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On integral equations in a Banach space

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Abstract. The paper deals with integral equations of the Urysohn–Volterra type in a Banach space, i.e., with equations

$$x(t) = \int_0^t f(s, t, x(s))\,ds, \tag{1}$$

where f: $[0,1]^2 \times E \to E$ is continuous, E is a Banach space, x: $[0, a) \to E$ is continuous and the integral is considered in the strong sense. Having the set of continuous functions f: $[0,1]^2 \times E \to E$ endowed with the topology of uniform convergence on bounded sets, the following theorem is true:

The set of functions $f \in X$ for which equation (1) *has one and only one unlimited solution* (for the definition see below) *is a generic set in X.*

The paper contains a detailed proof of this theorem and of some auxiliary results. The most important of them is Lemma 2 on the existence, uniqueness and continuous dependence of the solution of equation (1). The author also quotes Lasota and Yorke's results and Vidossich's results concerning the same problem, and compares briefly these results with the theorem of the paper.

1. Introduction. The purpose of this paper is to state a theorem concerning the problem of existence and uniqueness of solutions of an integral equation of Urysohn–Volterra type. To formulate this theorem we need some preliminary definitions.

Let E be a real Banach space and let X be the space of all continuous functions f defined on $U = [0,1]^2 \times E$ with values in E, endowed with the topology of uniform convergence on bounded sets. Further, let us consider an integral equation of Urysohn–Volterra type:

$$x(t) = r_0 + \int_0^t f(s, t, x(s))\,ds, \tag{1}$$

where f is in X and r_0 in E and the integral is interpreted in the strong sense. A solution of (1) is a continuous function $x(\cdot)$ defined on any interval starting at 0 and contained in $[0,1]$, with values in E.

A solution (or more generally: a continuous function with domain in $[0, 1]$ and values in E) is said to be *unlimited* (more precisely: *positivly unlimited*) iff it has no limit when its argument tends to the right end of its domain, unless it is defined on the whole interval $[0, 1]$.

The main result of this paper is the following

THEOREM 1. *The set of all functions $f \in X$ for which equation* (1) *has a unique unlimited solution is generic, i.e., its complement is a set of first category* (in the sense of the Baire Category Theorem).

This paper had been inspired by works of Orlicz [2] and of Lasota and Yorke [1]. The first part of this paper contains a detailed proof of the main theorem. The author has decided to present all the details because of their absence in almost all literature known to him and concerning integral equations in Banach spaces. In the second part we shall briefly compare our theorem with the result of Lasota and Yorke [1] and with results obtained by Vidossich [3] in an other way, with the use of general topological properties of function spaces.

2. Lemmas. For simplicity of proofs and without loss of generality we may assume that the topology in X is that of uniform convergence. One can easily show that no but little changes are necessary in the case of the topology of uniform convergence on bounded sets. The topology of uniform convergence in X is metrizable. Indeed, the function

$$d(f, g) = \sup_{(s,t,x)\in U} \frac{\|f(s, t, x) - g(s, t, x)\|}{1 + \|f(s, t, x) - g(s, t, x)\|},$$

where $\|\cdot\|$ is the norm in E, is a bounded metric in X consistent with the topology of uniform convergence.

For brevity sometimes we shall say "$x(\cdot)$ is a solution of f_0" instead of "$x(\cdot)$ is a solution of (1), letting f be f_0".

Let E_1 and E_2 be any Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Let U_1 be an open subset of E_1; a mapping $G\colon U_1 \to E_2$ is called *locally Lipschitzean* if for each $x_0 \in U_1$ there exist an open neighbourhood of x_0, say V_0, contained in U_1 and a number $L_0 > 0$ such that

$$\|G(x) - G(y)\|_2 \leqslant L_0 \|x - y\|_1 \quad \text{for } x, y \in V_0.$$

We shall also need the weaker notion of functions $f \in X$ locally Lipschitzean with respect to the variable in E (shortly: locally Lipschitzean). They are defined by the condition: for each $u_0 \in U = [0, 1]^2 \times E$ there exist an open neighbourhood V_0 of u_0 and a number $L > 0$ such that

$$\|f(s, t, x) - f(s, t, y)\| \leqslant L\|x - y\| \quad \text{for } (s, t, x), (s, t, y) \in V_0.$$

The result stated below for locally Lipschitzean functions remains true also for functions just defined.

LEMMA 1. *Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, and let $U_1 \subset E_1$ be open. Let $F\colon U_1 \to E_2$ be continuous and let $\delta > 0$. Then there exists a locally Lipschitzean function $G\colon U_1 \to E_2$ such that $\|F(x) - G(x)\|_2 < \delta$ for $x \in U_1$.*

(This is to say, the set of locally Lipschitzean functions is dense in the set of all continuous functions from U_1 to E_2 with the sup norm.)

A proof of this lemma (with the use of techniques of partitions of unity) is outlined in the above mentioned paper [1] and therefore it is omitted here.

LEMMA 2. *Let $f = f_0 \in X$ be locally Lipschitzean. Then:*

1° *Equation* (1) *has one and only one unlimited solution $x_0(t)$;*

2° *If, moreover, $\{f_n\}$ is a sequence in X uniformly convergent to f_0 and if $x_n(t)$ is an unlimited solution of the equation*

$$(1_n) \qquad x_n(t) = r_n + \int_0^t f_n\big(s, t, x_n(s)\big)\, ds \quad \text{for } n = 1, 2, \ldots$$

with $r_n \in E$ and $r_n \to r_0$, then $x_n \rightrightarrows x_0$.

Here the notation $x_n \rightrightarrows x_0$ means that, for every compact interval J contained in the domain of x_0, all but finitely many x_n are defined on J and the sequence $x_n(\cdot)$ tends uniformly to $x_0(\cdot)$ on J.

Proof of Lemma 2. Without loss of generality we may assume that $r_0 = 0$ and so $r_n \to 0$. We first prove assertion 1°.

Step 1. Using the Banach Fixed-Point Theorem we shall prove the existence and uniqueness of solution of f_0 in some right neighbourhood of 0.

Let Y be the set of all cotninuous functions $x\colon [0, 1] \to E$ with the sup norm. Let $c > 0$ be such that f_0 is Lipschitzean with a constant $L > 0$ for x with $\|x\| \leqslant c$ and for $0 \leqslant s,\ t \leqslant c$, and let $M = \sup\{\|f(s, t, 0)\|\colon s, t \in [0, 1]\}$. For $T > 0$ write $Y_T = \{x \in Y\colon \|x(t)\| \leqslant c,\ t \in [0, T]\}$. We shall choose $S > 0$ so small that the operator $F\colon Y \to Y$ given by the formula

$$(Fx)(t) = \int_0^t f\big(s, t, x(s)\big)\, ds$$

maps Y_S into itself and such that $F|Y_S$ is a contraction. At first, observe that Y_T is a non-empty closed set. From the inequality

$$\|Fx\| = \sup_{t \leqslant T} \|(Fx)(t)\| \leqslant \sup_{t \leqslant T} \int_0^t \big\|f(s, t, x(s))\big\|\, ds \leqslant T(M + Lc)$$

it follows that $F(Y_T) \subset Y_T$ if $T \leqslant c(M+Lc)^{-1}$. Furthermore, $F|Y_T$ is a contraction if $T < \min\{1/L, c\}$. Indeed,

$$\|Fx_1 - Fx_2\| = \sup_{t \leqslant T} \|(Fx_1)(t) - (Fx_2)(t)\|$$

$$\leqslant \sup_{t \leqslant T} \int_0 \big\|f(s, t, x_1(s)) - f(s, t, x_2(s))\big\| ds \leqslant TL \sup_{t \leqslant T} \|x_1(t) - x_2(t)\| \leqslant TL\|x_1 - x_2\|.$$

Thus it is enough to choose $S < \min\{1, c, c/(M+Lc), 1/L\}$. According to the Banach Fixed-Point Theorem there exists exactly one $x \in Y_S$ such that $(F|Y_S)(x) = x$.

Step 2. Let us define

$T_s = \sup\{T\colon x(\cdot)$ is defined on $[0, T]$ and is a solution of $f\}$.

There are two possible cases:

(I) $\lim\limits_{t \to T_s} x(t)$ does not exist or $T_s = 1$; then by definition $x\colon [0, T_s) \to E$ is unlimited.

(II) $\lim\limits_{t \to T_s} x(t)$ exists; then we set $x(T_s) = \lim\limits_{t \to T_s} x(t)$. In this case, applying the argument of Step 1 to the equation

$$x(t) = x(T_s) + \int_{T_s}^{t} f(s, t, x(s))\, ds$$

we may define a solution of (1) in the interval $[0, T_1)$ for some $T_1 > T_s$, and this contradicts the definition of T_s. So case (II) is impossible and 1° is proved.

Now we are going to prove assertion 2°.

Step 3. Under the assumptions of Lemma 2 we shall prove the existence of $\varepsilon > 0$ and $T > 0$ such that all but finitely many $x_n(\cdot)$ are defined in the interval $[0, T]$ and $\|x_n(t)\| \leqslant \varepsilon$ for $t \in [0, T]$.

Let $a = f_0(0, 0, 0)$; there is $\varepsilon > 0$ such that

$$\|a - f_0(s, t, x)\| \leqslant 1 \quad \text{for each } s \leqslant \varepsilon,\ t \leqslant \varepsilon,\ \|x\| \leqslant \varepsilon.$$

Since $f_n \rightrightarrows f_0$, we have $\|f_n(s, t, x)\| \leqslant \|a\| + 2$ for $s \leqslant \varepsilon$, $t \leqslant \varepsilon$, $\|x\| \leqslant \varepsilon$, and for sufficiently large n, say, for $n \geqslant n_0$. Let us set $K = \|a\| + 2$ and $T = \varepsilon/4K$. Choose $n_1 \geqslant n_0$ such that $\|r_n\| \leqslant \frac{1}{4}\varepsilon$ for $n \geqslant n_1$. We claim that for ε, K and T chosen above

(2) $x_n(\cdot)$ is defined on $[0, T]$ and we have $\|x_n(t)\| \leqslant \varepsilon$ for $t \in [0, T]$ and $n \geqslant n_1$.

If the claim is true, the proof in Step 3 is finished.

To prove the claim (2) let us write

$$\tau_n = \min\{T, \sup\{t\colon x_n(t) \text{ is defined}\}\} \quad \text{for } n \geqslant n_1.$$

Of course, $\|x_n(t)\| \leqslant \varepsilon$ in some interval $[0, t_n)$. Then for $n \geqslant n_1$ and $t \in [0, t_n)$

$$\|x_n(t)\| \leqslant \|r_n\| + \int_0^t \big\|f_n(s, t, x_n(s))\big\| ds \leqslant \tfrac{1}{4}\varepsilon + Kt \leqslant \tfrac{1}{2}\varepsilon.$$

This and the continuity of $x_n(\cdot)$ shows that

$$\|x_n(t)\| \leqslant \varepsilon \quad \text{for } t \in [0, \tau_n) \text{ and } n \geqslant n_1. \tag{3}$$

It remains to prove that $\tau_n = T$. Condition (3) implies

$$\|x_n(t_1) - x_n(t_2)\| \leqslant \int_{t_1}^{t_2} \big\|f_n(s, t, x_n(s))\big\| ds \leqslant K|t_1 - t_2| \quad \text{for } t_1 \leqslant t_2 \leqslant \tau_n;$$

i.e., $x_n(\cdot)$ is a Lipschitzean function (with the constant K) on the interval $[0, \tau_n)$. Thus there exists the limit of $x_n(t)$ at τ_n.

Now, if $\tau_n < T$ we obtain a contradiction with the assumption that $x_n(t)$ is unlimited. Thus we have $\tau_n = T$, which finishes the proof of (2).

Step 4. Since the constant T chosen in Step 3 depends only on ε for a fixed K, we obtain as an immediate corollary:

(4) *For every $c \leqslant \varepsilon$ there exist $T_c > 0$ and a positive integer $n(c)$ such that $\|x_n(t)\| \leqslant c$ for $t \in [0, T_c]$ and for $n \geqslant n(c)$.*

This will be useful in the proof of the uniform convergence of $x_n(\cdot)$ to $x_0(\cdot)$ in some right neighbourhood of 0.

Let $c \leqslant \varepsilon$ and $T_c > 0$ be such that $f(s, t, x)$ is Lipschitzean with a constant L for $s \leqslant T_c$, $t \leqslant T_c$, $\|x\| \leqslant c$ and such that (4) is true. Furthermore, let $\delta > 0$ and let n_0 be such that

$$\|f_n(s, t, x) - f_0(s, t, x)\| \leqslant \tfrac{1}{2}\delta \quad \text{and} \quad \|r_n\| \leqslant \tfrac{1}{2}\delta \quad \text{for } n \geqslant n_0.$$

Then

$$\begin{aligned}\|x_n(t) - x_0(t)\| &\leqslant \|r_n\| + \int_0^t \big\|f_n(s, t, x_n(s)) - f_0(s, t, x_0(s))\big\| ds \\ &\leqslant \|r_n\| + \int_0^t \big\|f_n(s, t, x_n(s)) - f_0(s, t, x_n(s))\big\| ds + \int_0^t \big\|f_0(s, t, x_n(s)) - \\ &\qquad - f_0(s, t, x_0(s))\big\| ds \leqslant \tfrac{1}{2}\delta + \int_0^t \tfrac{1}{2}\delta\, ds + \int_0^t L\|x_n(s) - x_0(s)\| ds.\end{aligned}$$

Hence, for $n \geqslant n_0$, $s \leqslant T_c$, $t \leqslant T_c$

$$\|x_n(t) - x_0(t)\| \leqslant \int_0^t \|x_n(s) - x_0(s)\| L\, ds + \delta.$$

Now, by virtue of the Gronwall inequality, we have

$$\|x_n(t)-x_0(t)\| \leqslant \delta \exp \int_0^t L ds = \delta e^{Lt} \leqslant \delta e^{L}.$$

This means (since δ is an arbitrary positive number) that

$$\lim_{n\to\infty} \left(\sup_{t\leqslant T_c} \|x_n(t)-x_0(t)\|\right) = 0$$

and consequently the sequence $x_n(\cdot)$ converges to $x_0(\cdot)$ uniformly on $[0, T_c]$.

Step 5. Clearly, repeating the above argument with regard to the problem

$$x_0(t) = x_0(T_c) + \int_{T_c}^{t} f_0\big(s, t, x_0(s)\big) ds,$$

$$x_n(t) = x_n(T_c) + \int_{T_c}^{t} f_n\big(s, t, x_n(s)\big) ds \quad \text{for } n = 1, 2, \ldots$$

we may prove the uniform convergence of $x_n(\cdot)$ to $x_0(\cdot)$ on some larger interval $[0, T]$ (with $T > T_c$).

Now let $t_0 = \sup\{t \in [0, 1]: x_0(s)$ is defined for $s \in [0, t]\}$ (this is to say, $x_0(\cdot)$ is an unlimited solution on $[0, t_0]$ or $[0, t_0)$). Further, let $t_s = \sup\{t' \in [0, 1]: x_n \rightrightarrows x_0$ on $[0, t']\}$. To end the proof of 2° it suffices to show that $t_s = t_0$. To this end, let us suppose, on the contrary, that $t_s < t_0$. Then the definition of t_s implies the existence of the smallest integer, say n_2, such that

$$\|x_n(t)-x_0(t)\| \leqslant \tfrac{1}{2} \quad \text{for } t \in [0, t_s - \tfrac{1}{2}t_s] \text{ and } n \geqslant n_2.$$

We now define a sequence $\{n_k\}$ for $k \geqslant 2$ as follows:

n_2 is defined above; for $k \geqslant 2$, n_{k+1} is the smallest of the positive integers which are greater than n_k and such that

$$\|x_n(t)-x_0(t)\| \leqslant \frac{1}{k+1} \quad \text{for } t \in \left[0, \frac{k}{k+1} t_s\right] \text{ and } n \geqslant n_{k+1}.$$

(Such an integer exists again by the definition of t_s.) Now let $\{t_n\}$ be the sequence given by the formula

$$t_n = \begin{cases} 0 & \text{for } n < n_2, \\ \dfrac{k-1}{k} t_s & \text{for } n_k \leqslant n < n_{k+1} \text{ and } k = 2, 3, \ldots \end{cases}$$

This sequence has the following properties:

$t_n \to t_s$; more precisely, $|t_n - t_s| < 1/k$ for $n \geqslant n_k$;

$$\|x_n(t)-x_0(t)\| \leqslant \frac{1}{k} \quad \text{for } n \geqslant n_k \text{ and } t \in [0, t_n].$$

We are going to prove that these properties imply the uniform convergence of $x_n(\cdot)$ to $x_0(\cdot)$ in some right neighbourhood of t_s, contrary to the definition of t_s.

To this effect let us denote

$$V(a, \beta, \gamma) = \{(s, t, x) \in U: |s-t_s| < a, |t-t_s| < \beta, \|x-x_0(t_s)\| < \gamma\}$$

and let us choose a number $\varepsilon > 0$ such that f_0 is Lipschitzean in $V(\varepsilon, \varepsilon, \varepsilon)$ and $\|f_0(u)\| \leqslant M+1$ for $u \in V(\varepsilon, \varepsilon, \varepsilon)$, where $M = \|f_0(t_s, t_s, x_0(t_s))\|$.

Since $f_n \rightrightarrows f_0$, there is a positive integer m such that $\|f_n(u)\| \leqslant M+2$ and $\|f_n(u)-f_0(u)\| < \varepsilon/8$ for $u \in V(\varepsilon, \varepsilon, \varepsilon)$ and $n \geqslant m$. Since $x_0(\cdot)$ is continuous, there is a number $\delta > 0$ such that $\delta < t_0-t_s$ and $\|x_0(s)-x_0(t_s)\|\varepsilon/8$ if $|s-t_s| < \delta$. Since f_0 is continuous, there is a number $\delta_1 > 0$ such that $\|f_0(s, t, x)-f_0(s, \bar{t}, x)\| < \varepsilon/8$ if $|t-t_s| < \delta_1$, $|\bar{t}-t_s| < \delta_1$. Then, if $n \geqslant m$, $|t-t_s| < \delta_1$, $|\bar{t}-t_s| < \delta_1$, $|s-t_s| < \varepsilon$ and $\|x-x_0(t_s)\| < \varepsilon$, we have

$$\|f_n(s, t, x)-f_n(s, \bar{t}, x)\| \leqslant \|f_n(s, t, x)-f_0(s, t, x)\| + \\ + \|f_0(s, t, x)-f_0(s, \bar{t}, x)\| + \|f_0(s, \bar{t}, x)-f_n(s, \bar{t}, x)\| < 3\varepsilon/8.$$

Now, take an integer k such that $k^{-1} < \varepsilon/8$ and write

$$n_0 = \min\{m, n_k\}, \qquad \eta = \min\left\{\delta, \delta_1, \frac{\varepsilon}{8(M+2)}\right\}.$$

Then the points $(t_n, t_n, x_n(t_n))$ are in $V(\eta, \eta, \varepsilon)$ for $n \geqslant n_0$. Indeed:

$$\|x_n(t_n)-x_0(t_s)\| \leqslant \|x_n(t_n)-x_0(t_n)\| + \|x_0(t_n)-x_0(t_s)\| < \varepsilon/4.$$

Since $x_n(\cdot)$ is unlimited, it is defined in some right neighbourhood of t_n. Moreover, there is $s_n > t_n$ such that the point $(s, t, t_n(s))$ belongs to $V(\eta, \eta, \varepsilon)$ for $s, t \in [t_n, s_n]$ (by the continuity of $x_n(\cdot)$).

Let us denote by t_n' the supremum of the numbers $s_n > t_n$ such that $x_n(s)$ is defined and $(s, t, x_n(s)) \in V(\eta, \eta, \varepsilon)$ for $s, t \in [t_n, s_n]$. We shall show that $t_n' = t_s+\eta$.

First, observe that for $n \geqslant n_0$ and $s, t \in [t_n, t_n')$

$$\|x_n(t)-x_0(t_s)\| \leqslant \|x_n(t_n)-x_0(t_s)\| + \int_{t_n}^{t} \|f_n(s, t, x_n(s))\| ds + \\ + \int_0^{t_n} \|f_n(s, t, x_n(s))-f_n(s, t_n, x_n(s))\| ds < \tfrac{1}{4}\varepsilon+2\eta(M+2)+\tfrac{3}{8}\varepsilon < \tfrac{7}{8}\eta\varepsilon. \tag{5}$$

In other words, if $(s, t, x_n(s)) \in V(\eta, \eta, \varepsilon)$, then also $(s, t, x_n(s)) \in V(\eta, \eta, \frac{7}{8}\varepsilon)$.

Further, one can see that $x_n(\cdot)$ has a limit at t_n'. So we may put $x_n(t_n') = \lim_{t\to t_n'} x_n(t)$.

Now, if we had $t'_n < t_s + \eta$, then, by virtue of (5), $(t'_n, t'_n, x_n(t'_n)) \in V(\eta, \eta, \varepsilon)$, and since $x_n(\cdot)$ is unlimited, it would be defined and remain in $V(\eta, \eta, \varepsilon)$ for arguments right to t'_n, contrary to the definition of t'_n. This contradiction implies that $x_n(\cdot)$ are defined on $[t_s, t_s+\eta]$ for $n \geqslant n_0$. Repeating the argument of Step 4 we can prove the uniform convergence of $x_n(\cdot)$ to $x_0(\cdot)$ on the interval $[0, t_s+\tau]$ for some $\tau > 0$, which contradicts the definition of t_s. This conclusion completes the proof of assertion 2° and, consequently, of Lemma 2.

LEMMA 3. *Let X be a metric space and let X_1 be a dense subset of X. Let φ be a non-negative real-valued function defined on X. Further, let*

$$(6) \qquad \varphi(f_n) \to 0 \quad \textit{if} \quad f_n \to f \quad \textit{for some } f \textit{ in } X_1.$$

Then the set $A = \{f \in X\colon \varphi(f) \neq 0\}$ is meager, i.e., is a set of first category in X.

Proof. Let us define $A_\varepsilon = \{f \in X\colon \varphi(f) \geqslant \varepsilon\}$. Then $A = \bigcup_{n=1}^{\infty} A_{1/n}$ and it suffices to prove that each A_ε is nowhere dense. The last condition is equivalent to the following one: for every ball $K_1 \subset X$ there exists a ball $K_2 \subset K_1$ such that $K_2 \cap A_\varepsilon = \emptyset$. So, let K_1 be a ball contained in X. Since X_1 is dense, there exists $f \in K_1 \cap X_1$. We claim that $K_2 = K(f, \delta)$ for some $\delta > 0$ has the desired property, i.e., that $K_2 \subset K_1 \setminus A_\varepsilon$. In other words, we claim that there exists $\delta > 0$ such that $\varphi(g) < \varepsilon$ for every $g \in K(f, \delta)$. Indeed, if not, then for each integer n there is a $g_n \in K(f, 1/n)$ such that $\varphi(g_n) \geqslant \varepsilon$, and this contradicts assumption (6).

3. Proof of Theorem 1. Let us recall that E is a Banach space, $U = [0,1]^2 \times E$ and X is the set of all continuous functions $f\colon U \to E$ endowed with the topology of uniform convergence. Let $\{U_n\}$ be a sequence of non-empty closed bounded subsets of U such that $U_n \subset \operatorname{int} U_{n+1}$ for $n = 1, 2, \ldots$ and such that $\bigcup_{n=1}^{\infty} U_n = U$.

For any $f \in X$ we define $W_n(f) = \{(s, t, x) \in U_n\colon \|f(s, t, x) - f(0, 0, 0)\| \leqslant n\}$. Note that $\bigcup_{n=1}^{\infty} W_n(f) = U$ for each $f \in X$. Furthermore, for every $f \in X$ and $n = 1, 2, \ldots$ the set $W_n(f)$ is a neighbourhood of 0 in U. Let us fix $f_0 \in X$. We shall write W_n instead of $W_n(f_0)$. Let $x(\cdot)$ be a solution of f for some $f \in X$ and write $\sigma_n(x(\cdot)) = \sup\{t' \geqslant 0\colon x(t') \text{ is defined and } (s, t, x(s)) \in W_n,\ 0 \leqslant s, t \leqslant t'\}$. If $x_1(\cdot)$, $x_2(\cdot)$ are solutions of equation (1) for some f_1, f_2, respectively, we shall write

$$I_n = I_n(x_1(\cdot), x_2(\cdot)) = [0, \min\{\sigma_n(x_1(\cdot)), \sigma_n(x_2(\cdot))\}]$$

and

$$\mu_n = \mu_n(x_1(\cdot), x_2(\cdot)) = \sup\{\|x_1(t) - x_2(t)\|\colon t \in I_n\}.$$

Let $B_\delta(f_0) = \{f \in X: d(f_0, f) \leqslant \delta\}$ ($d(\cdot, \cdot)$ stands for the metric in X). Let

$$T_\delta = T_\delta(f_0) = \{(x(\cdot), f): x(\cdot) \text{ is a solution for } f \in B_\delta(f_0)\}.$$

Let us set

$$\mu_{n,\delta}(f_0) = \sup\{\mu_n(x_1(\cdot), x_2(\cdot)): (x_i(\cdot), f_i) \in T_\delta(f_0),\ i = 1, 2\}$$

and

$$V_n(f_0) = \limsup_{\delta\to 0} \mu_{n,\delta}(f_0).$$

We shall prove some properties of the numbers $V_n(f_0)$.

(i) If $V_n(f) = 0$ for some $f \in X$ and for $n = 1, 2, \ldots$, then there exists at most one solution of f.

For, let $x_1(\cdot), x_2(\cdot)$ be solutions of f, let $t_1 > 0$ be such that both $x_1(\cdot)$ and $x_2(\cdot)$ are defined on $[0, t_1]$. If $s, t \in [0, t_1]$, then there exists a constant $K > 0$ such that

$$\|x_i(s)\| \leqslant K \quad \text{and} \quad \|f(s, t, x_i(s))\| \leqslant K \quad \text{for } i = 1, 2.$$

Thus one can find a positive integer n_0 such that $\|f(s, t, x_i(s)) - f(0, 0, 0)\| \leqslant n_0$ and

$$\{(s, t, x_i(s)): s, t \in [0, t_1], \|x_i(s)\| \leqslant K\} \subset U_{n_0} \quad \text{for } i = 1, 2.$$

But then $(s, t, x_i(s)) \in W_{n_0}$ for $s, t \in [0, t_1]$ and $i = 1, 2$. Hence $[0, t_1] \subset J_{n_0}(x_1(\cdot), x_2(\cdot))$. It follows that $x_1(s) = x_2(s)$ for $s \in [0, t_1]$, since $\mu_{n_0}(x_1, x_2) \leqslant V_{n_0}(f) = 0$.

(ii) If $V_n(f) = 0$ for some $f \in X$ and for $n = 1, 2, \ldots$, then there does exist an unlimited solution of f.

For, let us choose a sequence $\{f_i\} \subset X$ such that $f_i \to f$ in X and such that, for each i, f_i has an unlimited solution $x_i(\cdot)$. (This is possible by Lemma 1 and Lemma 2.1°.) At first, let us observe that if all but finitely many $x_i(\cdot)$ are defined at the point $t_1 \geqslant 0$ and if there is an integer n_0 such that $(s, t, x_i(s)) \in W_{n_0}$ for $s, t \in [0, t_1]$ and for all but finitely many i, then $\{x_i(\cdot)\}$ is a Cauchy sequence of functions defined on $[0, t_1]$; indeed, for i, j sufficiently great we have

$$\sup\{\|x_i(t) - x_j(t)\|: t \in [0, t_1]\} \leqslant \mu_{n_0,\delta}(f)$$

and $\mu_{n_0,\delta}(f) \to 0$ as $\delta \to 0$.

Therefore, U being complete (and so being $\overline{W}_{n_0}$), there exists a continuous function $x: [0, t_1] \to E$ such that $x_i(t)$ converges to $x(t)$ uniformly on $[0, t_1]$. Of course, $x(t)$ is a solution of f. So it remains to prove that

(a) there exists $t_1 > 0$ such that all but finitely many $x_i(\cdot)$ (hence also $x(\cdot)$) are defined on $[0, t_1]$;

(b) if $J = \bigcup\{[0, t_1]: x_i(t) \rightrightarrows x(t) \text{ on } [0, t_1]\}$, then $x(\cdot)$ defined on J is an unlimited solution of f.

To prove (a) we can use an argument similar to that applied in Step 3 of the proof of Lemma 2. For (b), let us consider the four possible cases.

1. $J = [0, 1]$, then $x(\cdot)$ is unlimited by definition.

2. $J = [0, 1)$ and $x(t)$ converges to some $x_0 \in E$ as $t \to 1$; then setting $x(1) = x_0$ we obtain an unlimited solution x: $[0, 1] \to E$.

3. $J = [0, b)$ and $x(t)$ has no limit as $t \to b$; again x: $[0, b) \to E$ is unlimited by definition.

4. $J = [0, b]$ with $b < 1$. This case is impossible. To show this, we can use an argument similar to that which has helped us to finish the proof of assertion 2° in Lemma 2, and we shall not do it here.

(iii) If $f \in X$ is locally Lipschitzean, then $V_n(f) = 0$ for $n = 1, 2, \ldots$ This is an immediate consequence of Lemma 2.

(iv) If $V_n(f) = 0$ for some $f \in X$ and for some integer n, then $V_n(g) \to 0$ as $g \to f$.

For, suppose, on the contrary, that there exist $\varepsilon > 0$ and a sequence $\{g_i\} \subset X$ such that $g_i \to f$ and $V_n(g_i) > \varepsilon$. By virtue of the definition of $V_n(g_i)$ there exist functions $f_{1,i}$ and $f_{2,i}$ with solutions $x_{1,i}$ and $x_{2,i}$, respectively, such that $\mu_n(x_{1,i}(\cdot), x_{2,i}(\cdot)) > \varepsilon$ for $i = 1, 2, \ldots$ Further, $\{f_{1,i}\}$ and $\{f_{2,i}\}$ can be chosen so that $d(g_i, f_{1,i}) \to 0$ and $d(g_i, f_{2,i}) \to 0$ as $i \to \infty$, again by the definition of $V_n(g_i)$. But then $d(f, f_{1,i}) \to 0$ and $d(f, f_{2,i}) \to 0$ as $i \to \infty$, and from the definition of $V_n(f)$ it follows that $V_n(f) \geqslant \varepsilon$, contrary to the assumption that $V_n(f) = 0$.

Now we are ready to prove Theorem 1. Let us apply Lemma 3 to the case where X is the set of all continuous functions f: $U \to E$ with the metrizable topology of uniform convergence, X_1 is the set of all locally Lipschitzean functions $f \in X$ and $\varphi(f) = V_n(f)$ for any integer n. By virtue of (iii) and (iv), all assumptions of Lemma 3 are satisfied. Therefore the set $T_n = \{f \in X\colon V_n(f) \neq 0\}$ is a set of first category in X for each positive integer n. Hence the set $T = \bigcup_{n=1}^{\infty} T_n$ is of first category, too. But if $f \in X \setminus T$, then by (i) and (ii) there exists one and only one unlimited solution of f, which we have had to prove.

4. Remarks. This part will be devoted to the comparison of our result with the results of Lasota and Yorke [1] and of Vidossich [3]. First of all we quote the main results of these papers.

In their paper [1] Lasota and Yorke proved the following

THEOREM 1. *Let $B \subset U$ be a countable union of compact sets. Let T be the set of $f \in X$ for which there is $u_0 \in B$ such that there exists no unlimited solution of the equation $x' = f(t, x)$ through u_0. In the space X, the set T is meager.*

Here E is a Banach space, $U = \boldsymbol{R} \times E$ and X is the set of all continuous functions f: $U \to E$ with the topology of uniform convergence.

Vidossich's paper [3] contains the following

THEOREM 2. *Let X be a complete metric space and M a set of continuous maps $X \to X$ endowed with any topology finer than the topology of uniform convergence on X. Let M_0 be any subset of M such that*

(a) *Every $f \in M_0$ has a unique fixed point x_f;*

(b) *For every $f \in M_0$ and every sequence $\{f_n\}_n$ in M converging in M to f, if x_n is a fixed point of f_n for every n, then $\lim_n x_n = x_f$.*

Then there exists a G_δ-set M^* in M such that $M^* \supseteq M_0$, every $f \in M^*$ has a unique fixed point x_f and $f \mapsto x_f$ is a continuous map $M^* \to X$.

The paper [3] contains also the following useful

COROLLARY. *Under the assumptions of Theorem 2, if M_0 is dense in M and if M is of second category, then "existence, uniqueness and approximation of fixed points by fixed points of members of M" is a generic property in M.*

(This means, the set of members of M having this property is generic).

The result of Lasota and Yorke and that of ours are independent (because they involve different classes of functions), however, the proof of our theorem follows the proof of Theorem 1 in [1].

From his main theorem, Vidossich obtained the Lasota and Yorke's result in some the special case in which the right-hand functions in the differential equation in question are bounded.

It seems that our result cannot be obtained by Vidossich's method, even under the assumption of boundedness.

References

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