ANNALES
POLONICI MATHEMATICI
LIII.1 (1991)

On the strong convergence to equilibrium for randomly perturbed dynamical systems

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Abstract. We study the asymptotic behaviour of discrete time dynamical systems in the presence of stochastic perturbations. We show a sufficient condition for the asymptotic stability of distributions of the state variables. This result is applied to a dynamical system appearing in mathematical biology.

Introduction. We consider the effect of stochastic perturbation on discrete time dynamical systems from a statistical point of view. Thus we study the behaviour of the sequence of distributions corresponding to a given system. Our aim is to establish simple criteria for the strong convergence of the distributions of the state variables to a stationary density.

In some places our definitions and proofs are similar to the results of [3] and [4]. There are, however, important differences. First, we prove the strong convergence to equilibrium without the assumption that the perturbations have an absolutely continuous distribution. This was the main assumption in [4]. Moreover, in [4] in order to establish the convergence it was assumed that the density distribution function of perturbations is positive on a sufficiently large set. Finally, we do not assume the existence of higher order moments of the right-hand side of the system under consideration. This assumption was used in [3] even in the proof of weak convergence.

On the other hand, we restrict ourselves to the case of additive perturbations. Thus we consider systems of the form $x_{n+1} = S(x_n) + \xi_n$, where x_n is the state variable and ξ_n is random vector. In this case the statements and proofs are much simpler. We believe, however, that our method can be extended to the general case $x_{n+1} = S(x_n, \xi_n)$.

1. Notations. Let m denote the standard Lebesgue measure on the d-dimensional Euclidean space $\boldsymbol{R}^d$ and let V be a closed subset of $\boldsymbol{R}^d$ such that $V+V \subset V$, $m(V) > 0$. Any probability measure F defined on Borel subsets of V will be called a *distribution*. If a distribution F is absolutely continuous with respect to m, we denote by dF/dm its Radon–Nikodym derivative. If F is an arbitrary Borel measure on V we denote by $\|F\|$ the total variation of F.

We shall consider the dynamical system

(1) $$x_{n+1} = S(x_n)+\xi_n, \qquad n = 0, 1, 2, \ldots,$$

where $S\colon V \to V$ is a Borel measurable transformation and $\{\xi_n\}$ is a sequence of identically distributed independent d-dimensional random vectors with values in V. We also assume that the initial value x_0 is a random vector independent of the sequence of perturbations $\{\xi_n\}$.

By G we denote the common distribution of ξ_n, i.e.

$$G(A) = \mathrm{Prob}(\xi_n \in A) \qquad \text{for } A \subset V,\ A \text{ Borelian.}$$

We say that G is *nonsingular* if in the Lebesgue decomposition of G

(2) $$G = G_0 + G_1 \qquad (G_0 \text{ abs. cont., } G_1 \text{ singular})$$

the absolutely continuous part G_0 is not vanishing ($\|G_0\| > 0$).

We say that a transformation $S\colon V \to V$ is *nonsingular* if S is Borel measurable and $m(A) = 0 \Rightarrow m(S^{-1}(A)) = 0$ for Borel subsets A of V.

Now return to the system (1) and denote by F_n the distribution of x_n,

$$F_n(A) = \mathrm{Prob}(x_n \in A) \qquad \text{for } A \subset V,\ A \text{ Borelian.}$$

It is easy to find a recurrence relation between F_{n+1} and F_n. In fact, the random vector $S(x_n)$ has distribution $F_n(S^{-1}(A))$, and x_{n+1}, being the sum of two independent random variables, has distribution given by the convolution formula

(3) $$F_{n+1}(A) = \int_V F_n(S^{-1}(A-y))G(dy) = \int_V G(A-S(y))F_n(dy).$$

In order to make formula (3) (and subsequent ones) correct we consider S in (3) to be a mapping from V into $\boldsymbol{R}^d$ and we extend G and F_n in a trivial way to the whole $\boldsymbol{R}^d$ ($G(\boldsymbol{R}^d \setminus V) = F_n(\boldsymbol{R}^d \setminus V) = 0$). In the special case when S is nonsingular and F_n is absolutely continuous with density $f_n = dF_n/dm$ we may rewrite (3) as

$$F_{n+1}(A) = \int_V \Big[\int_{S^{-1}(A-y)} f_n(x)\,dx\Big] G(dy) = \int_A \Big[\int_V P_S f_n(x-y) G(dy)\Big]\,dx,$$

where P_S is the Frobenius–Perron operator corresponding to S. It follows that F_{n+1} is absolutely continuous and its density f_{n+1} is given by

(4) $$f_{n+1}(x) = \int_V P_S f_n(x-y) G(dy) \qquad \text{for } x \in V.$$

Thus, for S nonsingular the existence of f_n implies the existence of f_{n+1}. In particular, if F_0 is absolutely continuous then all the densities $f_n = dF_n/dm$, $n = 0, 1, \ldots$, exist.

Now define the transition operators

(5) $$\mathscr{P}F(A) = \int_V G(A-S(y))F(dy),$$

(6) $$Pf(x) = \int_V P_S f(x-y) G(dy).$$

According to (3) the evolution of measures corresponding to the dynamical system (1) is given by the sequence $\{\mathscr{P}^n F\}$. When S is nonsingular and F is absolutely continuous formula (4) gives

$$\frac{d\mathscr{P}^n F}{dm} = P^n\left(\frac{dF}{dm}\right).$$

A distribution F satisfying $\mathscr{P}F = F$ will be called *stationary.* An absolutely continuous distribution is stationary if and only if its density $f = dF/dm$ satisfies $Pf = f$. Such a density is also called stationary.

2. Preliminary results. We start from a lemma showing that the singular part of F_n vanishes as $n \to \infty$. This resembles some properties of Harris operators [1].

LEMMA 1. *Assume that S and G are nonsingular. Then for every initial distribution F_0, in the Lebesgue decomposition*

(7) $$F_n = F_{n0} + F_{n1} \quad (F_{n0} \textit{ abs. const., } F_{n1} \textit{ singular})$$

of the sequence $F_n = \mathscr{P}^n F_0$ the singular part F_{n1} satisfies

(8) $$\lim_{n\to\infty} \|F_{n1}\| = 0.$$

Proof. Using (7) we may rewrite (3) in the form

$$F_{n+1}(A) = \int_V F_{n0}(S^{-1}(A-y))G(dy) + \int_V F_{n1}(S^{-1}(A-y))G(dy) = I_{n0}(A) + I_{n1}(A).$$

I_{n0} defines an absolutely continuous measure, namely

$$I_{n0}(A) = \int_V \Big[\int_{S^{-1}(A-y)} f_{n0}(x)dx\Big] G(dy) = \int_A \Big[\int_V P_S f_{n0}(x-y)G(dy)\Big] dx,$$

where $f_{n0} = dF_{n0}/dm$. Next, I_{n1} may be rewritten as follows:

$$I_{n1}(A) = \int_V F_{n1}(S^{-1}(A-y))G_0(dy) + \int_V F_{n1}(S^{-1}(A-y))G_1(dy) = J_{n0}(A) + J_{n1}(A).$$

Again, J_{n0} is absolutely continuous,

$$J_{n0}(A) = \int_V G_0(A - S(y))F_{n1}(dy) = \int_A \Big[\int_V g_0(x - S(y))F_{n1}(dy)\Big] dx.$$

Thus, J_{n1} contains the singular part of F_{n+1} and consequently

$$\|F_{n+1,1}\| \leqslant \|J_{n1}\| \leqslant \|F_{n1}\| \, \|G_1\|.$$

By an induction argument we obtain $\|F_{n1}\| \leqslant \|F_{01}\| \, \|G_1\|^n$. Since $\|G_1\| < 1$ this completes the proof. ■

We denote by $|\cdot|$ an arbitrary, not necessarily Euclidean, norm in $\boldsymbol{R}^d$. Our next lemma describes the behaviour of the first moment of x_n,

(9) $$E(|x_n|) = \int_V |x| F_n(dx).$$

LEMMA 2. *Assume that the first moments of ξ_n are finite,*

(10) $$\int_V |x| G(dx) < \infty,$$

and that S satisfies

(11) $$|S(x)| \leqslant \alpha|x| + \beta \quad \textit{for } x \in V,$$

where α, β are constants, $\alpha < 1$. Then there is a constant $k \geqslant 0$ such that

(12) $$\limsup_{n\to\infty} E(|x_n|) < k$$

for every solution $\{x_n\}$ of (1) *starting from an initial vector x_0 with finite first moment $(E(|x_0|) < \infty)$.*

Proof. From equation (1) it follows immediately that

$$E(|x_{n+1}|) \leqslant E(|S(x_n)|) + \mu,$$

where $\mu = \int_V |x| G(dx)$. Now using (11) we obtain

$$E(|x_{n+1}|) \leqslant \alpha E(|x_n|) + \beta + \mu,$$

which implies (12) with $k > (\beta+\mu)(1-\alpha)^{-1}$. ∎

Now define

$$|x|_q = \begin{cases} |x| & \text{for } |x| \geqslant q, \\ 0 & \text{for } |x| < q. \end{cases}$$

Our goal is to describe the asymptotic behaviour of

(13) $$E(|x_n|_q) = \int_V |x|_q F_n(dx)$$

for large q and n.

LEMMA 3. *Assume that S and G satisfy* (10) *and* (11). *Then for every $\varepsilon > 0$ there is a $q > 0$ such that*

(14) $$\limsup_{n\to\infty} E(|x_n|_q) < \varepsilon$$

for every solution $\{x_n\}$ of (1) *with finite first moment of x_0.*

Proof. Using (3) we may rewrite $E(|x_{n+1}|_q)$ in the form

$$E(|x_{n+1}|_q) = \iint_{V^2} |S(x)+y|_q F_n(dx) G(dy).$$

Let $\varepsilon > 0$. Choose $\gamma > 0$ such that $\alpha+\gamma < 1$ and define

$$V_q = \{(x, y) \in V^2\colon\ |y| \geq \gamma|x|,\ |S(x)+y| \geq q\},$$
$$W_q = \{(x, y) \in V^2\colon\ |y| < \gamma|x|,\ |S(x)+y| \geq q\}.$$

Then

$$I_{V_q} = \iint_{V_q} |S(x)+y| F_n(dx) G(dy) \leq \iint_{V_q} (\alpha|x|+\beta+|y|) F_n(dx) G(dy)$$
$$\leq \iint_{V_q} [(\alpha\gamma^{-1}+1)|y|+\beta] F_n(dx) G(dy).$$

The inequalities $|S(x)+y| \geq q$ and $|y| \geq \gamma|x|$ imply $|y| \geq aq-b$ with $a = (\alpha\gamma^{-1}+1)^{-1}$, $b = \beta(\alpha\gamma^{-1}+1)^{-1}$. Thus

$$I_{V_q} \leq \int_{|y| \geq aq-b} |y| G(dy).$$

The last integral tends to zero as $q \to \infty$. Analogously

$$I_{W_q} = \iint_{W_q} |S(x)+y| F_n(dx) G(dy) \leq \iint_{W_q} (\alpha|x|+\beta+|y|) F_n(dx) G(dy)$$
$$\leq \iint_{W_q} [(\alpha+\gamma)|x|+\beta] F_n(dx) G(dy).$$

The inequalities $|S(x)+y| \geq q$ and $|y| < \gamma|x|$ imply $|x| > (q-\beta)(\alpha+\gamma)^{-1}$ or $|x| \geq q$ for sufficiently large q, say $q \geq q_0$. Thus

$$I_{W_q} \leq (\alpha+\gamma) \int_V |x|_q F_n(dx) + \beta \int_{|x| \geq q} F_n(dx).$$

Using (12) and the Chebyshev inequality we may estimate the last term by $\beta k/q$ for sufficiently large n, $n \geq n_0$. Thus, finally,

$$E(|x_{n+1}|_q) \leq I_{V_q} + I_{W_q} \leq (\alpha+\gamma) E(|x_n|_q) + I_{V_q} + \beta k/q$$

for $q \geq q_0$ and $n \geq n_0$. Since I_{V_q} goes to zero as $q \to \infty$ we have $I_{V_q} + \beta k/q < \frac{1}{2}\varepsilon(1-\alpha-\gamma)$ for sufficiently large q, say $q \geq q_1 \geq q_0$. Therefore by induction

$$E(|x_{n+n_0}|_q) \leq (\alpha+\gamma)^n E(|x_{n_0}|_q) + \tfrac{1}{2}\varepsilon,$$

which implies (14).

3. Convergence results. We introduce notations which allow us to write the solution of (1) and its moments explicitly. Define

$$y^n = (y_1, \ldots, y_n), \qquad G_n(dy^n) = G(dy_1) \ldots G(dy_n) \qquad (y_i \in V),$$
$$S_n(x, y^n) = S(S_{n-1}(x, y^{n-1})) + y_n, \qquad S_1(x, y^1) = S(x) + y_1.$$

Then $x_n = S_n(x_0, \xi_0, \ldots, \xi_{n-1})$ and

$$E(|x_n|_q) = \int_{V^{n+1}} |S_n(x, y^n)|_q F_0(dx) G_n(dy^n). \tag{15}$$

Now consider two particular solutions $(\bar{x}_n)$ and $(\bar{\bar{x}}_n)$ of (1) corresponding to some constant initial values $\bar{x}_0$ and $\bar{\bar{x}}_0$. Our goal is to study the asymptotic behaviour of the mathematical expectation

$$u_n(\bar{x}, \bar{\bar{x}}) = \int_{V^n} |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| G_n(dy^n). \tag{16}$$

THEOREM 1. *Assume that* G *and* S *satisfy* (10), (11) *and that*

$$|S(x) - S(z)| < |x - z| \quad \text{for } x \neq z. \tag{17}$$

Then

$$\lim_{n\to\infty} u_n(\bar{x}, \bar{\bar{x}}) = 0 \quad \text{for } \bar{x}, \bar{\bar{x}} \in V. \tag{18}$$

Proof. From (17) it follows that the sequence $\{u_n\}$ is nonincreasing. To prove (18) assume that for some $\bar{x}$ and $\bar{\bar{x}}$

$$\lim_{n\to\infty} u_n(\bar{x}, \bar{\bar{x}}) = \sigma > 0. \tag{19}$$

Following [3] define

$$A_{rq} = \{(x, z) \in V^2 \colon |S(x) - S(z)| \geqslant r,\ |x| \leqslant q,\ |z| \leqslant q\},$$
$$\omega_{rq} = \max\{|x - z|^{-1} |S(x) - S(z)| \colon (x, z) \in A_{rq}\},$$
$$V_{rq} = \{y^n \in V^n \colon (S_n(\bar{x}, y^n), S_n(\bar{\bar{x}}, y^n)) \in A_{rq}\}, \qquad W_{rq} = V^n \setminus V_{rq}.$$

For sufficiently large q and small $r > 0$ the set A_{rq} is not empty. From (16) and (17) we obtain immediately

$$u_{n+1}(\bar{x}, \bar{\bar{x}}) \leqslant \omega_{rq} \int_{V_{rq}} |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| G_n(dy^n) + \int_{W_{rq}} |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| G_n(dy^n). \tag{20}$$

W_{rq} is contained in $W_q \cup U_q \cup Z_r$ where

$$W_q = \{y^n \colon |S_n(\bar{x}, y^n)| \geqslant q\}, \qquad U_q = \{y^n \colon |S_n(\bar{\bar{x}}, y^n)| \geqslant q\},$$
$$Z_r = \{y^n \colon |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| \leqslant r\}.$$

Thus, we have

$$\int_{W_{rq}} |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| G_n(dy^n) \leqslant r + \int_{W_q \cup U_q} |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| G_n(dy^n)$$
$$\leqslant r + 2\int_{W_q} |S_n(\bar{x}, y^n)| G_n(dy^n) + 2\int_{U_q} |S_n(\bar{\bar{x}}, y^n)| G_n(dy^n). \tag{21}$$

Using Lemma 3 with $\varepsilon = \sigma/8$ we can find a $q > 0$ and an integer n_0 such that the integrals over W_q and U_q are smaller than $\sigma/8$ for $n \geqslant n_0$. From (19), (20) and (21) it follows that

$$\sigma \leqslant \omega_{rq} \int_{V_{rq}} |S_n(\bar{x}, y^n) - S_n(\bar{\bar{x}}, y^n)| G_n(dy^n) + r + \tfrac{1}{2}\sigma$$

or

$$\tfrac{1}{2}\sigma \leqslant 2q\omega_{rq} G_n(V_{rq})+r \qquad \text{for } n \geqslant n_0.$$

Fixing an $r \in (0, \sigma/4)$ we obtain $G_n(V_{rq}) \geqslant \sigma/(8q\omega_{rq})$. On the other hand, from (20) it follows that

$$u_{n+1}(\bar{x}, \bar{x}) \leqslant u_n(\bar{x}, \bar{x})+(\omega_{rq}-1) \int_{V_{rq}} |S_n(\bar{x}, y^n)-S_n(\bar{x}, y^n)| G_n(dy^n)$$

or

$$u_{n+1}(\bar{x}, \bar{x})-u_n(\bar{x}, \bar{x}) \leqslant r(\omega_{rq}-1) G_n(V_{rq}) \leqslant \frac{r\sigma(\omega_{rq}-1)}{8q\omega_{rq}} \qquad \text{for } n \geqslant n_0.$$

Since $\omega_{rq} < 1$ and $u_{n+1}-u_n \to 0$ the last inequality leads to a contradiction. ∎

A result analogous to Theorem 1 was proved in [3] for dynamical systems of the form $x_{n+1} = S(x_n, \xi_n)$. However, in that proof the existence of the second moment of $|S(x, \xi_n)|$ was assumed. Due to Lemma 3 we released this assumption.

Now using the Komornik decomposition theorem [2] we may prove our main result.

THEOREM 2. *Assume that S and G are nonsingular and satisfy* (10), (11) *and* (17). *Then there exists a unique stationary distribution* $F_* = \mathscr{P}F_*$. *This distribution is absolutely continuous and*

$$\lim_{n\to\infty} \|\mathscr{P}^n F_0 - F_*\| = 0 \tag{22}$$

for every initial distribution F_0.

Proof. Assume first that the initial distribution F_0 is absolutely continuous. Then the densities $f_n = dF_n/dm$ are given by $f_n = P^n f_0$ with P defined by (6). According to (4) we have

$$P^n f_0(x) = \int_V P_S f_{n-1}(x-y) G(dy).$$

Using the decomposition (2) we may rewrite this equality in the form

$$P^n f_0(x) = \int_V g(x-y) P_S f_{n-1}(y) dy + \int_V P_S f_{n-1}(x-y) G_1(dy),$$

where $g = dG_0/dm$. Now integrating over a measurable set $C \subset V$ we obtain

$$\int_C P^n f_0(x) dx \leqslant \sup_y \int_C g(x-y) dx + \|G_1\|.$$

Define $\varepsilon = \tfrac{1}{3}(1-\|G_1\|)$ and choose $\delta > 0$ such that $\int_C g(x)dx \leqslant \varepsilon$ for $m(C) \leqslant \delta$. Then we have

$$\int_C P^n f_0(x) dx \leqslant 1-2\varepsilon \qquad \text{for } m(C) \leqslant \delta. \tag{23}$$

Now assume in addition that $E(|x_0|) < \infty$. According to Lemma 2 there exists an integer $n_0 = n_0(F_0)$ such that

$$E(|x_n|) = \int_V |x| P^n f_0(x)dx \leqslant k \quad \text{for } n \geqslant n_0.$$

Thus, by the Chebyshev inequality

$$\int_B P^n f_0(x)dx \leqslant k/q,$$

where $B = \{x\colon |x| \geqslant q\}$. Now setting $q = k/\varepsilon$ and using (23) we finally obtain

$$\int_{B\cup C} P^n f_0(x)dx \leqslant 1-\varepsilon \quad \text{for } m(C) \leqslant \delta,\ n \geqslant n_0.$$

Since the densities f_0 such that $E(|x_0|) < \infty$ form a dense subset of the set of all densities, the last inequality shows that the operator P is constrictive [2]. Thus, according to the Komornik decomposition theorem for every $f \in L^1$ the sequence $\{P^n f\}$ may be written in the form

$$P^n f = \sum_{i=1}^{r} \lambda_i(f) h_{\alpha^n(i)} + \varepsilon_n(f), \tag{24}$$

where $h_1, \ldots, h_r$ are densities with disjoint supports, $\lambda_1, \ldots, \lambda_r$ are linear functionals on L^1 and α^n is the n-th iterate of a permutation α of $1, \ldots, r$. The remainder $\varepsilon_n(f)$ converges strongly to zero ($\|\varepsilon_n(f)\| \to 0$) and $Ph_i = h_{\alpha(i)}$.

We claim that $r = 1$. Consider the scalar product

$$\langle z, P^n f\rangle = \int_V z(x) P^n f(x)dx \quad \text{for } f \in L^1,\ z \in L^\infty.$$

Using (6) and an induction argument we may rewrite it in the form

$$\langle z, P^n f\rangle = \int_{V^{n+1}} z(S_n(x, y^n)) f(x) G_n(dy^n)dx.$$

(This formula is also evident from the probabilistic point of view; namely both sides represent the mathematical expectation of $z(x_n)$ if x_0 has density distribution function f.) Now assume that z is bounded Lipschitzian and that $\bar{f}$ and $\bar{\bar{f}}$ are two densities with supports contained in a compact set K. For every integer $n \geqslant 1$ there exist $\bar{x}_n$, $\bar{\bar{x}}_n \in K$ such that

$$|\langle z, P^n \bar{f} - P^n \bar{\bar{f}}\rangle| \leqslant \int_{V^n} \left|z(S_n(\bar{x}, y^n)) - z(S_n(\bar{\bar{x}}, y^n))\right| G_n(dy^n).$$

Consequently

$$|\langle z, P^n \bar{f} - P^n \bar{\bar{f}}\rangle| \leqslant L u_n(\bar{x}, \bar{\bar{x}}_n),$$

where L is the Lipschitz constant for z and u_n is given by (16). Since for every n the function u_n is continuous and $u_{n+1} \leqslant u_n$ the convergence in (18) is uniform on K. Thus

$$\lim_{n\to\infty} \langle z, P^n \bar{f} - P^n \bar{\bar{f}}\rangle = 0. \tag{25}$$

Using an approximation argument it is easy to extend (25) to all $\bar{f}, \bar{\bar{f}} \in L^1$ and continuous z with compact support. In particular, choosing $\bar{f} = h_i$ and $\bar{\bar{f}} = h_j$ we obtain

$$\lim_{n\to\infty} \langle z, h_{\alpha^n(i)} - h_{\alpha^n(j)} \rangle = 0.$$

This shows that the sequence of measures

$$\mu_n(A) = \int_A (h_{\alpha^n(i)} - h_{\alpha^n(j)})\,dx \tag{26}$$

converges weakly to zero. Since the sequence (26) is periodic we conclude that the μ_n are all identically equal to zero. Thus we have $h_i = h_j$ for all i, j and $r = 1$ in (24). Now define $F_*(dx) = h_1\,dx$. From (24) with $r = 1$, (22) follows.

We have proved (22) in the case when F_0 is absolutely continuous. Using Lemma 1 we may easily extend this result to every initial distribution F_0. Finally, from (22) it follows that $\mathscr{P}F_* = F_*$ and that F_* is a unique fixed point of $\mathscr{P}$. ■

Remark 1. If G is absolutely continuous we may release, in the assumptions of Theorem 2, the condition that S is nonsingular. In this case for every initial distribution F_0 all the distributions F_n with $n \geqslant 1$ are absolutely continuous and the densities $f_n = dF_n/dm$ are given by $f_n = P^{n-1} f_1$, where

$$f_1(x) = \int_V g\big(x - S(y)\big) F_0(dy), \qquad Pf(x) = \int_V \big(x - S(y)\big) f(y)\,dy.$$

The proof goes without any substantial changes. It is even simpler, since the application of Lemma 1 is unnecessary.

Remark 2. Inequality (17) was used only to prove that $r = 1$ in the decomposition formula (24). If S and G satisfy all the assumptions of Theorem 2 except (17), then for every initial distribution F the sequence $\mathscr{P}^n F$ is asymptotically periodic. Namely from (24) and Lemma 1 it follows that

$$\mathscr{P}^n F = \sum_{i=1}^{r} \gamma_i(F) H_{\alpha^n(i)} + \varepsilon_n(F),$$

where H_i is a distribution with density h_i and $\gamma_i(F)$ a coefficient depending on F. The total variation of $\varepsilon_n(F)$ converges to zero as $n \to \infty$.

4. Applications. Interesting transition operators appear in the mathematical theory of the cell cycle (see [5], [6] and [7]). In [5] it was shown that the dynamics of a population of cells may be described by the operator

$$Pf(x) = \int_0^{\lambda(x)} k(x, y) f(y)\,dy \tag{27}$$

with the kernel

$$k(x, y) = -\frac{\partial}{\partial x} \exp\big[Q(y) - Q(\lambda(x))\big],$$

where Q and λ are given nondecreasing functions.

The operator (27) has a simple biological interpretation. Namely consider a population of proliferating cells and denote by f_n the density of the cell size distribution (at birth) in the n-th generation. Then $f_{n+1} = Pf_n$.

To simplify calculations we assume that the functions λ and Q are continuously differentiable and mapping $\boldsymbol{R}_+$ onto itself. Moreover, we assume that $Q'(x) > 0$, $\lambda'(x) > 0$ for $x \in \boldsymbol{R}_+$. In this case a straightforward calculation shows that P is the transition operator for the dynamical system

$$x_{n+1} = \lambda^{-1}\{Q^{-1}[Q(x_n)+\xi_n]\}, \quad n = 0, 1, \ldots, \tag{28}$$

where the ξ_n are independent random variables with common exponential density $g(x) = e^{-x}$ $(x \geqslant 0)$. The state variables x_n belong to the set $V = \boldsymbol{R}_+$. Substituting $\tilde{x}_n = Q[\lambda(x_n)]$ into (28) we obtain

$$\tilde{x}_{n+1} = Q[\lambda^{-1}(Q^{-1}(\tilde{x}_n))]+\xi_n, \quad n = 0, 1, \ldots \tag{29}$$

Denote by F_n and $\tilde{F}_n$ the distributions of the variables x_n and $\tilde{x}_n$ respectively. It is obvious that $F_n = \tilde{F}_n \circ Q \circ \lambda$. Set $S = Q \circ \lambda^{-1} \circ Q^{-1}$. If S satisfies (11) and (17), then according to Theorem 2 and Remark 1 there exists a unique stationary distribution $\tilde{F}_*$ for the system (29) and

$$\lim_{n\to\infty} \|\tilde{F}_n - \tilde{F}_*\| = 0 \tag{30}$$

for every initial distribution $\tilde{F}_0$. Now we may come back to the system (28) and define $F_* = \tilde{F}_* \circ Q \circ \lambda$. Then evidently $\|F_n - F_*\| = \|\tilde{F}_n - \tilde{F}_*\|$. Combining the last equality with (30) we finally obtain

$$\lim_{n\to\infty} \|F_n - F_*\| = 0 \tag{31}$$

for every initial distribution F_0. This shows that F_* is a unique stationary distribution for (28).

Acknowledgement. The authors would like to thank Dr. K. Łoskot for his helpful comments concerning the probabilistic interpretation of the operator (27). In particular, he introduced the dynamical system (28).

References

[1] S. R. Foguel, *The Ergodic Theory of Markov Processes*, Van Nostrand Reinhold, 1969.
[2] J. Komornik and A. Lasota, *Asymptotic decomposition of Markov operators*, Bull. Polish Acad. Sci. Math. 35 (1987), 321–327.
[3] A. Lasota and M. C. Mackey, *Stochastic perturbation of dynamical systems: The weak convergence of measures*, J. Math. Anal. Appl. 138 (1989), 232–248.
[4] —, —, *Noise and statistical periodicity*, Physica 28D (1987), 143–154.
[5] J. Tyrcha, *Asymptotic stability in a generalized probabilistic/deterministic model of the cell cycle*, J. Math. Biol. 26 (1988), 465–475.

[6] J. J. Tyson, *The coordination of cell growth and division – intentional or incidental?*, BioEssays 2 (1972), 72–77.
[7] –, *Mini review: Size control of cell division*, J. Theoret. Biol. 120 (1987), 381–391.

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Reçu par la Rédaction le 27.11.1989