1}(0) \arrow[r, "\pi(\tilde{K}_0,K)"'] & \mathcal{O}(C^{A(K)})/p_K^{-1}(0)
\end{tikzcd}$$

in which $\tilde{i}$ and i are canonical isomorphisms, it suffices to prove that $\pi(\tilde{K}_0, K_0)$ is s-nuclear.

Put

$$V = \{\sigma \in \mathcal{O}(C^{A(K)}) \colon p_{\tilde{K}_0}(\sigma) \leqslant 1\}.$$

Let V^0 denote the polar of V in $[\mathcal{O}(C^{A(K)})/p_{\tilde{K}_0}^{-1}(0)]'$. For every n and $\varphi \in C(V^0)$ write

$$\mu_n(\varphi) = (1/2\pi i)^n \prod_{j=1}^{n} (1 + r'_j/\varepsilon'_j) \int_0^{2\pi} \dots \int_0^{2\pi} \varphi(\delta_\omega)\, d\theta,$$

where $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j = (r'_j + \varepsilon'_j) e^{i\theta_j}$, $d\theta = d\theta_1 \dots d\theta_n$ and $\delta_\omega(\sigma) = \sigma(\omega)$ for $\sigma \in \mathcal{O}(C^{A(K)})$. Since $\sum_{j=1}^{n} r'_j/\varepsilon'_j < \infty$, we infer that $\{\mu_n\}$ is a bounded sequence of positive Radon measure on V^0. Hence there exists a net $\{\mu_{n_\alpha}\} \subset \{\mu_n\}$ converging to a positive Radon measure μ on V^0 in $C(V^0)$-topology.

Let $\sigma \in \mathcal{O}(C^{A(K)})$. By Liouville theorem it easy to see that there exists n_0 such that $\sigma \in \mathcal{O}(C^n)$ for $n > n_0$. Hence by Cauchy integral formula we have

$$|\sigma(z)| \leqslant \prod_{j=1}^{n} (1 + r'_j/\varepsilon'_j)/(2\pi)^n \underbrace{\int_0^{2\pi} \dots \int_0^{2\pi}}_{n} |\sigma(\delta_\omega)|\, d\theta$$

for all $z \in K_0$ and $n \geqslant n_0$. Thus

$$p_{K_0}(\sigma) \leqslant \int_{V^0} |\sigma(u')|\, d\mu$$

and therefore $\pi(\tilde{K}_0, K_0)$ is absolutely summing [11].

(b) For each $k \geqslant 1$ put $\tilde{K}_k = \prod_{j=1}^{\infty} \Delta(r'_j + \varepsilon'_j/2^k)$. Similarly as in (a) the maps $\pi(\tilde{K}_0, \tilde{K}_1), \dots, \pi(\tilde{K}_{k-1}, \tilde{K}_k)$ are absolutely summing. Combining this with the relation $\pi(\tilde{K}_0, K_0) = \pi(\tilde{K}_0, \tilde{K}_1) \dots \pi(\tilde{K}_k, K_0)$ we conclude by

Theorem 8.2.7 [11], that the map $\pi(\tilde{K}_0, K_0)$ belongs to class l^p for $p > 0$. Hence $\pi(\tilde{K}_0, K_0)$ is s-nuclear.

2. Extension and lifting of holomorphic maps. Some theorems on the extensions of holomorphic maps between Banach spaces have been established by Aron and Berner [1]. In [2] Boland proved that if F is a quotient space of a nuclear Frechet space E, then the restriction map $\mathcal{O}(E') \to \mathcal{O}(F')$ is surjective.

In this section we investigate the problems extension and lifting for holomorphic maps between Frechet and dual Frechet spaces.

THEOREM 2.1. *Let J be a continuous linear map from a Frechet space E onto a Frechet space F and let Ω be an open set in P', where P is a Frechet–Montel space. Then*

(i) *If either P or F is nuclear, then the map $\hat{J}$: $\mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega, F)$ induced by J is surjective.*

(ii) *If $G = \operatorname{Ker} J$ is nuclear, then the map $J\hat{\otimes}_\varepsilon \mathrm{id}$: $\mathcal{O}(\Omega, E\hat{\otimes}_\varepsilon Q) \to \mathcal{O}(\Omega, F\hat{\otimes}_\varepsilon Q)$ is surjective for all Frechet space Q.*

The proof of Theorem 2.1 is based on the following lemma [10]:

LEMMA 2.1. *Let $E = \varprojlim \{E_n, \omega_n^m\}$, and $F = \varprojlim \{F_n, \gamma_n^m\}$, where E_n and F_n are Frechet spaces, and let k be a natural number. Assume that for each n there exists a continuous linear map J_n from E_n into F_n satisfying the conditions:*

(L_1) $\gamma_n^m J_n = J_m \omega_n^m$ *for* $n \geqslant m$.

(L_2) $\operatorname{Im} J_{n-k} \supset \operatorname{Im} \gamma_n^{n-k}$ *for* $n > k$.

(L_3) $\omega_n^{n-k}(\operatorname{Ker} J_n)$ *is dense in* $\omega_{n-1}^{n-k}(\operatorname{Ker} J_{n-1})$ *for* $n > k$.

Then the map $J = \varprojlim J_n$: $E \to F$ is surjective.

Proof of Theorem 2.1. Since P is Frechet–Montel, P' is a k-space [7]. Hence $\mathcal{O}(\Omega, Q)$ is Frechet for every Frechet space Q.

(i) Let either P or F be nuclear. Then by Corollary 1.4 either $\mathcal{O}(\Omega)$ or F is nuclear. This yields the surjectivity of the map

$$\hat{J}\colon \mathcal{O}(\Omega, E) \equiv \operatorname{Hom}(E'_c, \mathcal{O}(\Omega)) \to \operatorname{Hom}(F'_c, \mathcal{O}(\Omega)) \equiv \mathcal{O}(\Omega, F),$$

where E'_c denotes the space E' equipped with the compact-open topology [5].

(ii) Let $G = \operatorname{Ker} J$ be nuclear and $\{U_n\}$ a decreasing basis of absolutely convex neighbourhoods of zero in E such that the canonical map θ_n^m: $G_n \doteq \widehat{G/p_{G\cap U_n}^{-1}(0)} \to \widehat{G/p_{G\cap U_m}^{-1}(0)}$ is nuclear for every $n > m$.

Since J is open, $\{JU_n\}$ forms a decreasing basis of neighbourhoods of zero in F. By J_n we denote the continuous linear map from $E_n = E(U_n)$ onto $F_n = F(JU_n)$ induced by J. Then

$$\mathcal{O}(\Omega, E\hat{\otimes}_\varepsilon Q) = \varprojlim \mathcal{O}(\Omega, E_n\hat{\otimes}_\varepsilon Q) \quad \text{and} \quad \mathcal{O}(\Omega, F\hat{\otimes}_\varepsilon Q) = \varprojlim \mathcal{O}(\Omega, F_n\hat{\otimes}_\varepsilon Q).$$

Thus in order to prove that $J\hat{\otimes}_\varepsilon \mathrm{id}$ is surjective it suffices to show that the maps $\widehat{J_n\hat{\otimes}_\varepsilon \mathrm{id}}$ satisfy the conditions of Lemma 2.1.

(L_1) is trivial.

(L_2) Let $g \in \mathcal{O}(\Omega, F_{n+1}\hat{\otimes}_\varepsilon Q)$. Since $\theta^n_{n+1}\colon G_{n+1} \to G_n$ is nuclear, by the Hahn–Banach theorem it can be extended to a continuous linear map $h\colon E_{n+1} \to G_n$. For each $v \in F_{n+1}$ put $dv = \omega^n_{n+1}u - hu$, where $u \leftarrow E_{n+1}$, $J_{n+1}u = v$. Obviously d is continuous linear. Hence, setting $f = (d\hat{\otimes}_\varepsilon \mathrm{id})g$, we get an $f \in \mathcal{O}(\Omega, E_n\hat{\otimes}_\varepsilon Q)$ such that $(J_n\hat{\otimes}_\varepsilon \mathrm{id})f = \big(\omega(JU_{n+1}, JU_n)\hat{\otimes}_\varepsilon \mathrm{id}\big)g$.

(L_3) Suppose we are given a $g \in \mathcal{O}(\Omega, G_n\hat{\otimes}_\varepsilon Q)$, an $\varepsilon > 0$, a compact set K in Ω and a continuous seminorm q on Q. Since θ^m_n is nuclear and $\mathrm{Im}\,\theta^m_n$ is dense in G_m for $n > m$, it follows that there exists a commutative diagram

$$\begin{tikzcd} G_{n+1} \arrow[r, "\theta^n_{n+1}"] & G_n \arrow[r, "\theta^{n-3}_n"] \arrow[d, "b"] & G_{n-3} \\ l^1 \arrow[u, "a"] \arrow[r, "\theta"'] & l^1 \arrow[r, "\gamma"'] & l^1 \arrow[u, "c"'] \end{tikzcd}$$

in which γ is compact and $\mathrm{Im}\,\gamma\theta$ is dense in $\mathrm{Im}\,\gamma$. Let $\{e_j\}$ be the canonical basis of l^1. By the compactness of $\overline{\{\gamma e_j\}}$ there exists a bounded set $\{x_j\}$ in l^1 such that

$$\|\gamma\theta x_j - \gamma e_j\| < \varepsilon \quad \text{for } j \geqslant 1.$$

For each $\xi \in l^1$ put

$$h\xi = \sum_{j=1}^{\infty} \xi_j x_j.$$

Then $h\colon l^1 \to l^1$ is continuous linear and

$$\|\theta^{n-3}_{n+1}ahb - \theta^{n-3}_n\| = \|c\gamma b\theta^n_{n+1}ahb - c\gamma b\| = \|c\gamma\theta hb - c\gamma b\| \leqslant \varepsilon\|c\|\,\|b\|.$$

Putting $f = (ahb\hat{\otimes}_\varepsilon \mathrm{id})g$ we get an $f \in \mathcal{O}(\Omega, G_{n+1}\hat{\otimes}_\varepsilon Q) = \mathrm{Ker}\,\widehat{(J_{n+1}\hat{\otimes}_\varepsilon \mathrm{id})}$ such that

$$\sup_{u\in K} \tilde{q}_{n-3}\big(\theta^{n-3}_{n+1}f(u) - \theta^{n-3}_n g(u)\big) \leqslant \varepsilon\|c\|\,\|b\| \sup_{u\in K} \tilde{q}_n(g(u)),$$

where $\tilde{q}_n$ denotes the seminorm on $G_n\hat{\otimes}_\varepsilon Q$ induced by q.

Hence condition (L_3) is fulfilled. The theorem is proved.

In [8] Gejler has proved that a Frechet space P is finite dimensional if and only if for every continuous linear map J from a Frechet space E onto a Frechet space F and for all $g \in \mathrm{Hom}(P, F)$ there exists an $f \in \mathrm{Hom}(P, E)$ such that $Jf = g$. Combining this with Theorem 2.1 we get the following

THEOREM 2.2. *A Frechet space P is finite dimensional if and only if, whenever E, F are Frechet spaces and $J\colon E \to F$ is a continuous linear surjection, then $\hat{J}\colon \mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega, F)$ is surjective for every open set Ω in P.*

THEOREM 2.3. *Let G be a closed subspace of a Frechet space E. Then there exists a continuous linear map θ from E into G'' such that $\theta|G = \mathrm{id}$ if and only if there exists a sequentially continuous linear extension map $T_{F'}\colon \mathcal{O}_b(G, F') \to \mathcal{O}_b(E, F')$ for all locally convex space F, where F' is quasi-complete.*

We need the following

LEMMA 2.2. *Let L be a locally convex space and let $B \in \mathscr{B}(L)$. Then $2B$ is $\sigma(L'', L')$-dense in B^{∞}.*

Proof. Since $L(B)$ is a normed space, B is $\sigma(L(B)'', L(B)')$-dense in the unit ball U of $L(B)''$. On the other hand, since $i_B''(2U) \supseteq B^{\infty}$, where i_B denotes the canonical embedding of $L(B)$ into L, we infer that $2B$ is $\sigma(L'', L')$-dense in B^{∞}.

LEMMA 2.3. *Let E be a Frechet space and let L be a locally convex space. Let $\mathscr{P}_n(E, L')$ denote the space of continuous n-homogeneous polynomials on E with values in L'. Then there is a sequentially continuous linear extension map $\theta_n\colon \mathscr{P}_n(E, L') \to \mathscr{P}_n(E'', L')$ such that*

$$\sup\{|v\theta_n f(u_1'', \ldots, u_n'')|\colon v \in D,\ u_i'' \in B^{\infty}\} \leqslant ((2n)^n/n!)\sup\{|vf(u)|\colon v \in D,\ u \in B\}$$

for all bounded sets D and B in L and E, respectively.

Proof. Let $f \in \mathscr{P}_n(E, L')$ and let $\check{f}\colon E^n = E \times \ldots \times E \to L'$ denote the n-linear map associated with f. Define a continuous n-linear map $f^1\colon E'' \times E^{n-1} \to L'$ by

$$f^1(u_1'', \ldots, u_n)(v) = u_1''(f_{(u_2, \ldots, u_n)}(v)),$$

where $f_{(u_2, \ldots, u_n)}\colon E \to L'\colon f_{(u_2, \ldots, u_n)}(u_1)v = f(u_1, \ldots, u_n)v$. Then by Lemma 2.2 we have

$$\begin{aligned}\sup\{|f^1(u_1'', u_2, \ldots, u_n)v|\colon v \in D,\ u_1'' \in B^{\infty},\ u_2, \ldots, u_n \in B\} \\ = \sup\{|u_1''(f_{(u_2, \ldots, u_n)}(v))|\colon v \in D,\ u_1'' \in B^{\infty},\ u_2, \ldots, u_n \in B\} \\ \leqslant \sup\{|u_1(f_{(u_2, \ldots, u_n)}(v))|\colon v \in D,\ u_1 \in 2B,\ u_2, \ldots, u_n \in B\} \\ = 2\sup\{|f(u_1, \ldots, u_n)v|\colon v \in D,\ u_1, \ldots, u_n \in B\}.\end{aligned}$$

Similarly we define a continuous n-linear map $f^2\colon E'' \times E'' \times E^{n-2} \to L'$ by

$$f^2(u_1'', u_2'', u_3, \ldots, u_n)v = u_2'' f^1_{(u_1'', u_3, \ldots, u_n)}(v).$$

By Lemma 2.2 we have

$$\begin{aligned}\sup\{|f^2(u_1'', u_2''; u_3, \ldots, u_n)v|\colon v \in Du_2'',\ u_1'' \in B^{\infty},\ u_3, \ldots, u_n \in B\} \\ = \sup\{|u_2'' f^1(u_1'', u_3, \ldots, u_n)v|\colon v \in D,\ u_2'', u_1'' \in B^{\infty},\ u_3, \ldots, u_n \in B\} \\ \leqslant \sup\{|u_2 f_{(u_1, \ldots, u_n)}(v)|\colon v \in D,\ u_2, u_1 \in 2B,\ u_3, \ldots, u_n \in B\} \\ = 2^2\sup\{|f(u_1, \ldots, u_n)v|\colon v \in D,\ u_1, \ldots, u_n \in B\}.\end{aligned}$$

Continuing this process, we get a continuous n-linear map $\theta_n f: E''^n \to L'$ such that

$$\begin{aligned}(2.1)\quad \sup\{|(\theta_n f)(u_1'', \ldots, u_n'')v|: v\in D,\ u_1'', \ldots, u_n'' \in B^\infty\}\\ \leqslant 2^n \sup\{|f(u_1, \ldots, u_n)v|: v\in D,\ u_1, \ldots, u_n\in B\}\\ \leqslant (2^n n^n/n!) \sup\{|f(u)v|: v\in D,\ u\in B\}.\end{aligned}$$

Obviously $\theta_n: \mathscr{P}_n(E, L') \to \mathscr{P}_n(E'', L')$ is linear and $\theta_n f|E = f$ for all $f\in\mathscr{P}_n(E, L')$. By (2.1) and in view of the fact that each bounded countable set in E'' is equicontinuous [12], θ_n is sequentially continuous.

Proof of Theorem 2.3. Let $\theta: E \to G''$ be a continuous linear map such that $\theta|G = \mathrm{id}$ and let $\hat{\theta}$ denote the continuous linear map from $\mathcal{O}_b(G'', F')$ into $\mathcal{O}_b(E, F')$ induced by θ. Then $\hat{\theta} f|G = f|G$ for all $f\in\mathcal{O}_b(G'', F')$. Hence it suffices to show that there exists a sequentially continuous linear map $\gamma: \mathcal{O}_b(G, F') \to \mathcal{O}_b(G'', F')$ such that $\gamma f|G = f$ for $f\in\mathcal{O}_b(G, F')$.

Let $f\in\mathcal{O}_b(G, F')$. Consider the Taylor expansion of f at zero

$$f(u) = \sum_{n=0}^{\infty} P_n f(u) \quad \text{for } u\in G.$$

Let $B\in\mathscr{B}(G)$. Since $f|G(B)\in\mathcal{O}_b(G(B), F')$ we have

$$\left(\limsup_n \sqrt[n]{p_{(D,B)} P_n f}\right)^{-1} = \infty$$

for $D\in\mathscr{B}(F)$, where

$$p_{(D,B)}(P_n f) = \sup\{|(P_n f u)v|: v\in D,\ u\in B\}.$$

Hence, by Lemma 2.3,

$$(2.2)\qquad \left(\limsup_n \sqrt[n]{p_{(D,B)}\, \theta_n P_n f}\right)^{-1} = \infty.$$

Thus the series $\sum_{n=0}^{\infty} \theta_n P_n f$ converges to an element $\gamma f\in\mathcal{O}_b(G'', F')$. Obviously $\gamma f|G = f$ for $f\in\mathcal{O}_b(G, F')$.

Let $\{f^n\}\to 0$ in $\mathcal{O}_b(G, F')$ and let $B''\in\mathfrak{B}(G'')$. Since G is Frechet, every bounded countable set in G'' is equicontinuous ([11], Corollary 1, p. 153), and we may assume that $B'' = B^\infty$ for some $B\in\mathfrak{B}(G)$. Then for all $D\in\mathfrak{B}(F)$

we have

$$\sup\{|\sum_{k=0}^{\infty} \theta_k P_k f^n(u'')v|: v\in D, u''\in B^{\infty}\}$$
$$\leqslant \sum_{k=0}^{\infty} \sup\{|\theta_k P_k f^n(u'')|: v\in D, u''\in B^{\infty}\} \leqslant \sum_{k=0}^{\infty} (2^k k^k/k!)\, p_{(D,B)}(P_k f^n)$$
$$\leqslant \sum_{k=0}^{\infty} ((k/3)^k/k!)\, p_{(D,3B)}(f^n) \to 0.$$

Hence γ is sequentially continuous.

The converse statement is trivial.

If G' is bornological, then each bounded set in G'' is equicontinuous [12]; thus the following is an immediate consequence of Theorem 2.3.

COROLLARY 2.1. *Let G be a closed subspace of a Frechet space E such that G' is bornological. Then the following are equivalent:*

(i) *There exists a continuous linear map θ: $E \to G''$ such that $\theta|G = \mathrm{id}$.*

(ii) *There exists a continuous linear extension map β: $G' \to E'$.*

(iii) *There exists a continuous linear extension map T: $\mathcal{O}_b(G) \to \mathcal{O}_b(E)$.*

(iv) *There exists a continuous linear extension map $T_{F'}$: $\mathcal{O}_b(G, F') \to \mathcal{O}_b(G'', F')$ for any locally convex space F such that F' is quasi-complete.*

COROLLARY 2.2. *Let S be a σ-compact space and let E be a Frechet space containing $C(S)$ as a subspace. Then for every locally convex space F with F' quasi-complete there exists a continuous linear extension map $T_{F'}$: $\mathcal{O}_b(C(S), F') \to \mathcal{O}_b(E, F')$.*

Proof. By Corollary 2.1 it suffices to show that there exists a continuous linear map θ: $E \to C(S)''$ such that $\theta|C(S) = \mathrm{id}$ and $C(s)'$ is bornological.

We write $S = \bigcup_{n=1}^{\infty} S_n$, where $S_n \subset \mathrm{Int}\, S_{n+1}$ and S_n are compact. Then $C(S)' = \bigcup_{n=1}^{\infty} C(S_n)'$ and $r_{n+1}^{n'}$ is imbedding for $n \geqslant 1$, where r_{n+1}^{n}: $C(S_{n+1}) \to C(S_n)$ denote the restriction map. For each $n \geqslant 1$ define a continuous linear map P_{n+1}^{n}: $C(S_{n+1})' \to C(S_n)'$ by $P_{n+1}^{n}\mu = \mu|S_n$. Then $P_{n+1}^{n} r_{n+1}^{n'} = \mathrm{id}$. Hence $P_{n+1}^{n'}$ is a continuous linear map from $C(S_n)''$ into $C(S_{n+1})''$ such that $r_{n+1}^{n''} P_{n+1}^{n'} = \mathrm{id}$. Combining this with the relation $C(S)'' = \varprojlim C(S_n)''$, we see that $C(S)'' \equiv \prod_{n=1}^{\infty} C(S_n)''$. Since $C(S_n)''$ is a $P\lambda_n$-space for $n \geqslant 1$, [9], there exists a continuous linear map θ: $E \to C(S)''$ which extends the canonical embedding $C(S) \to C(S)''$. Since $C(S_n)'$ is a subspace of $C(S_{n+1})'$ and since each linear map from $C(S)'$ into a locally convex space which is continuous on all bounded subsets of $C(S)'$ is continuous [12], it follows that $C(S)'$ is bornological.

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