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Topological transversality. Applications to initial-value problems

by J. W. LEE and D. O'REGAN (Corvallis, Oregon)

Abstract. A simple fixed point analysis is used to examine the dependence of the interval of existence for an initial-value problem upon its initial data and the nonlinearity in the differential equation. These results refine and extend a theorem of Wintner, and are shown to be best possible for the class of problems considered.

1. Introduction. The basic existence theorem for the initial value problem

$$y' = f(t, y) \qquad y(0) = r, \tag{1.1}$$

where $f\colon Z \to \mathbf{R}^n$ is continuous and Z is the cylinder $[0, T] \times \mathbf{R}^n$, guarantees that a solution exists for $t > 0$ and near 0. Familiar examples show that the interval of existence can be arbitrarily short, depending on the initial value r and the nonlinear behavior of f. In the present note, we examine the dependence of the interval of existence of f and r. Our results are closely related to earlier work of Wintner and of Conti which has focused on global existence in the future, $t > 0$. See, for example, Hartman [5] and Corduneanu [1].

The analysis in these works rests on two key ideas: (i) the initial value problem has a (maximal) solution $y(t)$ such that $(t, y(t))$ tends to the boundary (i.e., leaves any compact set) of the cylinder Z; (ii) the construction of a comparison, initial value problem known to have a global solution which dominates $y(t)$. The comparison in (ii) reveals that $y(t)$ is bounded. Since $(t, y(t))$ tends to the boundary of $[0, T] \times \mathbf{R}^n$, it follows that the solution extends across the entire interval from 0 to T. The comparison involved in (ii) is typically based on the inequality $|f(t, y)| \leqslant \psi(|y|)$, where $\psi > 0$ is continuous and

$$\int_0^\infty \frac{du}{\psi(u)} = \infty.$$

The function $\psi(u)$ delimits the growth of f on the cylinder. With these assumptions on $\psi(u)$, the solution $y(t)$ to (1.1) extends from 0 to T provided

$|f(t, y)| \leqslant \psi(|y|)$ holds on $[0, T] \times \boldsymbol{R}^n$. Consequently, if this inequality actually holds on $[0, \infty) \times \boldsymbol{R}^n$, a solution to (1.1) exists for all $t > 0$.

We obtain these conclusions and some interesting additional results by essentially different and more natural arguments. Our analysis is based on fixed point arguments and the use of a priori bounds. This point of view leads naturally to the study of the dependence of the interval of existence of a solution to (1.1) upon r and f. Additionally, it automatically produces best possible results. Our arguments yield simultaneously the existence of a solution and the maximal interval of existence for the class of initial value problems considered. In contrast, the approach based on (i) and (ii) first requires a local existence result, then prolongation to a maximal solution, and finally construction of a comparison problem to establish the interval of existence of solutions to (1.1). The rather technical details required to carry out this program, especially when $f(t, y)$ is not locally Lipschitz, are replaced by a simple fixed point analysis.

2. Maximal intervals of existence for classes of initial value problems. The use of a priori bounds to establish existence theorems for boundary value problems is well known, dating back to S. Bernstein at the turn of this century and the systematic developments of Leray and Schauder in the 20's and 30's. See, for example, Dugundji and Granas [2] or Granas, Guenther and Lee [3], [4]. These techniques also apply to initial value problems; however, this fact seems to have been largely overlooked. Specializing Theorem 2.1 in [4] for initial value problems, we have

THEOREM 2.1. *Let f: $[0, T] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be continuous. Suppose there is a constant B such that $|y(t)|$, $|y'(t)| \leqslant B$ for t in $[0, T]$ for each solution $y(t)$ to*

$$(2.1)_\lambda \qquad \begin{aligned} y' &= \lambda f(t, y), \quad 0 \leqslant t \leqslant T. \\ y(0) &= 0. \end{aligned}$$

for any $0 \leqslant \lambda \leqslant 1$.

Then the initial value problem

$$(2.1) \qquad \begin{aligned} y' &= f(t, y), \quad 0 \leqslant t \leqslant T, \\ y(0) &= 0, \end{aligned}$$

has a solution y in $C^1[0, T]$.

Theorem 2.1 in [4] is formulated for a scalar equation; however, the proof extends immediately to the case of systems as in the present formulation.

In view of Theorem 2.1 we obtain immediately

THEOREM 2.2 *Let ψ: $[0, \infty) \to (0, \infty)$ be continuous, and assume*

$$(2.2) \qquad |f(t, y)| \leqslant \psi(|y|)$$

for all $(t, y) \in [0, T] \times \mathbf{R}^n$. *Then the initial-value problem* (2.1) *has a solution in* $C^1[0, T]$ *for each*

$$T < T_\infty = \int_0^\infty \frac{du}{\psi(u)}. \tag{2.3}$$

Moreover, this result is best possible in the sense that the initial value problem

$$\begin{aligned} y' &= \hat{f}(t, y), \quad 0 \leqslant t \leqslant \hat{T}, \\ y(0) &= 0, \end{aligned} \tag{2.4}$$

with $\hat{f}(t, y) = (\psi(|y|), 0, \ldots, 0)$ *and for which* (2.2) *holds can have a solution only if* $\hat{T} < T_\infty$.

Remark. If $T_\infty = \infty$, Theorem 2.2 is called *Wintner's theorem* [5], [6].

Proof. To prove existence of a solution in $C^1[0, T]$ we apply Theorem 2.1. To establish the a priori bounds for $(2.1)_\lambda$, let $y(t)$ be a solution to $(2.1)_\lambda$. Then

$$|y'| \leqslant |\lambda f(t, y)| \leqslant \psi(|y|).$$

Now if $|y(t)| \neq 0$ we have $|y|' = y \cdot y'/|y| \leqslant |y'|$ and the inequality above yields

$$|y|' \leqslant \psi(|y|)$$

at any point t, where $y(t) \neq 0$. Suppose $y(t) \neq 0$ for some t in $[0, T]$. Since $y(0) = 0$, there is an interval $[a, t]$ in $[0, T]$ such that $|y(\sigma)| > 0$ on $a < \sigma \leqslant t$ and $y(a) = 0$.

Then the previous inequality implies

$$\int_a^t \frac{|y(\sigma)|' d\sigma}{\psi(|y(\sigma)|)} \leqslant t - a,$$

$$\int_0^{|y(t)|} \frac{du}{\psi(u)} \leqslant t - a \leqslant T < T_\infty = \int_0^\infty \frac{du}{\psi(u)}.$$

This inequality implies there is a constant M_0 such that $|y(t)| \leqslant M_0$. Now $(2.1)_\lambda$ gives

$$|y'(t)| \leqslant \max_{[0,T] \times [-M_0, M_0]} |f(t, u)| = M_1.$$

So, $|y(t)|$, $|y'(t)| \leqslant B = \max(M_0, M_1)$, and existence of a solution to (2.1) is established.

Finally, $y(t) = (y_1(t), \ldots, y_n(t))$ solves (2.4) if and only if $y_2(t) = \ldots = y_n(t) \equiv 0$ and $y_1' = \psi(|y_1|)$, $y_1(0) = 0$. Clearly, $y_1'(t) > 0$, so $y_1(t) \geqslant 0$ and integration yields

$$\int_0^{\hat{T}} \frac{y_1'(\sigma)\,d\sigma}{\psi(y_1(\sigma))} = \hat{T}.$$

Thus,

$$\hat{T} = \int_0^{y_1(\hat{T})} \frac{du}{\psi(u)} < \int_0^{\infty} \frac{du}{\psi(u)} = T_\infty,$$

which completes the proof of Theorem 2.2.

Theorem 2.1 also holds for the inhomogeneous boundary condition $y(0) = r$. See Theorem 2.4 in [4]. So trivial adjustments in the proof above yield,

THEOREM 2.3. *Let $f(t, y)$ and $\psi(u)$ satisfy the hypotheses in Theorem 2.2. Then the initial value problem*

$$\text{(2.5)} \qquad \begin{aligned} y' &= f(t, y), \quad 0 \leqslant t \leqslant T, \\ y(0) &= r \end{aligned}$$

has a solution $y(t)$ in $C^1[0, T]$ for each

$$T < T_\infty = \int_{|r|}^{\infty} \frac{du}{\psi(u)}.$$

Moreover, this result is best possible as described in Theorem 2.2.

A few examples illustrate these results.

EXAMPLE 1. (Linear and sublinear growth.) Suppose

$$|f(t, y)| \leqslant A(t)|y|^p + B(t), \qquad p \leqslant 1,$$

for bounded functions $A(t), B(t) \geqslant 0$. If A_0 and B_0 are upper bounds for $A(t)$ and $B(t)$, then

$$|f(t, y)| \leqslant A_0|y|^p + B_0 = \psi(|y|)$$

and

$$T_\infty = \int_{|r|}^{\infty} \frac{du}{A_0 u^p + B_0} = \infty.$$

Consequently, the IVP (2.5) has a solution on $[0, T]$ for all $T > 0$.

EXAMPLE 2. (Polynomial growth.) Suppose

$$|f(t, y)| \leqslant A(t)|y|^m + B(t)$$

for $m > 1$ and for bounded functions $A(t), B(t) \geqslant 0$. If A_0 and B_0 are upper bounds for $A(t)$ and $B(t)$, then

$$|f(t, y)| \leqslant A_0|y|^m + B_0 = \psi(|y|), \qquad T_\infty = \int_{|r|}^{\infty} \frac{du}{A_0 u^m + B_0},$$

and the IVP (2.5) has a solution on $[0, T]$ for any $T < T_\infty$. In the case of zero initial data, $r = 0$, we have

$$T_\infty = \int_0^\infty \frac{du}{A_0 u^m + B_0} = \frac{\pi \csc(\pi/m)}{m A_0^{1/m} B_0^{(m-1)/m}}.$$

EXAMPLE 3. (Estimate on the time before shocks.) The first order quasilinear partial differential equation

$$a(x, y, u) u_x + b(x, y, u) u_y = c(x, y, u)$$

with suitable assumptions on the coefficients a, b, and c can be solved by the method of characteristics. If the solution surface $u = u(x, y)$ is to contain the smooth, initial curve

$$x_0 = x_0(s), \quad y_0 = y_0(s), \quad u_0 = u_0(s),$$

where $0 \leqslant s \leqslant 1$ is a parameter, then the characteristic IVP is

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{du}{dt} = c, \quad x(0, s) = x_0(s),$$

$$y(0, s) = y_0(s), \quad u(0, s) = u_0(s),$$

where $x = x(t, s)$, etc., and s is regarded as a parameter in the initial-value problem. The solution to the initial value problem yields the solution surface by expressing $u(t, s)$ in terms of x and y after solving $x = x(t, s)$, $y = y(t, s)$ for $t = t(x, y)$ and $s = s(x, y)$. In the region about the initial curve, where this can be done, a smooth solution surface results. Suppose we have an estimate on the growth rate of the coefficients in the partial differential equation; say,

$$|f(x, y, z)| = |(a(x, y, z), b(x, y, z), c(x, y, z))| \leqslant \psi(|(x, y, z)|).$$

Then by Theorem 2.3 no shocks can develop up to time

$$T < T_\infty = \int_{|r|}^\infty \frac{du}{\psi(u)}, \quad \text{where } r = \max_{0 \leqslant s \leqslant 1} |(x_0, (s), y_0(s), u_0(s))|,$$

assuming of course that t and s are expressible as functions of x and y as required above.

EXAMPLE 4. (Slightly greater than linear growth.) As Wintner notes in his paper, if $\psi(u) \sim u \ln u$ as $u \to \infty$, then $T_\infty = \infty$ so global existence in obtained for this slightly greater than linear growth. On the other hand, if $\psi(u) \sim u(\ln u)^2$ as $u \to \infty$, then the improper integral is convergent and $T_\infty < \infty$.

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DEPARTMENT OF MATHEMATICS, OREGON STATE UNIVERSITY
CORVALLIS, OREGON, U.S.A.

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