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Some remarks concerning the paper of S. Gołąb and A. Jakubowicz *

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§ 1. Associate with any point $P(x^1, \ldots, x^n)$ of a space L_n the space of its bivectors. This space will be regarded as a (generalized) Klein projective space K_N $(N = \frac{1}{2}n(n-1)-1)$ and every object of K_N will be termed a K-object. The generalized Kronecker Deltas [1] and [2]

$$\delta_P^{\lambda\nu} = \delta_P^{[\lambda\nu]}, \qquad \delta_{\lambda\mu}^P = \delta_{[\lambda\nu]}^P$$

project a bivector $h^{\lambda\nu}(h_{\lambda\nu})$ of L_n into a K-point (K-plane) the *homogeneous* coordinates of which are

$$X^P = (*)\delta_{\lambda\nu}^P h^{\lambda\nu} \qquad (X_P = (*)\delta_P^{\lambda\nu} h_{\lambda\nu})\ (^1)\,. \tag{1}$$

The transformation rule is

$$X^{P'} = \Delta_P^{P'} X^P\,, \qquad X_{P'} = \Delta_{P'}^P X_P\,,$$
$$\Delta_P^{P'} = \delta_P^{\lambda\nu}\delta_{\omega'\mu'}^{P'} A_\lambda^{\omega'} A_\nu^{\mu'}\,.$$

Thus for instance

$$R_{P\lambda}{}^{\nu} = \delta_P^{\omega\nu} R_{\omega\mu\lambda}{}^{\nu}\,, \qquad R_{\omega\mu\lambda}{}^{\nu} = \delta_{\omega\mu}^P R_{P\lambda}{}^{\nu}\,.$$

§ 2. We choose an arbitrary but fixed index Q_0, and an arbitrary but fixed K-plane X_P, and require

$$Q_0' = Q_0\,, \qquad X_{Q_0'} = \Delta_{Q_0'}^P X_P \neq 0\,. \tag{2}$$

This is a restriction imposed on coordinate transformation. Using it, we define the *non-homogeneous* coordinates $\underset{Q_0}{h_P}$ of X_P by

$$\underset{Q_0}{h_P} \overset{\text{def}}{=} X_P / X_{Q_0}\,. \tag{3a}$$

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* See, this fascicle, pp. 161-165.

(1) $(*)$ stands for an appropriate numerical factor. In the sequel we shall leave this symbol out. X stands for the Greek χ (hi).

The transformation rule of $\underset{Q_0}{h_P}$ is obviously

$$\underset{Q_0}{h_{P'}} = \varphi \underset{Q_0}{h_P} \Delta^P_{P'} = \frac{\underset{Q_0}{h_P} \Delta^P_{P'}}{\underset{Q_0}{h_R} \Delta^R_{Q_0'}}, \tag{3b}$$

where

$$\varphi = \frac{1}{\underset{Q_0}{h_R} \Delta^R_{Q_0'}} .$$

§ 3. The K-connection $\Gamma_{P\xi}{}^S$ can be obtained by means of the requirement that $\delta_P{}^{\lambda\nu}$ (or $\delta_{\lambda\nu}{}^P$) are covariant constant

$$D_\xi \delta_P{}^{\lambda\nu} = \Gamma_{\alpha\xi}{}^\lambda \delta_P{}^{\alpha\nu} + \Gamma_{\alpha\xi}{}^\nu \delta_P{}^{\lambda\alpha} - \Gamma_{P\xi}{}^R \delta_R{}^{\lambda\nu} = 0 .$$

Multiplying this equation by $\delta_{\lambda\nu}{}^S$ and taking into account

$$\delta_R{}^{\lambda\nu} \delta_{\lambda\nu}{}^S = \delta_R{}^S$$

one obtains

$$\Gamma_{P\xi}{}^S = 2\delta_{\lambda\nu}{}^S \Gamma_{\alpha\xi}{}^\nu \delta_P{}^{\lambda\alpha} . \tag{4}$$

In order to find the covariant derivative of $\underset{Q_0}{h_P}$ we use the following definition:

$$\underset{Q_0}{D_\xi h_P} = D_\xi \frac{X_P}{X_{Q_0}} \overset{\text{def}}{=} \frac{(D_\xi X_P) X_{Q_0} - X_P D_\xi X_{Q_0}}{(X_{Q_0})^2} \tag{4a}$$

$$= \partial_\xi \underset{Q_0}{h_P} - \underset{Q_0}{h_R} (-\Gamma_{Q_0\xi}{}^R \underset{Q_0}{h_P} + \Gamma_{P\xi}{}^R) .$$

If one introduces the symbols

$$\Lambda_{P\xi}{}^R \overset{\text{def}}{=} \Gamma_{P\xi}{}^R - \Gamma_{Q_0\xi}{}^R \underset{Q_0}{h_P} ,$$

then

$$D_\xi \underset{Q_0}{h_P} = \partial_\xi \underset{Q_0}{h_P} - \Lambda^R_{P\xi} \underset{Q_0}{h_R} . \tag{4b}$$

Remark. According to (3a) we have

$$\underset{Q_0}{h_{Q_0}} = 1$$

and this equation is invariant with respect to coordinate transformations (cf. (3b)). Hence, we must have

$$D_\xi \underset{Q_0}{h_{Q_0}} = 0$$

as it follows immediately from (4a), (4b). In particular

$$\Lambda_{Q_0\xi}{}^R = \Gamma_{Q_0\xi}{}^R - \Gamma_{Q_0\xi}{}^R \underset{Q_0}{h_{Q_0}} = \Gamma_{Q_0\xi}{}^R - \Gamma_{Q_0\xi}{}^R = 0 .$$

§ 4. From

$$R_{\omega\mu\lambda}{}^{\nu} = \underset{Q_0}{h_{\omega\mu}} R_{Q_0\lambda}{}^{\nu}$$

one obtains for symmetric connection $\Gamma_{\lambda\mu}{}^{\nu}$

(15) $$R_{Q_0[\lambda}{}^{\nu} \underset{Q_0}{h_{\omega\mu]}} = 0$$

and this implies

(6a) $$\underset{Q_0}{h_{\omega\mu}} = \underset{Q_0}{\overset{[1}{h_\omega}} \underset{Q_0}{\overset{2]}{h_\mu}},$$

(6b) $$R_{Q_0\lambda}{}^{\nu} = \underset{Q_0}{\overset{1}{h_\lambda}} \underset{1}{V^\nu} + \underset{Q_0}{\overset{2}{h_\lambda}} \underset{2}{V^\nu}.$$

Similarly from

(7) $$V_\xi R_{\omega\mu\lambda}{}^{\nu} = \underset{Q_0}{k_{\xi\omega\mu}} R_{Q_0\lambda}{}^{\nu}$$

we get for symmetric connection

$$\underset{Q_0}{k_{[\xi\omega\mu]}} = 0$$

and this implies

$$\underset{Q_0}{k_{\xi\omega\mu}} = \underset{Q_0}{k_\xi} \underset{Q_0}{k_{[\omega}} \underset{Q_0}{l_{\mu]}}.$$

These results suggest that the holonomy group is probably perfect. If this is so, then there are $n-2$ linearly independent absolute parallel vector fields.

References

[1] V. Hlavatý, *Geometry of Einstein's unified field theory*, Groningen, Noordhoff (1958).

[2] – and R. S. Mishra, *Classification of space-time curvature tensor. I. Introduction*, Tensor, N.S. 14 (1963), pp. 138-168.

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