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Convex functions of higher orders in Euclidean spaces

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1. Throughout this paper R^m denotes the space of all points $a = (a_1, a_2, \ldots, a_m)$, where a_i are arbitrary real numbers ($i = 1, 2, \ldots, m$). For arbitrary $a \in R^1$, $b \in R^m$ and a set $A \subset R^m$ we write

$$aA + b = \{x\colon\ x = aa + b,\ a \in A\}.$$

The symbol $m(A)$ denotes the m-dimensional Lebesgue measure of the set $A \subset R^m$ and $|a-b|$ the Euclidean distance between points $a, b \in R^m$. Further, let

$$R^m_+ \stackrel{\mathrm{df}}{=} \{h \in R^m \setminus \{0\}\colon \text{ the first non-zero coordinate of } h \text{ is positive}\}.$$

Thus we have $R^m_+ \cup (-R^m_+) \cup \{0\} = R^m$ and $R^m_+ \cap (-R^m_+) = \emptyset$.

Definition 1. Let f be a real-valued function defined on an open convex set $\mathscr{D} \subset R^m$. We say that f is *convex of n-th order* iff

$$\Delta_h^{n+1} f(x) \geqslant 0 \ (^1) \tag{1}$$

holds for all $x \in \mathscr{D}$ and $h \in R^m_+$ such that $x + ih \in \mathscr{D}$ for $i = 1, 2, \ldots, n+1$.

This definition requires some words of comment. Namely, in the case of $m = 1$, it reduces to the well-known definition of a convex function of n-th order in a single variable. Moreover, the choice of the set R^m_+ as the set of admissible values on h is inessential in the case of n odd. In fact, a simple calculation shows that

$$\Delta_{-h}^{n+1} f(x) = (-1)^{n+1} \Delta_h^{n+1} f(y), \qquad y = x - (n+1)h$$

and thus

$$\Delta_{-h}^{n+1} f(x) = \Delta_h^{n+1} f(y)$$

for n odd.

(1) $\Delta_h^0 f(x) \stackrel{\mathrm{df}}{=} f(x)$; $\Delta_h^{k+1} f(x) \stackrel{\mathrm{df}}{=} \Delta_h^k f(x+h) - \Delta_h^k f(x)$.

However, if n is an even number, then $\Delta_{-h}^{n+1}f(x) = -\Delta_h^{n+1}f(y)$, and letting h vary in the whole R^m, we shall reduce the class of functions considered to the class of polynomial functions of n-th order, i.e. functions fulfilling the condition $\Delta_h^{n+1}f(x) \equiv 0$. This is the reason why the set R_+^m appears in Definition 1.

It is also easily seen that one can replace R_+^m by any other set $A \subset R^m$ such that $A \cup (-A) \cup \{0\} = R^m$ and $A \cap (-A) = \emptyset$. Anyhow, R_+^m seems to be the most natural set of such form.

One can also ask if there exist, for even n, convex functions of n-th order which are not polynomial functions. It is not difficult to construct a certain class of such functions. For, let

$$f(x_1, x_2, \ldots, x_m) = f_1(x_1) + f_2(x_2) + \ldots + f_m(x_m),$$

where $f_i\colon R^1 \to R^1$ for $i = 1, 2, \ldots, m$, f_1 is convex of n-th order and not polynomial, and $f_2, f_3, \ldots, f_m$ are arbitrary polynomial functions of n-th order in a single variable.

For example, in the case of $m = 2$, we can take

$$f(x_1, x_2) = e^{x_1} + x_2^2.$$

This function is convex of order 2 and not polynomial, since

$$\Delta_{(h_1,h_2)}^3 f(x_1, x_2) = \Delta_{h_1}^3 e^{x_1} + \Delta_{h_2}^3 x_2^2 = e^{x_1}(e^{h_1}-1)^3 \geqslant 0$$

for $(h_1, h_2) = h \in R_+^2$, and $\Delta_{(h_1,h_2)}^3 f(x_1, x_2) > 0$ whenever $h_1 > 0$.

Popoviciu [7], in the second part of his paper, is also concerned with the problem of a generalization of the notion of convex function of higher order to the case of $m = 2$. In this case our definition is equivalent to the second of the two definitions considered in [7]; however, in [7] the author restricts himself to n odd.

The purpose of the present paper is to prove that a convex function of n-th order (in the sense of Definition 1) bounded (bilaterally) on a set $T \subset \mathscr{D}$ such that the suitable set $H^k(T)$ (defined in section 5) is of positive Lebesgue measure, or of the second category with the Baire property, is continuous in $\mathscr{D}$. An analogous result has earlier been proved by Ciesielski [1] in the case of $k = 0$, $m = 1$. The present note yields simultaneously a new proof of Z. Ciesielski's theorem.

2. Suppose that points $a = (a_1, a_2, \ldots, a_m)$, $b = (b_1, b_2, \ldots, b_m)$, $a \neq b$, belong to $\mathscr{D}$.

DEFINITION 2. We say that $a < b$ iff $(b-a) \in R_+^m$.

Remark 1. If points $a, x, b \in \mathscr{D}$, $a < b$, are collinear and pairwise different, then x lies between a and b iff $a < x < b$.

The proof is obvious.

In the sequel the notion of divided difference investigated by T. Popoviciu will play an important part. But it must be slightly modified, since we are concerned with $m \geqslant 1$.

DEFINITION 3. Let points $x_1, x_2, \ldots, x_{k+1} \in \mathscr{D}$ be collinear and such that $x_1 < x_2 < \ldots < x_{k+1}$. We may assign to them linear coordinates $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_{k+1}$, respectively, where $\lambda_i = |x_i - x_1|$ for $i = 1, 2, \ldots, k+1$. We put

$$[\lambda_1; f] \overset{\mathrm{df}}{=} f(x_1),$$

$$[\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; f] \overset{\mathrm{df}}{=} \frac{[\lambda_2, \lambda_3, \ldots, \lambda_{k+1}; f] - [\lambda_1, \lambda_2, \ldots, \lambda_k; f]}{\lambda_{k+1} - \lambda_1}.$$

We may write $[\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; f]$ in the form

$$\frac{U(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; f)}{V(\lambda_1, \lambda_2, \ldots, \lambda_{k+1})},$$

where

$$U(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; f) \overset{\mathrm{df}}{=} \begin{vmatrix} 1 & \lambda_1 & & \lambda_1^{k-1} & f(x_1) \\ & & & & . \\ 1 & \lambda_{k+1} & & \lambda_{k+1}^{k-1} & f(x_{k+1}) \end{vmatrix}$$

and

$$V(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \overset{\mathrm{df}}{=} \begin{vmatrix} 1 & \lambda_1 & & \lambda_1^{k-1} & \lambda_1^{k} \\ & & & & . \\ 1 & \lambda_{k+1} & & \lambda_{k+1}^{k-1} & \lambda_{k+1}^{k} \end{vmatrix},$$

Following Popoviciu, let us note that

LEMMA 1. *If collinear points $x_1 < x_2 < \ldots < x_{n+2} \in \mathscr{D} \subset R^m$ are so chosen that $x_2, x_3, \ldots, x_{n+1}$ divide rationally the segment $\overline{x_1; x_{n+2}}$ (i.e. the segment $\overline{x_k; x_{k+1}}$ is commensurable with $\overline{x_1; x_{n+2}}$ for $k = 1, 2, \ldots, n+1$) and if $f\colon \mathscr{D} \to R^1$ is convex of n-th order, then for their linear coordinates λ_i $(i = 1, 2, \ldots, n+2)$ the inequality*

$$[\lambda_1, \lambda_2, \ldots, \lambda_{n+2}; f] \geqslant 0$$

holds.

The proof is formally the same as in [7].

Let us fix points $x, y \in \mathscr{D}$, $x < y$, and pass through them the straight line l. Further, let us take points $x_1 < x_2 < \ldots < x_n \in l \cap \mathscr{D}$, $x_n < x$, in such a way that $x_2, x_3, \ldots, x_n, x$ divide rationally the segment $\overline{x_1; y}$. Let the suitable linear coordinates be $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda < \varkappa$, respectively ($\lambda$ and $\varkappa$ being the coordinates of x and y respectively). By Lemma 1 we have

$$U(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda, \varkappa; f) \geqslant 0$$

(since $V(\lambda_1, \ldots, \lambda_n, \lambda, \varkappa) > 0$ in view of the fact that the sequence $\lambda_1, \ldots, \lambda_n, \lambda, \varkappa$ is strictly increasing).

Expanding this determinant with respect to the last column, we obtain:

$$\sum_{i=1}^{n}(-1)^{n-i}f(x_i)\quad V(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n, \lambda, \varkappa)-$$
$$-f(x)\quad V(\lambda_1, \lambda_2, \ldots, \lambda_n, \varkappa)+f(y)\quad V(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda) \geqslant 0$$

(we assume here that $\lambda_{n+1} = \lambda$). Thus

$$(2)\qquad f(x) \leqslant f(y)\frac{V(\lambda_1, \ldots, \lambda_n, \lambda)}{V(\lambda_1, \ldots, \lambda_n, \varkappa)}+ \\ +\sum_{i=1}^{n}(-1)^{n-i}f(x_i)\frac{V(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda, \varkappa)}{V(\lambda_1, \ldots, \lambda_n, \varkappa)}$$

and

$$(3)\qquad f(y) \geqslant f(x)\frac{V(\lambda_1, \ldots, \lambda_n, \varkappa)}{V(\lambda_1, \ldots, \lambda_n, \lambda)}- \\ -\sum_{i=1}^{n}(-1)^{n-i}f(x_i)\frac{V(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda, \varkappa)}{V(\lambda_1, \ldots, \lambda_n, \lambda)}.$$

Inequalities (2) and (3) will be useful in the proof of the following

LEMMA 2. *Let $f\colon \mathscr{D} \to R^1$ be a convex function of n-th order. If there exists a point $x_0 \in \mathscr{D}$ such that f is bounded in a neighbourhood of x_0, then it is also bounded in a neighbourhood of every point $z_0 \in \mathscr{D}$.*

Proof. Let $K(x, \varrho)$ denote the ball with the center x and radius ϱ and assume that f is bounded on a ball $K = K(x_0, \eta)$.

Let us take an arbitrary point $z_0 \in \mathscr{D} \setminus K(x_0, \eta)$ and pass the straight line l through x_0 and z_0. Suppose that $x_0 < z_0$ (in the case of $z_0 < x_0$ the proof is analogous). Since $\mathscr{D}$ is open, we can find a point $p \in \mathscr{D} \cap l$ such that $z_0 < p$.

Let us consider a cone S with the vertex at p and such that $K(x_0, \frac{1}{2}\eta)$ is inscribed in S. We shall show that f is bounded on the ball $K(z_0, \mu)$ centered at z_0 and inscribed in S.

In fact, let us take an arbitrary point $z \in K(z_0, \mu)$ and pass the straight line l' through z and p. The segment $K \cap l'$ is of a length greater than $\eta\sqrt{3}$. Now, we choose points $x_1 < x_2 < \ldots < x_n \in K \cap l'$ such that $x_2, x_3, \ldots, x_n, z$ divide rationally the segment $\overline{x_1; p}$ and

$$(4)\qquad \min|x_{i+1}-x_i| \geqslant \frac{1}{2n}\eta\sqrt{3}\quad \text{for } i = 1, 2, \ldots, n-1.$$

We put $x = z$, $y = p$ in (2). Then $f(x_i)$ vary in a bounded set, since f is bounded on K, and the Vandermonde determinants occurring in (2) are bounded above and below by positive constants independent of z (in virtue of (4)). Hence f is bounded above on $K(z_0, \mu)$.

Similarly, taking points $x_1, x_2, \ldots, x_{n+1} \in K \cap l'$ such that $x_2, x_3, \ldots, \ldots, x_{n+1}$ divide rationally the segment $\overline{x_1; z}$ and fulfil condition (4) ($i = 1, 2, \ldots, n$), and making use of relation (3) with $y = z$ and $x = x_{n+1}$, we infer that f is bounded below on $K(z_0, \mu)$. Thus f is bounded (bilaterally) on $K(z_0, \mu)$.

LEMMA 3. *Let $f\colon \mathcal{D} \to R^1$ be a convex function of n-th order. If f is bounded in a neighbourhood of a point $x_0 \in \mathcal{D}$, then it is continuous at x_0.*

Proof. It is enough to show that

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = \lim_{\substack{x \to x_0 \\ x_0 < x}} f(x) = f(x_0). \tag{5}$$

Let f be bounded on a ball $K = K(x_0, \eta)$. We take an arbitrary point $x \in K$, $x < x_0$, and pass the straight line l through x and x_0. Next, we choose two sets of points belonging to $K \cap l$:

1° $x_1 < x_2 < \ldots < x_n$ such that $x_n < x$ and $x_2, x_3, \ldots, x_n, x$ divide rationally the segment $\overline{x_1; x_0}$. We assign to the points $x_1, x_2, \ldots, x_n, x, x_0$ the linear coordinates $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda, \lambda_0$, respectively;

2° $x'_1 < x'_2 < \ldots < x'_{n-1} < x'_{n+1}$ such that $x'_{n-1} < x < x_0 < x'_{n+1}$ and $x'_2, x'_3, \ldots, x'_{n-1}, x, x_0$ divide rationally the segment $\overline{x'_1; x'_{n+1}}$. We assign to the points $x'_1, x'_2, \ldots, x'_{n-1}, x, x_0, x'_{n+1}$ the linear coordinates $\lambda'_1, \lambda'_2, \ldots, \ldots, \lambda'_{n-1}, \lambda', \lambda'_0, \lambda'_{n+1}$, respectively.

One can assume that $x_1 = x'_1$, which implies that $\lambda = \lambda'$ and $\lambda_0 = \lambda'_0$. It is obvious that $x \to x_0$ is equivalent to $\lambda \to \lambda_0$. Moreover, by Lemma 1,

$$U(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda, \lambda_0; f) \geqslant 0$$

and hence (compare (2)), writing λ as λ_{n+1}, we have

$$f(x_0) - f(x) \geqslant f(x_0) \cdot \left[1 - \frac{V(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda)}{V(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_0)}\right] - \tag{6}$$
$$-(\lambda_0 - \lambda) \sum_{i=1}^{n} (-1)^{n-i} \frac{f(x_i)}{\lambda_0 - \lambda} \frac{V(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n, \lambda, \lambda_0)}{V(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_0)}.$$

Similarly, since

$$U(\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}, \lambda, \lambda_0, \lambda'_{n+1}; f) \geqslant 0.$$

we have (writing λ as λ'_n)

$$(7)\quad f(x_0)-f(x) \leqslant f(x_0)\cdot\left[1-\frac{V(\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}, \lambda, \lambda'_{n+1})}{V(\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}, \lambda_0, \lambda'_{n+1})}\right]+$$

$$+(\lambda_0-\lambda)\sum_{i=1}^{n-1}(-1)^{n-i}\frac{f(x_i)}{\lambda_0-\lambda}\,\frac{V(\lambda'_1, \ldots, \lambda'_{i-1}, \lambda'_{i+1}, \ldots, \lambda'_{n-1}, \lambda, \lambda_0, \lambda'_{n+1})}{V(\lambda'_1, \ldots, \lambda'_{n-1}, \lambda_0, \lambda'_{n+1})}+$$

$$+f(x_{n+1})\cdot\frac{V(\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}, \lambda, \lambda_0)}{V(\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}, \lambda_0, \lambda'_{n+1})}.$$

As in proof of Lemma 2, we may (by a convenient choice of the points x_i and x'_i) make the Vandermonde determinants occurring in (6) and (7) bounded away from zero and infinity. Moreover, $V(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n, \lambda, \lambda_0)$ and $V(\lambda'_1, \ldots, \lambda'_{i-1}, \lambda'_{i+1}, \ldots, \lambda'_{n-1}, \lambda, \lambda_0, \lambda'_{n+1})$ contain the factor $(\lambda_0-\lambda)$. Consequently, the expressions under the $\sum$ sign in (6) and (7) remain bounded, and hence

$$\lim_{\substack{x\to x_0\\ x<x_0}}[f(x)-f(x_0)] = 0.$$

Interchanging the roles of x and x_0 (λ and λ_0), we conclude by the same argument that

$$\lim_{\substack{x\to x_0\\ x_0<x}}[f(x_0)-f(x)] = 0,$$

which completes the proof.

Lemma 2 and Lemma 3 imply the following

THEOREM 1. *If $f\colon \mathcal{D}\to R^1$ is convex of n-th order and if there exists a ball $K(x_0, \eta)\subset\mathcal{D}$ on which f is bounded, then f is continuous in $\mathcal{D}$.*

3. For a given set $B\subset R^m$, let B^* denote the set of all points of density of B. The following theorem, due to J. H. B. Kemperman, will be used:

THEOREM 2. *For fixed sets $B_1, B_2, \ldots, B_k\subset R^m$ with a positive Lebesgue measure, and for arbitrary points $b_i\in B_i^*$, $i = 1, 2, \ldots, k$, there exist positive numbers δ and ε such that $|x_i-b_i|<\delta$, $i = 1, 2, \ldots, k$, implies* $m\big((B_1-x_1)\cap(B_2-x_2)\cap\ldots\cap(B_k-x_k)\big)>\varepsilon$ (see [3]).

A similar result (in a somewhat stronger form) was also obtained by S. Kurepa ([5], Lemma 1).

An analogous theorem may be proved for sets of the second category with the Baire property. At first let us note that if a set $B\subset R^m$ has the Baire property and is of the second category, then there exist points $b\in B$ and a $\mu>0$ such that $B\cap K(b, \mu)$ is residual in $K(b, \mu)$. Let B^{**} denote the set of all points b with the above property.

We say that a set $B \subset R^m$ is of the second category at a point b iff for every neighbourhood U_b of b the set $U_b \cap B$ is of the second category. The set of all points at which B is of the second category will be denoted by $D(B)$. The properties of the operation D may be found e.g. in [4].

Now, we have the following

THEOREM 3. *For fixed sets $B_1, B_2, \ldots, B_k \subset R^m$ of the second category with the Baire property, and for arbitrary points $b_i \in B_i^{**}$, $i = 1, 2, \ldots, k$, there exists a positive number δ such that $|x_i - b_i| < \delta$, $i = 1, 2, \ldots, k$, implies $D\big((B_1 - x_1) \cap (B_2 - x_2) \cap \ldots \cap (B_k - x_k)\big) \neq \emptyset$.*

Proof. $0 \in (B_i - b_i)^{**}$ for $i = 1, 2, \ldots, k$. Thus, for i fixed, there exists a ball $K_i = K(0, \varepsilon_i)$, $\varepsilon_i > 0$, such that $K_i \setminus (B_i - b_i)$ is of the first category. Let $K = K(0, \varepsilon)$ be the smallest of the balls K_i, $i = 1, 2, \ldots, k$. We put $\delta = \varepsilon/2k$.

Let us take arbitrary x_i such that $|x_i - b_i| < \delta = \varepsilon/2k$. We write

$$d_i = b_i - x_i, \quad C_i = K \cap (B_i - b_i), \quad i = 1, 2, \ldots, k.$$

By hypothesis, C_i is residual in K and hence the set $E_i \stackrel{\mathrm{df}}{=} C_i + d_i$ is residual in $K + d_i$. It is easily seen that

$$E_i \subset B_i - x_i. \tag{8}$$

Let us put $P_i \stackrel{\mathrm{df}}{=} K \cap (K + d_i)$ and $Q_i \stackrel{\mathrm{df}}{=} K \setminus (K + d_i)$. We have $P_i \cup Q_i = K$ for $i = 1, 2, \ldots, k$. Thus, for i fixed,

$$K \setminus E_i = (P_i \setminus E_i) \cup (Q_i \setminus E_i).$$

Since $P_i \setminus E_i$ is of the first category, we have

$$D(K \setminus E_i) = D(Q_i \setminus E_i) \subset D(Q_i) \subset \bar{Q}_i.$$

Hence

$$\bar{K} \setminus \bigcup_{i=1}^{k} D(K \setminus E_i) \neq \emptyset. \tag{9}$$

On the other hand,

$$K = \Big(K \cap \bigcap_{i=1}^{k} E_i\Big) \cup \bigcup_{i=1}^{k} (K \setminus E_i),$$

and so

$$\bar{K} = D(K) = D\Big(K \cap \bigcap_{i=1}^{k} E_i\Big) \cup \bigcup_{i=1}^{k} D(K \setminus E_i).$$

Thus, by (8)

$$\bar{K} \setminus \bigcup_{i=1}^{k} D(K \setminus E_i) \subset D\Big(K \cap \bigcap_{i=1}^{k} E_i\Big) \subset D\Big(\bigcap_{i=1}^{k} (B_i - x_i)\Big),$$

and in view of (9) we obtain our assertion.

COROLLARY 1. *Let $T \subset R^m$ be of positive Lebesgue measure or of the second category with the Baire property. If $0 \in T^*$ resp. $0 \in T^{**}$, then there exists a positive number $\delta > 0$ such that for a fixed positive integer n and for every $\sigma_1, \sigma_2, \ldots, \sigma_{n+1}$ belonging to $K(0, \delta)$ we have*

$$(10)\quad \left(-\frac{1}{n+1}T+\sigma_{n+1}\right)\cap\left(-\frac{1}{n}T+\sigma_n\right)\cap\ldots\cap(-T+\sigma_1)\cap \\ \cap(T-\sigma_1)\cap\ldots\cap\left(\frac{1}{n}T-\sigma_n\right)\cap\left(\frac{1}{n+1}T-\sigma_{n+1}\right)\neq\emptyset.$$

4. For a set $T \subset R^m$ and a positive integer n fixed, we define the set $H(T)$ as follows:

$$H(T) \stackrel{\mathrm{df}}{=} \{x \in R^m\colon \text{there exists an } h \in R^m \text{ such that } x-ih,\ x+ih \in T, \\ \text{for } i = 1, 2, \ldots, n+1\}.$$

Evidently $T \subset H(T)$ and, if T is contained in a convex set $\mathscr{D}$, then so is also $H(T)$.

THEOREM 4. *If $T \subset R^m$ is of positive Lebesgue measure or of the second category with the Baire property, then there exist a point $x_0 \in T$ and a number $\delta > 0$ such that $K(x_0, \delta)$ is contained in $H(T)$.*

Proof. Without loss of generality we can assume that $0 \in T^*$ (or $0 \in T^{**}$, respectively). Since the hypotheses of Corollary 1 are fulfilled, we put $\sigma_i = (1/i)z$ for $i = \pm 1, \pm 2, \ldots, \pm(n+1)$ and for an arbitrarily fixed point z of $K(0, \delta)$. Thus

$$\bigcap_{i=1}^{n+1}\left(-\frac{1}{i}T+\frac{1}{i}z\right)\cap\bigcap_{i=1}^{n+1}\left(\frac{1}{i}T-\frac{1}{i}z\right)\neq\emptyset,$$

and so there exists an $h \in R^m$ such that

$$h \in -\frac{1}{i}T+\frac{1}{i}z \quad\text{and}\quad h \in \frac{1}{i}T-\frac{1}{i}z \quad \text{for } i = 1, 2, \ldots, n+1,$$

i.e.

$$z-ih \in T \quad\text{and}\quad z+ih \in T \quad \text{for } i = 1, 2, \ldots, n+1,$$

which means that $z \in H(T)$. This completes the proof.

THEOREM 5. *If $f\colon \mathscr{D} \to R^1$ is convex of n-th order, and if f is bounded on a set $T \subset \mathscr{D}$, then it is also bounded on a set $H(T)$.*

Proof. By (1)

$$(11)\quad \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}f(x+jh) \geqslant 0$$

for all $x \in \mathscr{D}$ and $h \in R^m_+$ such that $x-ih,\ x+ih \in \mathscr{D}$, $i = 1, 2, \ldots, n+1$.

Let us take an arbitrary $x \in H(T)$ and suppose that $|f(t)| \leqslant M$ for $t \in T$. There exists an $h \in R^m$ such that $x+ih \in T$ for $i = 1, 2, \ldots, n+1$. If $h = 0$, then this means that $x \in T$, and thus $|f(x)| \leqslant M$. If $h \neq 0$, then, replacing if necessary h by $-h$, we may assume that $h \in R^m_+$. Now, we must distinguish two cases.

1° n is even. We have

$$f(x) \leqslant \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f(x+ih),$$

and putting $x-h$ in the place of x, we get

$$f(x) \geqslant \frac{1}{n+1} \left\{ \sum_{i=2}^{n+1} (-1)^{n-i} \binom{n+1}{i} f\big(x+(i-1)h\big) + f(x-h) \right\}.$$

By a simple calculation, from the fact that $|f(x+ih)| \leqslant M$ for $i = -1, 1, 2, \ldots, n$, we obtain

$$|f(x)| \leqslant (2^{n+1}-1)M.$$

2° n is odd. We have

$$f(x) \geqslant \sum_{i=1}^{n+1} (-1)^{n-i} \binom{n+1}{i} f(x+ih)$$

and

$$f(x) \leqslant \frac{1}{n+1} \left\{ \sum_{i=2}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} f\big(x+(i-1)h\big) + f(x-h) \right\},$$

whence, in view of $x+ih \in T$ for $i = -1, 1, \ldots, n$, we have also

$$|f(x)| \leqslant (2^{n+1}-1)M,$$

which ends the proof.

5. We define the iterates of the operation H:

$$H^0(T) \overset{\mathrm{df}}{=} T, \quad H^{k+1}(T) \overset{\mathrm{df}}{=} H\big(H^k(T)\big), \quad k = 0, 1, 2, \ldots$$

From Theorems 1, 4 and 5 we obtain the following theorem, which is the main result of the present paper.

THEOREM 6. *Let f be a real-valued function, convex of n-th order, defined on an open convex domain $\mathscr{D} \subset R^m$. Suppose that there exist a positive integer k and a set $T \subset \mathscr{D}$ such that $H^k(T)$ is of positive Lebesgue measure, or of the second category with the Baire property. If f is bounded on T, then f is continuous in $\mathscr{D}$.*

Proof. In fact, in view of Theorem 5, f is also bounded on $H^{k+1}(T)$. Moreover, Theorem 4 implies (according to our hypothesis on $H^k(T)$) that $H^{k+1}(T)$ contains a ball. Now, our assertion results from Theorem 1.

In the case of $k = 0$ and $m = 1$ Theorem 6 was proved by Ciesielski [1]. However, our proof is different.

It follows from S. Kurepa's theorem (see [6]) that a function f fulfilling condition (1) and bounded on a set of positive Lebesgue measure is also bounded on a certain ball (the author makes use of this result in order to prove Z. Ciesielski's theorem once more). Our proof of this fact is obtained by using a different method, which permits us to give a unified proof for sets of positive Lebesgue measure as well as for sets of the second category with the Baire property. On the other hand, a generalization of Ciesielski's theorem ($m=1$) consisting in replacing the set T by $H^k(T)$ is essential. A suitable example may be found in [2].

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