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Extension of separately holomorphic functions defined in non-open sets in the infinite dimensional case

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Abstract. The main result of this paper is the following generalization of the Siciak theorem on the extension of separately holomorphic functions. Let E_k be a real topological vector space, let $\tilde{E}_k$ denote the complexification of E_k, let V_k be open in E_k, let U_k be an open neighbourhood of V_k in $\tilde{E}_k$ and further let $X := (V_1 \times U_2) \cup (U_1 \times V_2)$ be a cross in $\tilde{E}_1 \times \tilde{E}_2$, $k = 1, 2$. Then there exists an open neighbourhood U of $V_1 \times V_2$ in $\tilde{E}_1 \times \tilde{E}_2$ such that

1° every separately G-holomorphic function in X is continuable to a G-holomorphic function in U ($U = \tilde{E}_1 \times \tilde{E}_2$ if $U_k = \tilde{E}_k$, $k = 1, 2$);

2° if E_1, E_2 are metrizable and $\tilde{E}_1 \times \tilde{E}_2$ is a Baire space (or $\tilde{E}_1 \times \tilde{E}_2$ is a Baire space and the restriction of function f to $V_1 \times V_2$ is continuous), then every separately holomorphic function in X is continuable to a holomorphic function in U.

1. Introduction. The aim of this paper is to present a generalization of the Siciak theorem on the extension of separately analytic functions defined in a cross-set in C^n ([6]) to the case when the cross lies in the Cartesian product of infinite dimensional Hausdorff topological vector spaces (t.v.s.) over the field C.

In the sequel K denotes either the field of complex numbers C or the field of real numbers R and the index k is equal either 1 or 2. Let E, E_1 be t.v.s. over K, let $N(E)$ denote the set of all balanced and absorbing neighbourhoods of 0 in E ([5]) and let E' denote the vector space of all continuous linear forms of E into the field K. If E is a t.v.s. over R, then $\tilde{E} := E+iE$ endowed with the product-topology from $E \times E$, denotes the complexification of E ([1]). We identify: $E = E+i\cdot 0$. Note that $\widetilde{E \times E_1} = \tilde{E} \times \tilde{E}_1$.

A mapping $f\colon E \to E_1$ is called a *homogeneous polynomial of degree n* iff there exists an n-linear symmetrical mapping $\hat{f}\colon E^n \to E_1$ such that $f(x) = \hat{f}(x, \ldots, x)$, $x \in E$.

Let us denote by $\mathscr{C}(E, E_1)$ the v.s. of all continuous functions of E into E_1 and by $Q^n(E, E_1)$ the v.s. of all homogeneous polynomials of E into E_1 of degree n. We put $P^n(E, E_1) := Q^n(E, E_1) \cap \mathscr{C}(E, E_1)$ and $P^0(E, E_1)$

$:= Q^0(E, E_1) := E_1$. If E is a t.v.s. over $\boldsymbol{R}$, $f \in Q^n(E, E_1)$, then $\tilde{f}(x+iy) := \sum_{s=0}^{n} i^s \binom{n}{s} f(\underbrace{x, \ldots, x}_{s}, \underbrace{y, \ldots, y}_{n-s})$, $x+iy \in \tilde{E}$. The function $\tilde{f}$ is called the *complexification* of the homogeneous polynomial f. Obviously, $\tilde{f} \in Q^n(\tilde{E}, \tilde{E}_1)$ and $\tilde{f} \in P^n(\tilde{E}, \tilde{E}_1)$ if $f \in P^n(E, E_1)$.

We say that a locally convex space (shortly l.c.s.) F over K is *sequentially-complete* if for every Cauchy sequence $\{x_n, n \in N\} \subset F$ there exists an element $x \in F$ such that $x_n \to x$. (A sequence $\{x_n, n \in N\}$ is called *Cauchy* iff for every $q \in \Gamma(F)$, $q(x_m - x_n) \to 0$ as $m, n \to \infty$, where $\Gamma(F)$ denotes the family of seminorms in F determining the topology of F.) If F is a s-c. l.c.s. (i.e., F is a sequentialy complete locally convex space) over $\boldsymbol{R}$, then its complexification $\tilde{F}$ is also a s-c. l.c.s.

Definition 1. A function f: $U \to E_1$ is called *G-analytic in an open subset U of E* iff for every $x \in U$ there exists a sequence $f_n \in Q^n(E, E_1)$ such that

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h)$$

for all h in a neighbourhood of $0 \in E$.

We write $f \in GA(U, E_1)$ when $K = \boldsymbol{R}$ ($f \in GA(U)$ if $E_1 = \boldsymbol{R}$) or $f \in GH(U, E_1)$ when $K = \boldsymbol{C}$ ($f \in GH(U)$ if $E_1 = \boldsymbol{C}$).

Definition 2. A function f: $U \to E_1$ is called *analytic in U* iff f is continuous and for every point $x \in E$ there exists a sequence $f_n \in P^n(E, E_1)$ such that

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h)$$

for all h in a neighbourhood of $0 \in E$.

We write $f \in GA(U, E_1)$ when $K = \boldsymbol{R}$ ($f \in GA(U)$ if $E_1 = \boldsymbol{R}$) or $f \in H(U, E_1)$ when $K = \boldsymbol{C}$ ($f \in H(U)$ if $E_1 = \boldsymbol{C}$).

Definition 3. A function f: $U \to E_1$ is called *weakly-analytic* (*weakly-G-analytic*) iff for every $u \in E_1$ the function $u \circ f$ is analytic (G-analytic).

We write $f \in WA(U, E_1)$ when $K = \boldsymbol{R}$ or $f \in WH(U, E_1)$ when $K = \boldsymbol{C}$.

Remark. The above definitions have the same meaning as in [2]. In the case where E is a t.v.s. over $\boldsymbol{C}$ we usually prefer the term "holomorphic" (respectively G-holomorphic, weakly-holomorphic) in place of the term "analytic".

Definition 4. Let $\tilde{E}_k$ be the complexification of real t.v.s. E_k, let V_k be an open subset of E_k, let U_k be an open subset of $\tilde{E}_k$ and further let V_k be contained in U_k. Then the set

$$X = (V_1 \times U_2) \cup (U_1 \times V_2)$$

is said to be a *cross* in the space $\tilde{E}_1 \times \tilde{E}_2$.

DEFINITION 5. Let X be a cross in $\tilde{E}_1 \times \tilde{E}_2$ and let F be a s-c. l.c.s. We introduce the notion of the family of separately G-holomorphic (respectively holomorphic) functions defined in X:

$$GH(X, F) := \{f\colon X \to F;\ \text{(i)}\ \forall x \in V_1\ f(x, \cdot) \in GH(U_2, F),$$
$$\text{(ii)}\ \forall y \in V_2\ f(\cdot, y) \in GH(U_1, F)\},$$
$$H(X, F) := \{f\colon X \to F;\ \text{(i)}\ \forall x \in V_1\ f(x, \cdot) \in GH(U_2, F),$$
$$\text{(ii)}\ \forall y \in V_2\, f(\cdot, y) \in H(U_1, F)\}.$$

The main result of this paper is the following:

THEOREM. *For every cross X there exists an open subset U of $\tilde{E}_1 \times \tilde{E}_2$ with the following properties*:

1° $V_1 \times V_2 \subset U$.

2° *For any $f \in GH(X, F)$ there exists exactly one function $\hat{f} \in GH(U, F)$ such that: $\hat{f} = f$ in $U \cap X$, for every $q \in \Gamma(F)$, $\sup\{q \circ \hat{f}(z)\colon z \in U\} \leqslant \sup\{q \circ f(z)\colon z \in X\}$.*

3° *If $X := (V_1 \times \tilde{E}_2) \cup (\tilde{E}_1 \times V_2)$, then U can be chosen as the whole of $\tilde{E}_1 \times \tilde{E}_2$, i.e., every function $f \in GH(X, F)$ can be uniquely continued to a G-entire function $\hat{f} \in GH(\tilde{E}_1 \times \tilde{E}_2, F)$.*

4° *If $\tilde{E}_1 \times \tilde{E}_2$ is a Baire space, $f \in GH(X, F)$ and f is continuous in $V_1 \times V_2$, then $\hat{f} \in H(U, F)$ ($\hat{f} = f$ in $U \cap X$).*

5° *If $\tilde{E}_1 \times \tilde{E}_2$ is a Baire space, E_1, E_2 are metrizable and $f \in H(X, F)$, then $\hat{f} \in H(U, F)$ ($\hat{f} = f$ in $U \cap X$).*

2. Some elementary properties of analytic functions in topological vector spaces. In the sequel F will denote a s-c. l.c.s. over K and U will be an open subset of a t.v.s. E over K.

DEFINITION 6. We say that a function $f\colon U \to F$ has the *p-th Gateaux differential at a point* $x_0 \in U$ iff:

1° For every $h \in E$ the mapping

$$f_h\colon D \ni t \to f(x_0 + th) \in F,$$

defined in a neighbourhood D of $0 \in K$, has the p-th derivative at 0,

2° the mapping

$$\delta^l_{x_0} f\colon\ E \ni h \to \frac{d^l}{dt^l} f(x_0 + th)\bigg|_{t=0} \in F$$

is a homogeneous polynomial of degree l, $l = 1, \ldots, p$. The polynomial $\delta^l_{x_0} f$ is called the *l-th G-differential at* x_0.

We write $f \in G^p(U, F)$, if f has the p-th G-differential at every point $x_0 \in U$.

LEMMA 1 (Proposition 5.5 in [2], p. 93).

(i) *$f \in GH(U, F)$ iff for every point $x \in U$ there exists non-empty $V \in N(E)$ such that for every affine line passing trough x*

$$f|_{L \cap (x+V)} \in H(L \cap (x+V), F);$$

(ii) *$f \in GH(U, F)$ iff f is weakly G-holomorphic in U;*

(iii) *If $f \in GH(U, F)$, then $f \in G^\infty(U, F)$ and*

$$f(x+h) = \sum_{n=0}^{\infty} \delta^n_x f(h)$$

for $h \in W$, where W is the maximal balanced neighbourhood of $0 \in E$ such that $x+W \subset U$.

LEMMA 2 (Theorem 6.1 in [2], p. 97, and Theorem 6.3 in [2], p. 98).

(i) *$f \in H(U, F)$ iff $f \in GH(U, F)$ and f is continuous;*

(ii) *If E is metrizable, then: $f \in H(U, F)$ iff $f \in WH(U, F)$;*

(iii) *If E is a Baire space and U is connected, then: $f \in H(U, F)$ iff $f \in GH(U, F)$ and for every seminorm $q \in \Gamma(F)$ there exists an open and non-empty subset $W \subset U$ such that $q \circ f$ is bounded in W.*

LEMMA 3 (Proposition 6.6 in [2], p. 102). *Let U be an open connected subset of the complexification $\tilde{E}$ of a t.v.s. E over $\mathbf{R}$. If $f \in GH(U, F)$ (or $f \in H(U, F)$) and $f(x) = 0$ for x belonging to an open non-empty subset of $E \cap U$, then $f = 0$.*

As a simple corollary from Corollary 5.1 ([2]), Theorem 5 ([1]), Proposition 5.2.1° ([2]), Proposition 6a ([1]) and Lemma 5 below we get:

LEMMA 4. *If U is an open subset of a t.v.s E which is a Baire space over $\mathbf{R}$ and $f \in \mathscr{C}(U, F) \cap GA(U, F)$, then there exist an open subset $\tilde{U}$ of $\tilde{E}$ and a holomorphic function $\hat{f} \in H(\tilde{U}, F)$ such that $\hat{f} = f$ in $U \subset \tilde{U}$.*

LEMMA 5 (a version of the Vitali theorem). *Assume that:*

(1) *V is an open subset of a t.v.s. E over $\mathbf{R}$,*

(2) *U is an open connected subset of $\tilde{E} := E+iE$ and U contains V;*

(3) *$f_n \in H(U, F)$, $n \in N$, is a sequence of functions such that for every $q \in \Gamma(F)$ the sequence $\{q \circ f_n \colon n \in N\}$ is locally uniformly bounded;*

(4) *for each $x \in V$ the sequence $f_n(x)$ is convergent to an element of F.*

Then there exists $f \in H(U, F)$ such that, for every $q \in \Gamma(F)$, $q \circ (f_n(x) - f(x)) \to 0$ uniformly on compact subsets of U ($n \to \infty$, $x \in U$).

Proof. Let W be an open balanced neighbourhood of a point $a \in V$ such that $W \cap E \subset V$ and $\sup\{q \circ f_n(x)\colon x \in W, n \in N\} < \infty$. We observe that for every affine line L if $L \cap W \neq \emptyset$, then a non-empty interval is contained in $L \cap W \cap V$. Therefore, by the classical Vitali theorem, the sequence $\{f_n\colon n \in N\}$ converges in the open set $W \cap U$. Now the assertion of Lemma 5 follows from Proposition 6.2 in [2].

We will need the following technical lemmas.

LEMMA 6 (Lemma 2.1 in [2]). *Let D be an open subset of K. If $f\colon D \to F$ is a mapping such that for every $u \in F'$ the function $u \circ f$ has the $(n+1)$-th derivative at $t_0 \in D$, then f has the n-th derivative at t_0.*

LEMMA 7 (II in [6], p. 54). *Let $D_l := \{w_l \in C\colon |w_l + \sqrt{w_l^2 - 1}| < R_l\}$ $(R_l > 1)$ be an ellipse in the complex w_l-plane with foci -1 and $+1$. Let*

$$X := (D_1 \times F_2 \times \ldots \times F_n) \cup \ldots \cup (F_1 \times \ldots \times F_{n-1} \times D_n),$$

where F_l denotes the interval $[-1, 1]$ in the w_l-plane. Let $f\colon X \to C$ be holomorphic with respect to $w_l \in D_l$ for every fixed

$$(u_1, \ldots, u_{l-1}, u_{l+1}, \ldots, u_n) \in F_1 \times \ldots \times F_{l-1} \times F_{l+1} \times \ldots \times F_n.$$

Then f is continuable to a holomorphic function $\hat{f}$ in

$$D := \left\{w \in C^n;\ \sum_{l=1}^{n} \frac{\lg|w_l + \sqrt{w_l^2 - 1}|}{\lg R_l} < 1\right\};$$

D is the envelope of holomorphy of X and $\sup\{|f(w)|\colon w \in X\} = \sup\{|\hat{f}(w)|\colon w \in D\}$.

3. Proof of the main theorem.

LEMMA I. *Assume that*

$$U_k \in N(\tilde{E}_k), \quad V_k := U_k \cap E_k, \quad X := (V_1 \times RU_2) \cup (RU_1 \times V_2), \quad R > 1,$$

$A_k \in N(E_k)$ is such that $A_k + A_k + A_k + A_k \subset V_k$, B_k is a maximal open balanced subset of $A_k + iA_k$,

$$W := \frac{R^{1/8} - 1}{2}(B_1 \times B_2) \in N(\tilde{E}_1 \times \tilde{E}_2).$$

Then for every $f \in GH(X)$ there exists exactly one function $\hat{f} \in GH(W)$ such that $\hat{f} = f$ in $W \cap X$ and $\sup\{|\hat{f}(z)|\colon z \in W\} \leqslant \sup\{|f(z)|\colon z \in X\}$.

Proof of Lemma I. Let us fix $f \in GH(X)$ and $x := (x_1, x_2, x_3, x_4) \in A_1^4$, $y := (y_1, y_2, y_3, y_4) \in A_2^4$. For $s, t \in C^4$ we put $sx := s_1 x_1 + s_2 x_2 + s_3 x_3 + s_4 x_4$, $ty := t_1 y_1 + t_2 y_2 + t_3 y_3 + t_4 y_4$, $w := (s, t) \in C^8$. Let

$$D := \{w \in C^8\colon |s| < 1, |t| < R, \operatorname{im} s = 0\} \cup \{w \in C^8\colon |s| < R, |t| < 1, \operatorname{im} t = 0\},$$

where for $z \in C^n$, $|z| := \max\{|z_j|\colon j = 1, \ldots, n\}$.

Let us put

(1) $$g_{xy}(w) := f(sx, ty), \quad w = (s, t) \in D.$$

Note that, in particular, the function g_{xy} is defined in the cross

$$T := (D_1 \times F_2 \times \ldots \times F_8) \cup \ldots \cup (F_1 \times \ldots \times F_7 \times D_8),$$

where $F_j := [-1, 1]$, $D_j := \{w_j \in C: |w_j + \sqrt{w_j^2 - 1}| < R\}$, $j = 1, \ldots, 8$. Owing to the assumptions imposed on f, the function g_{xy} is separately holomorphic on the cross T. Thus by Lemma 7 there exists exactly one function $\hat{g}_{xy} \in H(P)$ such that

(2) $\hat{g}_{xy} = g_{xy}$ in $P \cap D$

and $\sup\{|\hat{g}_{xy}(w)|: w \in P\} \leqslant \sup\{|g_{xy}(w)|: w \in D\}$,

where $P := \{w \in C^8: |w| \leqslant r\}$, $2r := R^{1/8} - 1$.

Observe that

(3) $$\hat{g}_{xy}(s, t) = \hat{g}_{xy}(s_0, t) \quad \text{if} \quad sx = s_0 x_0.$$

For the proof of (3), let s, s_0 be fixed and write

$$h(t) := \hat{g}_{x_0 y}(s_0, t) - \hat{g}_{xy}(s, t) \in H(|t| \leqslant r);$$

hence by (1) we obtain

$$h(t) = g_{x_0 y}(s_0, t) - g_{xy}(s, t) = f(s_0 x_0, ty) - f(sx, ty) = 0$$

for $|t| < 1$, $\operatorname{im} t = 0$ and so $h \equiv 0$.

In view of (3) we get

(4) $$\hat{g}_{xy}(s, t) - \hat{g}_{x_0 y_0}(s_0, t_0) \quad \text{if} \quad sx = s_0 x_0, \quad ty = t_0 y_0.$$

We define

(5) $$\hat{f}(rx_1 + irx_2, ry_1 + iry_2) := \hat{g}_{xy}(r, ir, 0, 0, r, ir, 0, 0)$$

for $x_1, x_2 \in A_1$, $y_1, y_2 \in A_2$.

By dint of (4) the function $\hat{f}$ is well defined in the open balanced set $W := rB_1 \times rB_2$, where B_k is the maximal open balanced subset of $A_k + iA_k \subset \tilde{E}_k$.

We see that for every fixed $a = r(a_1 + ia_2)$, $u = r(u_1 + iu_2) \in B_1$ and $b = r(b_1 + ib_2)$, $v = r(v_1 + iv_2) \in B_2$ the function

$$\hat{f}((a, b) + z(u, v)) = \hat{g}_{xy}(r, ir, zr, izr, r, ir, zr, izr),$$

where $x := (a_1, a_2, u_1, u_2)$, $y := (b_1, b_2, v_1, v_2)$, $|z| < 1$, is holomorphic in the unit disc. So by Lemma 1 (i) we get $\hat{f} \in GH(W)$. By (1), (2), (4) and (5) we have $\hat{f} = f$ in $W \cap X$ and from this, an account of Lemma 3, we derive that $\hat{f}$ is unique, in view of (1) the inequality $\sup\{|\hat{f}(z)|: z \in W\} \leqslant \sup\{|f(z)|: z \in X\}$ holds true. Q.E.D.

LEMMA II. *The assumptions are the same as in Lemma* I. *We assert that for every function $f \in GH(X, F)$ there exists the unique function $\hat{f} \in GH(W, F)$ such that*:

$$\hat{f} = f \quad \text{in } X \cap W,$$

for every $q \in \Gamma(F)$ the inequality $\sup\{q \circ \hat{f}(z)\colon z \in W\} \leqslant \sup\{q \circ f(z)\colon z \in X\}$ *holds true.*

Proof of Lemma II. For every $u \in F$ the function $f_u := u \circ f$ belongs to $GH(X)$, and so by Lemma I there exists a function $\hat{f}_u \in GH(U)$ such that $\hat{f}_u = f_u$ in $W \cap X$. According to Lemma 1 (iii) we may write the Taylor series for $\hat{f}_u$ at the point $0 \in W$

$$\hat{f}_u(z) = \sum_{l=0}^{\infty} \frac{1}{l!} \delta_0^l \hat{f}_u(z), \qquad z \in W \in N(\tilde{E}_1 \times \tilde{E}_2). \tag{1}$$

Since $\hat{f}_u = f_u$ in $W \cap X$, from Definition 6 gives the equality

$$\delta_0^l f_u(x) = \frac{d^l}{dt^l} \hat{f}_u(tx)\Big|_{t=0} = \frac{d^l}{dt^l} f_u(tx)\Big|_{t=0},$$

where $x \in E_1 \times E_2$, $l = 0, 1, \ldots$

By virtue of Lemma 6, there exists the l-th derivative of the function f; we put

$$\delta_0^l f(x) := [f(tx)]^{(l)}|_{t=0}, \qquad l = 0, 1, \ldots$$

$\delta_0^l f$ is the l-th G-differential of f, because for every $u \in F'$ the mapping $u \circ \delta_0^l f = \delta_0^l f_u = \delta_0^l \hat{f}_u|_{E_1 \times E_2} \in Q^l(E_1 \times E_2, K)$ and consequently $\delta_0^l f \in Q^l(E_1 \times E_2, F)$, $l = 0, 1, \ldots$ We put

$$\delta_0^l \hat{f}(z) := \widehat{\delta_0^l f}(z), \qquad z \in \tilde{E}_1 \times \tilde{E}_2,\ l = 0, 1, \ldots$$

Since, for every $u \in F'$, $u \circ \delta_0^l \hat{f} = \delta_0^l \hat{f}_u$ we have the equality

$$u \circ \sum_{l=0}^{\infty} \frac{1}{l!} \delta_0^l \hat{f} = \sum_{l=0}^{\infty} \frac{1}{l!} \delta_0^l \hat{f}_u.$$

Because a series of homogeneous polynomials from a t.v.s. E to a s-c. l.c.s. F converges in an open set W iff it weakly converges in W (Proposition 5.6 in [2]), it is legitimate to define

$$\hat{f}(z) := \sum_{l=0}^{\infty} \frac{1}{l!} \delta_0^l f(z) \quad \text{for } z \in W.$$

By Theorem 5.1 ([2]), $\hat{f} \in GH(W, F)$ and since $u \circ \hat{f} = \hat{f}_u$ we have $\hat{f} = f$ in $W \cap X$. If $M := \sup\{q \circ f(z)\colon z \in X\}$, then by Lemma I

$$\sup\{|u \circ \hat{f}(z)|\colon z \in W\} \leqslant \sup\{|u \circ f(z)|\colon z \in X\} \leqslant M$$

for every $u \in F'$ such that $|u(y)| \leqslant q(y)$ for $y \in F$. Hence by the Hahn–Banach theorem we get $q \circ f(z) \leqslant M$ for $z \in W$. Q.E.D.

LEMMA III. *Under the same assumptions as in Lemma* I, *if additionaly* E_1, E_2 *are metrizable and* $\tilde{E}_1 \times \tilde{E}_2$ *is a Baire space, then for every* $f \in H(X, F)$ *there exists a unique function* $\hat{f} \in H(W, F)$ *such that* $\hat{f} = f$ *in* $X \cap W$.

Proof of Lemma III. By Lemma II there exists a function $\hat{f} \in GH(W, F)$ such that $\hat{f} = f$ in $X \cap W$. Let us take a fixed $u \in F'$. According to Lemma 2 it is sufficient to show the boundedness of the function $g := u \circ f$ in some open non-empty subset of the domain W (because $\tilde{E}_1 \times \tilde{E}_2$ is a Baire metrizable space).

Let d_k be a metric in the space E_k, $\tilde{d}_k(x+iy) := d_k(x)+d_k(y)$ be the metric in the space $\tilde{E}_k$, $B_k(a, r) := \{x \in E_k\colon d_k(a, x) < r\}$ and $\tilde{B}_k(a, r) := \{z \in \tilde{E}_k\colon \tilde{d}_k(a, z) < r\}$. The function g belongs to $GH(W)$ and $g|_{W \cap X} \in H(W \cap X)$. Let $A_{ml} := \{x \in V_1\colon |g(x, y)| \leqslant m\}$ for $y \in B_2(0, 1/l)\}$. Since g is separately continuous on $V_1 \times V_2$, there exists $A_{ml} \neq \emptyset$. If $x_n \in A_{ml}$ and $x_n \to x_0 \in V_1$, then

$$\sup\{|g(x_n, y)|\colon d_2(0, y) < 1/l,\ n \in N\} \leqslant m, \quad g(x_n, y) \to g(x_0, y)$$

for $n \to \infty$, $y \in B_2(0, 1/l)$.

By Lemma 5 (Vitali theorem) and by Lemma 3 we have $g(x_n, y) \to g(x_0, y)$ for $n \to \infty$, $y \in B_2(0, 1/l)$ and consequently $|g(x_0, y)| \leqslant m$ for $y \in B_2(0, 1/l)$. Therefore A_{ml} is closed in V_1 and $\bigcup_{m,l \in N} A_{ml} = V_1$. From the Baire property of V_1 we derive the inclusion $B_1(a_1, p) \subset \operatorname{int} A_{m_0 l_0} \neq \emptyset$.

Write

$$C_{lm} := \{y \in B_2(0, 1/l)\colon |g(x, y)| \leqslant m \text{ for } x \in B_1(a_1, 1/l)\}.$$

By an analogous argument we show that $B_2(a_2, s) \subset \operatorname{int} C_{lm} \neq \emptyset$.

Let us put $r := \min\{p, s, 1/l, 1/l_0\}$ and $M := \max\{m, m_0\}$ and let us consider the cross $Y := B_1(a_1, r) \times \tilde{B}_2(a_2, r) \cup \tilde{B}_1(a_1, r) \times B_2(a_2, r)$. We obtain $\sup\{|g(z)|\colon z \in Y\} \leqslant M$ and by Lemma 1 we derive the existence of an open balanced set T such that $\sup\{|g(z)|\colon z \in T\} \leqslant M$. Since W is an open connected set and $\tilde{E}_1 \times \tilde{E}_2$ is a Baire space, $g \in H(W)$. Q.E.D.

Proof of the theorem. For $X = (V_1 \times U_2) \cup (U_1 \times V_2)$, where $V_k \in \operatorname{top} E_k$, $V_k \subset U_k \in \operatorname{top} \tilde{E}_k$ we put

$$U := \bigcup_{a \in V_1 \times V_2} (a + W(a)),$$

$W(a)$ being the same as in Lemma I, i.e., for $a = (a_1, a_2)$ the set $W(a)$ is defined to be equal to $\frac{1}{2}(R^{1/8}-1)(\tilde{B}_1(a_1) \times \tilde{B}_2(a_2))$; here $\tilde{B}_k(a_k)$ is the maximal open balanced subset of $A_k(a_k)+iA_k(a_k)$, where $A_k(a_k) \in N(E_k)$ and

$$A_k(a_k)+A_k(a_k)+A_k(a_k)+A_k(a_k) \subset V_k \subset R^{-1}(U_k \cap E_k), \quad R > 1.$$

It is obvious that $V_1 \times V_2 \subset U \in \mathrm{top}(\tilde{E}_1 \times \tilde{E}_2)$. If $U_k = \tilde{E}_k$ for $k = 1, 2$, then $W(a)$ can be chosen to be the whole space $\tilde{E}_1 \times \tilde{E}_2$ and consequently $U = \tilde{E}_1 \times \tilde{E}_2$. The above remarks end the proof of assertions 1°, 2°, 3° of the theorem.

As regards assertion 4°, observe that if $E \times E_1$ is a Baire space, then E and E_1 are Baire spaces, too. (The converse is false: even if E is a metrizable Baire space, $E \times E$ needs not be a Baire space [3].) Since $\tilde{E}$ is viewed in the product topology of $E \times E$, it follows that if $\tilde{E}_1 \times \tilde{E}_2 = \widetilde{E_1 \times E_2}$ is a Baire space, then E_k, $\tilde{E}_k$ and $E_1 \times E_2$ are Baire spaces.

Therefore by Lemma II we get $\hat{f} \in GH(U, F)$ and by the construction of U we see that for every connected component A of U it is true that $B := A \cap (V_1 \times V_2)$ is non-void. In virtue of Lemma 4, for every B there exists an open non-empty set $\tilde{B}$ and a functions $\tilde{f} \in H(\tilde{B}, F)$ such that the restrictions of $\tilde{f}$ to B is equal to f. Now by Lemma 3 and Lemma 2 we get $\hat{f} \in H(A, F)$ for each connected component A of U.

Assertion 5° is an immediate consequence of Lemma III. The proof of the theorem is completed.

Remark 1. The continuation of a separately holomorphic function f defined in a cross of the form $X = (A \times U_1) \cup (U_1 \times U_2)$, where A satisfies the L-condition at some point $a \in U_1$ and f is bounded on an open subset of $U_1 \times U_2$, was studied in [4].

Remark 2. Observe that the assumption: $\tilde{E}$ is the complexification of a real t.v.s. E does not essentially restrict generality in the above considerations, because every complex t.v.s. $\tilde{E}$ may be slipt into the real and imaginary parts (i.e. $\tilde{E} = E + iE$, where E is a t.v.s. over $\mathbf{R}$, $E \subset \tilde{E}$ and $E \cap iE = \{0\}$).

References

[1] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings in topological vector spaces*, Studia Math. 39 (1971), p. 59–76.

[2] — and —, *Analytic functions in topological vector spaces*, ibidem 39 (1971), p. 77–112.

[3] M. R. Krom, *Cartessian products of metric Baire spaces*, Proc. Amer. Math. Soc. 42 (1973), p. 588—594.

[4] Nguyen Thanh Van, *Fonctions séparément analytiques et prolongement analytique faible en dimension infinie*, Ann. Polon. Math. 33 (1976), p. 71–83.

[5] W. Rudin, *Functional analysis*, New York 1973.

[6] J. Siciak, *Analyticity and separate analyticity of functions defined on lower dimensional subset of C^n*, Zeszyty Naukowe UJ 13 (1969), p. 53–70.

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