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On a generalization of the Perron integral on one-dimensional intervals

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Introduction

The integral studied in the present paper is a generalization of the one-dimensional Perron integral. We call it an *H-integral* and denote by $(H)\int_a^b f\,dt$, where H specifies a system of pointed intervals used in the definition. The integral is defined as a certain limit of the sums

$$S(f, \Delta) = \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1}),$$

where $x_0 < a = t_1 < x_1 < t_2 < \ldots < x_{k-1} < t_k = b < x_k$. The intervals $[x_{j-1}, x_j]$ forming a covering (rather than a partition) of the interval $[a, b]$, the integral has some unexpected properties. For example, it is possible that $(H)\int_a^b f\,dt$ exists but $(H)\int_a^c f\,dt$ does not for some $c \in (a, b)$. For some choices of the set H we have $(H)\int_{-1}^{1} dx/x = 0$.

In our paper [4], such examples were presented and the transformation of the integral for a special choice of the set H was discussed. In the general case, the main results were given without proofs. The aim of the present paper is to give brief proofs of the results announced in [4], Section 3.

1. Preliminaries

Let $[a, b]$ be an interval. Then any finite set

$$\Delta = \{(t_j, [x_{j-1}, x_j]); j = 1, \ldots, k\}$$

such that

$$x_0 < t_1 = a < x_1 < t_2 < \ldots < t_{k-1} < x_{k-1} < t_k = b < x_k$$

is called a *covering* of $[a, b]$.

If δ is a gauge on $[a, b]$, i.e., δ: $[a, b] \to (0, +\infty)$, and

$$[x_{j-1}, x_j] \subset B(t_j, \delta(t_j)), \quad j = 1, \ldots, k,$$

then Δ is said to be *δ-fine*. (Here and in the sequel, $B(t, r) = (t-r, t+r)$.)

Write

$$J = J[a, b] = \{(t,[x, y]);\ t \in [a, b],\ x < t < y\},$$

$$\mathrm{Sym} = \mathrm{Sym}[a, b] = \{(t,[x, y]) \in J;\ t = \tfrac{1}{2}(x+y)\}.$$

Let $H = H[a, b]$ be a set such that $\mathrm{Sym} \subset H \subset J$ and

(1) for every $(t,[x, y]) \in H$ there is $\xi > 0$ such that $(t,[x+h, y-h]) \in H$ for every h, $|h| < \xi$.

A covering Δ such that $\Delta \subset H$ will be called an *H-covering*.

1.1. Remark. Let $K > 0$, $\varrho \geqslant 1$ be constants. Then

$$AS_{K,\varrho} = \{(t,[x, y]) \in J;\ 0 < t-x < y-t+K(y-t)^\varrho,\ 0 < y-t < t-x+K(t-x)^\varrho\}$$

satisfies $\mathrm{Sym} \subset AS_{K,\varrho} \subset J$ and has property (1). (Cf. [4], Note 3.2.)

1.2. Remark. Given a set

$$\Xi = \{(\tau_i,[\xi_{i-1}, \xi_i]);\ i = 1, \ldots, l\} \subset H$$

and a gauge δ on $[\tau_1, \tau_l]$ such that

$$\xi_0 < \tau_1 < \xi_1 < \tau_2 < \ldots < \tau_{l-1} < \xi_{l-1} < \tau_l < \xi_l, \quad [\xi_{i-1}, \xi_i] \subset B(\tau_i, \delta(\tau_i));$$

then there exists $\eta > 0$ such that

$$\xi_0+h < \tau_1 < \xi_1-h < \tau_2 < \ldots < \tau_{l-1} < \xi_{l-1}(-1)^{l-1}h < \tau_l < \xi_l+(-1)^l h,$$

$$[\xi_{i-1}+(-1)^{i-1}h,\ \xi_i+(-1)^i h] \subset B(\tau_i, \delta(\tau_i))$$

for $i = 1, \ldots, l$ provided $|h| < \eta$. (This follows from (1) and from the fact that Ξ is finite.)

The set

$$\Xi_h = \{(\tau_i,[\xi_{i-1}+(-1)^{i-1}h,\ \xi_i+(-1)^i h]);\ i = 1, \ldots, l\}$$

will be called an *h-modification* (or briefly a *modification*) of the set Ξ.

In particular, if Ξ is a δ-fine H-covering of $[a, b]$, then its h-modification (with h sufficiently small) is a δ-fine H-covering of $[a, b]$ as well. This fact will be frequently used in the proofs throughout the paper without further notice.

2. Definition and main properties of the H-integral

2.1. DEFINITION. A function f: $[a, b] \to R$ is called *H-integrable* (on $[a, b]$) if there is $q \in R$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$|q - S(f, \Delta)| < \varepsilon$$

for every δ-fine H-covering Δ of $[a, b]$, where

$$S(f, \Delta) = \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1}).$$

The number q is the H-integral of f over $[a, b]$ and we write

$$q = (H)\int_a^b f(t)dt = (H)\int_a^b f\, dt.$$

2.2. Remark. If $H = J$, then the H-integral is the Perron integral. Indeed, let f: $[a, b] \to R$, $\gamma \in R$. It is well known (cf. [2], Section 1.2; [3], Theorem 3.5; [1], Appendix A, Proposition 4.3) that the Perron integral of f exists and $\gamma = (P)\int_a^b f dt$ iff for every $\varepsilon > 0$ there exists such a gauge δ on $[a, b]$ that

$$\left|\gamma - \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1})\right| \leqslant \varepsilon$$

holds for every sequence

$$(*) \qquad a = x_0 \leqslant t_1 \leqslant x_1 \leqslant \ldots \leqslant x_{k-1} \leqslant t_k \leqslant x_k = b$$

satisfying

$$t_j - \delta(t_j) < x_{j-1}, \qquad x_j < t_j + \delta(t_j).$$

It can be assumed without loss of generality that $a < t - \delta(t)$, $t + \delta(t) < b$ for $t \in (a, b)$, so that we have in addition $t_1 = a$, $t_k = b$. It is not difficult to prove that the same concept of integral is obtained if $(*)$ is replaced by

$$x_0 < a = t_1 < x_1 < \ldots < x_{k-1} < t_k = b < x_k,$$

and this modified concept is at the same time the H-integral for $H = J$.

2.3. PROPOSITION. *A function f is H-integrable if and only if for every $\varepsilon > 0$ there is a gauge δ such that*

$$|S(f, \Delta_1) - S(f, \Delta_2)| < \varepsilon$$

for any two δ-fine H-coverings Δ_1, Δ_2.

Proof is standard.

2.4. Remark. Let δ be a gauge on $[a, b]$, let $d \in (a, b)$. The point d is said to be *δ-reachable from a* (more precisely, *δ-H-reachable from a*) if there is a set

$$\theta = \{(\sigma_i, [\vartheta_{i-1}, \vartheta_i]);\ i = 1, \ldots, m\} \subset H$$

such that

$$\vartheta_0 < \sigma_1 = a < \vartheta_1 < \sigma_2 < \ldots < \vartheta_{m-1} < \sigma_m < \vartheta_m = d,$$

$$[\vartheta_{i-1}, \vartheta_i] \subset B(\sigma_i, \delta(\sigma_i)). \tag{2}$$

(Notice that θ is a covering of $[a, \sigma_m]$ but not of $[a, d]$.)

Similarly, d is called *δ-reachable from b* if θ satisfies (2) and

$$d = \vartheta_0 < \sigma_1 < \vartheta_1 < \sigma_2 < \ldots < \vartheta_{m-1} < \sigma_m = b < \vartheta_m.$$

The set θ will be called a *δ-chain from a to d* (or, as the case may be, from b to d). Lemma 2.4 [4] asserts that the set of points $d \in (a, b)$ which are not δ-reachable from either a or b is at most countable.

Indeed, let s be the supremum of all $c \in (a, b)$ such that in (a, c) there are at most countably many points not δ-reachable from a. We have $s \geqslant a + \delta(a)$ since every $x \in (a, a + \delta(a))$ is δ-reachable from a (it suffices to put $\vartheta_0 = 2a - x < \sigma_1 = a < \vartheta_1 = x$). Assume $s < b$. Then in the interval $(s, \min(s + \delta(s), b))$ there exist uncountably many points not δ-reachable from a. Let x be such a point. Then $y = s - (x - s) = 2s - x$ cannot be δ-reachable from a since otherwise we could extend the corresponding chain from a to y by the pair $(s, [2s - x, x]) \in \mathrm{Sym}$, thus obtaining a δ-chain from a to x. But there are only countably many points $y < s$ not δ-reachable from a, which is a contradiction.

2.5. Remark. It follows from [4], Lemma 2.3 or from Remark 2.4 above that for every gauge δ there exists a δ-fine H-covering. In fact, there always exists a δ-fine Sym-covering. Indeed, by Remark 2.4 we can find a point d, $b - \delta(b) < d < b$, which is δ-Sym-reachable from a. By adding the element $(b, [d, 2b - d])$ to the corresponding chain we obtain a δ-fine Sym-covering of $[a, b]$.

2.6. Theorem. *Let $f\colon [a, b] \to R$ be H-integrable. Denote by $E = E_f$ the set of all $c \in (a, b)$ such that $f|_{[a,c]}$ is not H-integrable. Then E is at most countable.*

Proof. Set $\varepsilon_j = 2^{-j}$ and find the corresponding gauge δ_j from the definition of the H-integral $(H)\int_a^b f\,dt$. Denote by W_j the set of all points $c \in (a, b)$ which are not δ_j-reachable from either a or b, and put $W = \bigcup_{j=1}^{\infty} W_j$. By Remark 2.4 the set W is at most countable.

Let $c \in (a, b) \setminus W$. Then $(H)\int_a^c f\,dt$ exists.

Indeed, let $\varepsilon > 0$. Find j such that $\frac{1}{2}\varepsilon > 2^{-j}$. Since c is δ_j-reachable from

b, there exists a set

$$\theta = \{(\sigma_i, [u_{i-1}, u_i]);\ i = 1, \ldots, m\} \subset H$$

such that

$$c = u_0 < \sigma_1 < u_1 < \sigma_2 < \ldots < \sigma_{m-1} < u_{m-1} < \sigma_m = b < u_m,$$
$$[u_{i-1}, u_i] \subset B(\sigma_i, \delta_j(\sigma_i)).$$

For this set, find $\eta > 0$ such that its every h-modification with $0 < h < \eta$ is a δ_j-fine H-covering of $[\sigma_1, b]$ (cf. Remark 1.2).

Further, choose h, $0 < h < \eta$, so that

$$2h \sum_{i=1}^{m} |f(\sigma_i)| < 2^{-j},$$

and a gauge $\tilde{\delta}_j$ such that

$$\begin{aligned} &\tilde{\delta}_j(t) < \min(\delta_j(t), c-t) && \text{for } a \leqslant t < c,\\ &\tilde{\delta}_j(c) < \min(h, \delta_j(c)),\\ &\tilde{\delta}_j(t) = \delta_j(t) && \text{for } c < t \leqslant b. \end{aligned}$$

Let Δ^1, Δ^2 be $\tilde{\delta}_j$-fine H-covering of $[a, c]$ where

$$\Delta^p = \{(t_j^p, [x_{j-1}^p, x_j^p]);\ j = 1, \ldots, k_p\}, \qquad p = 1, 2.$$

Write $\varkappa_p = x_{k_p}^p - c$; then $0 < \varkappa_p < h$. Construct $\varkappa_p$-modifications of the set θ for $p = 1, 2$, and denote them by θ^1, θ^2. By the choice of h and $\tilde{\delta}_j$, the sets $\Delta^p \cup \theta^p$, $p = 1, 2$, are δ_j-fine H-covering of $[a, b]$. Evidently,

$$S(f, \Delta^p \cup \theta^p) = S(f, \Delta^p) + S(f, \theta^p)$$

and

$$\begin{aligned} |S(f, \theta^1) - S(f, \theta^2)| &= \Big|\sum_{i=1}^{m} f(\sigma_i)\big[(u_i + (-1)^i\varkappa_1 - u_{i-1} - (-1)^{i-1}\varkappa_1)\\ &\qquad - (u_i + (-1)^i\varkappa_2 - u_{i-1} - (-1)^{i-1}\varkappa_2)\big]\Big|\\ &\leqslant \sum_{i=1}^{m} |f(\sigma_i)|\, 2(\varkappa_1 - \varkappa_2) \leqslant 2h \sum_{i=1}^{m} |f(\sigma_i)| < \tfrac{1}{2}\varepsilon. \end{aligned}$$

Consequently, we have

$$|S(f, \Delta^1) - S(f, \Delta^2)| \leqslant |S(f, \Delta^1 \cup \theta^1) - S(f, \Delta^2 \cup \theta^2)| + |S(f, \theta^1) - S(f, \theta^2)| < \varepsilon,$$

and the desired integrability (over $[a, c]$) follows by Proposition 2.3.

2.7. Theorem. *Let $a < c < b$ and let two of the integrals in the equality*

$$(3) \qquad (H)\int_a^c f\, dt + (H)\int_c^b f\, dt = (H)\int_a^b f\, dt$$

exist. Then the third integral exists as well and the equality holds.

Proof. Consider the case where the first and the last integral exist. Let $\varepsilon > 0$, and find gauges δ_1, δ_2 on $[a, c]$, $[a, b]$, respectively, corresponding to ε in the sense of Definition 2.1. Without loss of generality we can and will assume that

$$\delta_2(t) \leqslant \delta_1(t) \quad \text{for } t \in [a, c],$$
$$\delta_2(t) \leqslant |t-c| \quad \text{for } t \in [a, b] \setminus \{c\},$$
$$2|f(c)|\delta_2(c) < \varepsilon.$$

Let Δ be a δ_2-fine H-covering of $[c, b]$, where

$$\Delta = \{(t_j, [x_{j-1}, x_j]);\ j = 1, \dots, k\}.$$

Let $h > 0$ be such that the h-modification Δ_h of Δ is a δ_2-fine H-covering of $[c, b]$, and $x_0 + h$ is δ_2-reachable from a. (Existence of such an h follows from Remarks 1.2 and 2.4.) Moreover, let h be so small that

$$2h \sum_{j=1}^{k} |f(t_j)| < \varepsilon.$$

Let θ be the first set from Remark 2.4 with $d = x_0 + h$ and $\delta = \delta_2$. Then the set $\theta \cup \Delta_h$ is a δ_2-fine H-covering of $[a, b]$, and the set $\theta \cup \{(c, [x_0+h, x_1-h])\}$ is a δ_1-fine H-covering of $[a, c]$. Consequently,

$$\begin{aligned}
\Big|S(f, \Delta) - (H)\int_a^b f\,dt + (H)\int_a^c f\,dt\Big| &\leqslant |S(f, \Delta) - S(f, \Delta_h)| \\
&\quad + \Big|S(f, \theta \cup \{(c, [x_0+h, x_1-h])\}) - (H)\int_a^c f\,dt\Big| \\
&\quad + |f(c)|(x_1 - h - x_0 - h) + \Big|S(f, \theta \cup \Delta_h) - (H)\int_a^b f\,dt\Big| \\
&\leqslant 2h \sum_{j=1}^{k} |f(t_j)| + \varepsilon + 2|f(c)|\delta_2(c) + \varepsilon < 4\varepsilon
\end{aligned}$$

which proves the existence of the integral $(H)\int_c^b f\,dt$ as well as the validity of equality (3).

Proofs of the other cases are analogous.

2.8. Remark. Let $(H)\int_a^b f\,dt$ exist, let $c, d \in [a, b]$, $c < d$. If $c, d \in [a, b] \setminus E$, then by Theorems 2.6, 2.7 the integral $(H)\int_c^d f\,dt$ exists. Conversely, if $(H)\int_c^d f\,dt$ exists, then by Theorem 2.6 there is $\tau \in (c, d)$ such that the integrals $(H)\int_a^\tau f\,dt$

and $(H)\int_c^\tau f\,dt$ exist, and by Theorem 2.7 we obtain $c \notin E$ and similarly $d \notin E$.

3. Indefinite H-integral and its properties

In this section let $f\colon [a, b] \to R$ be H-integrable, and define

$$F(a) = 0, \qquad F(t) = (H)\int_a^t f(s)ds \qquad \text{for } t \in [a, b] \setminus E,$$

where E is the (at most countable, cf. Theorem 2.6) set of points $c \in (a, b)$ such that $(H)\int_a^c f\,dt$ does not exist.

3.1. THEOREM. *The function F is continuous on $[a, b] \setminus E$.*

Proof. Let $c \in [a, b] \setminus E$, let (c_n) be an increasing sequence, $c_n \notin E$, $\lim_{n\to\infty} c_n = c$. Let $\varepsilon > 0$. Find the gauge δ on $[a, c]$ corresponding to the definition of the integral $(H)\int_a^c f\,dt$, assuming without loss of generality that

$$\delta(t) < c - t \quad \text{for } t < c, \quad |f(c)|\delta(c) < \varepsilon. \tag{4}$$

Let k be such an integer that $c_k > c - \delta(c)$. Let δ_k be the gauge from Definition 2.1 corresponding to ε and $(H)\int_a^{c_k} f\,dt$. Without loss of generality let us assume that $\delta_k(t) \leqslant \delta(t)$ for $t \in [a, c_k]$. Let Δ be a δ_k-fine H-covering of $[a, c_k]$ whose last element is $(c_k, [x_{k-1}, x_k])$. We have $x_k < c$ by (4). Hence $\Delta \cup \{(c, [x_k, 2c - x_k])\}$ is a δ-fine covering of $[a, c]$. Consequently,

$$\Big|(H)\int_a^c f\,dt - (H)\int_a^{c_k} f\,dt\Big| \leqslant \Big|(H)\int_a^c f\,dt - S(f, \Delta \cup \{(c, [x_k, 2c - x_k])\})\Big|$$
$$+ |f(c)| 2\delta(c) + \Big|S(f, \Delta) - (H)\int_a^{c_k} f\,dt\Big| < 3\varepsilon,$$

which proves the theorem.

Before formulating a converse result, we will prove the version of the Saks–Henstock lemma corresponding to the H-integral.

3.2. THEOREM (Saks–Henstock lemma). *Let $f\colon [a, b] \to R$ be H-integrable, $\varepsilon > 0$. Let δ be the gauge from Definition* 2.1 *and let*

$$\{(t_j, [u_j, v_j]);\ j = 1, \ldots, k\} \subset H \tag{5}$$

satisfy

$$[u_j, v_j] \subset B(t_j, \delta(t_j)); \tag{6}$$

$$a \leqslant u_1 < t_1 < v_1 \leqslant u_2 < t_2 < \ldots < v_{k-1} \leqslant u_k < t_k < v_k \leqslant b \tag{7}$$

and $u_j, v_j \notin E$ *for* $j = 1, \dots, k$. *Then*

$$\sum_{j=1}^{k} \Big((H)\int_{u_j}^{v_j} f\, dt - f(t_j)(v_j - u_j)\Big) \leqslant \varepsilon. \tag{8}$$

Proof. Let us first make two remarks. First, we may and will assume, without loss of generality, that all inequalities in (7) are strict. Indeed, we can pass from the points u_j, v_j to $u'_j > u_j$, $v'_j < v_j$ so that conditions (5)–(7) are fulfilled with the new points, and the "error" made by replacing u_j, v_j by u'_j, v'_j on the left-hand side of (8) is arbitrarily small. Second, notice that by Remark 2.8 the H-integrals of f over $[u_j, v_j]$ and $[v_j, u_{j+1}]$ exist for $j = 1, \dots, k$ ($k-1$, respectively).

Since the proof of Theorem 3.2 is technically rather complicated, we first prove a lemma.

3.3. Lemma. *Let* δ, t_j, u_j, v_j *be from Theorem* 3.2, *let* $\varrho > 0$. *Then there exists a* δ*-fine H-covering* Ω *of* $[a, b]$,

$$\Omega = \{(\tau_i, [\omega_{i-1}, \omega_i]);\ i = 1, \dots, l\}$$

and integers $0 < m_1 < m_2 < \dots < m_k$ *such that*

$$\tau_{m_j} = u_j < \omega_{m_j} < \tau_{m_j+1} = t_j < \omega_{m_j+1} < v_j = \tau_{m_j+2}, \tag{9}$$

$$\omega_{m_j} - u_j = v_j - \omega_{m_j+1} < \varrho.$$

Proof. Denote $s_j = \frac{1}{2}(v_j + u_{j+1})$ for $j = 1, \dots, k-1$, $s_k = b$. Set

$$\delta'(s_j - \lambda) = \delta'(s_j + \lambda) = \min\big(\delta(s_j - \lambda), \delta(s_j + \lambda)\big)$$
$$\text{for} \quad 0 \leqslant \lambda \leqslant s_j - v_j,\ j = 1, \dots, k-1,$$

$$\delta'(t) = \delta(t) \quad \text{otherwise, } t \in [a, b].$$

For $j = 1, \dots, k-1$ find σ_j, $0 < \sigma_j < \delta'(s_j)$ such that

$$v_j < s_j - \sigma_j < s_j + \sigma_j < u_{j+1} \text{ and } s_j - \sigma_j,\ s_j + \sigma_j \notin E$$

(cf. Theorem 2.6). Then there exists h, $0 < h < \varrho$, satisfying

$$h < \xi(t_j, [u_j, v_j]) \quad \text{(cf. (1))},\ h < s_j - \sigma_j - v_j, \quad h < \delta'(u_j), \quad h < \delta'(v_j)$$

for $j = 1, \dots, k$, and such that the points $v_j + h$ are δ'-reachable from $s_j - \sigma_j$ and $u_1 - h$ is δ'-reachable from a (cf. Remark 2.4).

Now we will construct the desired covering.

By the choice of h there exists a δ'-chain from a to $u_1 - h$; let it consist of points

$$\omega_0,\ \tau_1 = a,\ \omega_1, \dots, \tau_{m_1-1},\ \omega_{m_1-1} = u_1 - h.$$

Put $\tau_{m_1} = u_1$, $\omega_{m_1} = u_1 + h$, $\tau_{m_1+1} = t_1$, $\omega_{m_1+1} = v_1 - h$, $\tau_{m_1+2} = v_1$, $\omega_{m_1+2} = v_1 + h$. Again by the choice of h, the last point is δ'-reachable from $s_1 - \sigma_1$.

Suppose we have found the points of Ω up to a point $\omega_{m_j+2} = v_j + h$ in such a way that (9) is fulfilled. Then there is a δ'-chain from $s_j - \sigma_j$ to ω_{m_j+2}, and a "symmetric" chain from $s_j + \sigma_j$ to $u_{j+1} - h$. (Here "symmetric" means that the points of the latter chain are symmetric about s_j to the corresponding points of the former.) The two chains together with the element $(s_j, [s_j - \sigma_j, s_j + \sigma_j]) \in \mathrm{Sym} \subset H$ filling the gap between them extend our construction up to the point $u_{j+1} - h = \omega_{m_{j+1}-1}$. Put $\tau_{m_{j+1}} = u_{j+1}$, $\omega_{m_{j+1}} = u_{j+1} + h$, $\tau_{m_{j+1}+1} = t_{j+1}$, $\omega_{m_{j+1}+1} = v_{j+1} - h$, $\tau_{m_{j+1}+2} = v_{j+1}$, $\omega_{m_{j+1}+2} = v_{j+1} + h$. Thus we have proceeded from step j to step $j+1$ in our construction. Repeating the procedure, we extend the covering Ω to the whole interval $[a, b]$. It is seen directly from the construction that Ω has the required properties.

3.4. Proof of Saks–Henstock lemma. Let $\eta > 0$. Find gauges φ_j on $[v_j, u_{j+1}]$ for $j = 0, 1, \ldots, k$ (denoting $v_0 = a$, $u_{k+1} = b$) such that

$$\left|(H)\int_{v_j}^{u_{j+1}} f\,dt - S(f, \Phi_j)\right| < \eta/(k+1) \tag{10}$$

for every φ_j-fine H-covering Φ_j of $[v_j, u_{j+1}]$.

Let $\tilde{\delta}$ be a gauge on $[a, b]$ such that

$$\begin{aligned}
&\tilde{\delta}(t) \leqslant \delta(t) && \text{for} \quad t \in [a, b],\\
&\tilde{\delta}(t) < |t - u_j| && \text{for} \quad t \neq u_j,\\
&\tilde{\delta}(t) < |t - v_j| && \text{for} \quad t \neq v_j,\\
&\tilde{\delta}(t) \leqslant \varphi_j(t) && \text{for} \quad t \in [v_j, u_{j+1}],
\end{aligned}$$

$$\tilde{\delta}(u_j) < \delta(t_j) - (t_j - u_j), \qquad \tilde{\delta}(v_j) < \delta(t_j) - (v_j - t_j);$$

$$\max(\tilde{\delta}(u_j), \tilde{\delta}(v_j)) < \xi = \xi(t_j, [u_j, v_j])$$

(with ξ from formula (1)), and

$$\tilde{\delta}(u_j)\,|f(t_j)| < \eta/2(k+1),$$

$$\tilde{\delta}(v_j)\,|f(t_j)| < \eta/2(k+1).$$

Let Ω be the partition from Lemma 3.3 (corresponding to the gauge $\tilde{\delta}$ instead of δ). Since Ω is a δ-fine H-covering of $[a, b]$ we have

$$\left|(H)\int_a^b f(t)dt - S(f, \Omega)\right| < \varepsilon.$$

Further, denote

$$\Phi_j = \{(\tau_p, [\omega_{p-1}, \omega_p]);\ p = m_j + 2, m_j + 3, \ldots, m_{j+1}\},$$

$$j = 0, 1, \ldots, k,\ m_0 = -1,\ m_{k+1} = l.$$

Then Φ_j is a φ_j-fine H-covering of $[v_j, u_{j+1}]$ and hence satisfies (10). Obviously we have

$$\sum_{j=1}^{k} \left((H)\int_{u_j}^{v_j} f\,dt - f(t_j)(v_j-u_j)\right) = (H)\int_a^b f\,dt - S(f, \Omega)$$

$$-\sum_{j=0}^{k} \left((H)\int_{v_j}^{u_{j+1}} f\,dt - S(f, \Phi_j)\right) - \sum_{j=1}^{k} f(t_j)(v_j - \omega_{m_j+1} - u_j + \omega_{m_j}),$$

and consequently,

$$\left|\sum_{j=1}^{k} \left((H)\int_{u_j}^{v_j} f\,dt - f(t_j)(v_j-u_j)\right)\right| < \varepsilon + 2\eta.$$

Since $\eta > 0$ has been arbitrary, the proof is complete.

3.5. Remark. The assertion of the Saks–Henstock lemma can be modified to

$$\sum_{j=1}^{k} |F(v_j) - F(u_j) - f(t_j)(v_j-u_j)| \leqslant 2\varepsilon.$$

This is obtained by dividing the set of $(t_j, [u_j, v_j])$ in two groups according to the sign of the corresponding summand, and applying the lemma in the original form to each group separately.

Now we can prove a converse of Theorem 3.1.

3.6. Theorem. *Let $f\colon [a, b] \to R$ be H-integrable, $d \in (a, b)$. If there exists a finite limit $\lim_{c \nearrow d} F(c) = q \in R$ for $c \in (a, d)\setminus E$, then $(H)\int_a^d f\,dt$ exists and equals q.*

Proof. Let $\varepsilon > 0$ and let δ be the gauge on $[a, b]$ corresponding to ε and $(H)\int_a^b f\,dt$. Let $\tilde{\delta}$ be a gauge on $[a, b]$ such that

$$2|f(a)|\tilde{\delta}(d) < \varepsilon, \quad 2|f(d)|\tilde{\delta}(d) < \varepsilon,$$

$$|F(x) - q| < \varepsilon \quad \text{for any} \quad x \notin E,\ d - \tilde{\delta}(d) < x < d,$$

$$|F(x)| < \varepsilon \quad \text{for any} \quad x \notin E,\ a < x < a + \tilde{\delta}(a)$$

(cf. Theorem 3.1),

$$\tilde{\delta}(x) \leqslant \delta(x) \quad \text{for all } x \in [a, b].$$

Let Δ be a $\tilde{\delta}$-fine H-covering of $[a, d]$, where

$$\Delta = \{(t_j, [x_{j-1}, x_j]);\ j = 1, \ldots, k\}.$$

Find a $\tilde{\delta}$-fine modification Δ_h (cf. Remark 1.2) such that $x'_j = x_j + (-1)^{k-1-j}h \notin E$ for $j = 1, \ldots, k$, $h > 0$ and

(11) $$2h \sum_{j=1}^{k} |f(t_j)| < \varepsilon.$$

By the Saks-Henstock lemma we have

$$(12)\qquad \Big|\sum_{j=2}^{k-1}\big(f(t_j)(x_j'-x_{j-1}')-(H)\int_{x_{j-1}'}^{x_j'} f\,dt\big)\Big|\leqslant\varepsilon.$$

Consequently, using (11), (12) and the properties of the gauge $\tilde{\delta}$ we obtain

$$\begin{aligned}\Big|\sum_{j=1}^{k} f(t_j)(x_j-x_{j-1})-q\Big| &\leqslant \varepsilon+\Big|\sum_{j=1}^{k} f(t_j)(x_j'-x_{j-1}')-q\Big|\\ &\leqslant \varepsilon+2\tilde{\delta}(a)|f(a)|+\Big|\sum_{j=2}^{k-1}\big[f(t_j)(x_j'-x_{j-1}')-(H)\int_{x_{j-1}'}^{x_j'} f\,dt\big]\Big|\\ &\quad +2\tilde{\delta}(d)|f(d)|+\Big|(H)\int_{x_1'}^{x_{k-1}'} f\,dt-q\Big|\\ &\leqslant 4\varepsilon+|F(x_{k-1}')-q|+\Big|(H)\int_a^{x_1'} f\,dt\Big|\leqslant 6\varepsilon,\end{aligned}$$

which proves the theorem.

The next theorem strengthens the result on continuity of the function F, asserting that it has a derivative equal to f almost everywhere in $[a, b]$. The symbol $m(M)$ stands for the Lebesgue measure of the set M.

3.7. THEOREM. *There is a set $M\subset[a, b]$, $m(M)=0$, such that for every $\varepsilon>0$ and $t\in[a, b]\setminus M$ there is $\vartheta=\vartheta(t)>0$ such that*

$$(13)\qquad |F(y)-F(x)-f(t)(y-x)|<\varepsilon|y-x|$$

for every x, y such that

$$(14)\qquad (t, [x, y])\in H,\quad [x, y]\subset B(t, \vartheta(t)),\quad x, y\notin E.$$

Proof. For $\vartheta>0$, $t\in[a, b]\setminus E$ define

$$\Phi_\vartheta(t)=\inf\big(F(y)-F(x)\big)/(y-x),\qquad \Phi_*(t)=\sup_\vartheta \Phi_\vartheta(t),$$

$$\Phi^\vartheta(t)=\sup\big(F(y)-F(x)\big)/(y-x),\qquad \Phi^*(t)=\inf_\vartheta \Phi^\vartheta(t),$$

where the infimum or supremum is taken over all x, y satisfying (14). Denote

$$P_n=\{t\in(a, b);\ \Phi_*(t)\leqslant f(t)-n^{-1}\},$$

$$Q_n=\{t\in(a, b);\ \Phi^*(t)\geqslant f(t)+n^{-1}\},$$

$$M=\bigcup_{n=1}^{\infty}(P_n\cup Q_n).$$

If $m(M)=0$, the theorem holds. Assume $m_e(M)>0$. Then there exists, say, an index p such that $m_e(P_p)=\sigma>0$ (m_e denotes the outer Lebesgue measure; the proof is analogous if $m_e(Q_q)>0$ for some q).

Choose $0 < \varepsilon < \sigma/4p$ and find the gauge δ corresponding to ε by Definition 2.1. For $t \in P_p$ set

$$\mathscr{S}(t) = \{(x, y) \subset [a, b];\ x, y \notin E,\ (t, [x, y]) \in H,\ [x, y] \in B(t, \delta(t)),$$
$$(F(y) - F(x))(y-x)^{-1} < f(t) - 1/2p\}.$$

Then $\bigcup_{t \in P_p} \mathscr{S}(t)$ covers P_p in the sense of Vitali; hence there exists its finite disjoint subsystem of (x_i, y_i), $i = 1, \ldots, r$, such that

$$\sum_{i=1}^{r} (y_i - x_i) \geqslant \sigma/2.$$

We may apply the Saks–Henstock lemma to this subsystem, which yields

$$\left|\sum_{i=1}^{r} (F(y_i) - F(x_i) - f(t_i)(y_i - x_i))\right| \leqslant \varepsilon < \sigma/4p;$$

on the other hand, from the definition of $\mathscr{S}(t)$ we have

$$\left|\sum_{i=1}^{r} (F(y_i) - F(x_i) - f(t_i)(y_i - x_i))\right| > \sum_{i=1}^{r} \frac{1}{2p}(y_i - x_i) \geqslant \frac{\sigma}{4p},$$

a contradiction.

3.8. Theorem. *Let F be defined as above, put $F(t) = F(a)$ for $t < a$ and $F(t) = F(b)$ for $t > b$. Let $C \subset [a, b]$, $m(C) = 0$. Then for every $\varepsilon > 0$ there is a gauge δ on C such that for any finite system $\{(\tau_j, [\xi_j, \eta_j]); j = 1, \ldots, r\} \subset H$ such that $\tau_j \in C$, $[\xi_j, \eta_j] \subset B(\tau_j, \delta(\tau_j))$; $\xi_j, \eta_j \notin E$ and the intervals $[\xi_j, \eta_j]$ do not overlap, the inequality*

$$\sum_{j=1}^{r} |F(\eta_j) - F(\xi_j)| < \varepsilon \tag{15}$$

holds.

Proof. Denote

$$C_n = \{t \in C;\ n-1 \leqslant |f(t)| < n\}.$$

There exist open sets $G_n \supset C_n$, $m(G_n) < \varepsilon_n = \varepsilon(3 \cdot 2^n n)^{-1}$; for $t \in C_n$ choose $\delta_1(t)$ such that $B(t, \delta_1(t)) \subset G_n$. Since $[\xi_j, \eta_j]$ do not overlap, we have

$$\sum_{j=1}^{r} |f(\tau_j)|(\eta_j - \xi_j) = \sum_{n=1}^{\infty} \sum_{\tau_j \in C_n} |f(\tau_j)|(\eta_j - \xi_j) \leqslant \sum_{n=1}^{\infty} n m(G_n) < \varepsilon/3. \tag{16}$$

For $\varepsilon/3$ find the gauge δ_2 from Definition 2.1 and put $\delta(t) = \min(\delta_1(t), \delta_2(t))$ for $t \in C$, $\delta(t) = \delta_2(t)$ otherwise. Using the modified version of the Saks–Henstock lemma from Remark 3.5 we obtain

$$\sum_{j=1}^{r} |F(\eta_j) - F(\xi_j) - f(\tau_j)(\eta_j - \xi_j)| < \tfrac{2}{3}\varepsilon$$

that is,

$$\sum_{j=1}^{r} |F(\eta_j)-F(\xi_j)| < \tfrac{2}{3}\varepsilon + \sum_{j=1}^{r} |f(\tau_j)|(\eta_j-\xi_j) < \varepsilon$$

by inequality (16).

The next theorem shows that the properties from the two preceding theorems characterize the indefinite H-integral.

3.9. THEOREM. *Let E be an at most countable subset of $[a, b]$, let $f\colon [a, b] \to R$, $F\colon [a, b]\backslash E \to R$. Extend F by $F(t) = F(a)$ for $t < a$, $F(t) = F(b)$ for $t > b$.*

Assume that

(i) *for almost all t and all $\varepsilon > 0$ there exists $\vartheta(t)$ such that* (13) *holds for all x, y satisfying* (14);

(ii) *if $C \subset [a,b]$, $m(C) = 0$, then for every $\varepsilon > 0$ there is a gauge δ on C such that* (15) *holds provided τ_j, ξ_j, η_j satisfy the assumptions of Theorem* 3.8.

Then $(H)\int_a^b f\,dt$ exists and equals $F(b)-F(a)$.

Proof. Let C_1 be the set of $t\in[a, b]$ for which (i) is not fulfilled, $C = C_1 \cup E$. Then $m(C) = 0$. Find δ from (ii) and define $\delta_1(t) = \min(\delta(t), \vartheta(t))$ for $t\in C$, $\delta_1(t) = \vartheta(t)$ otherwise. Let $\varepsilon > 0$ and let Δ be a δ_1-fine H-partition of $[a, b]$,

$$\Delta = \{(t_j, [x_j, y_j]);\ j = 1, \ldots, k\}.$$

Similarly as in the proof of Theorem 3.6, modify Δ so that $F(x_j')$, $F(y_j')$ are defined for all j's and $S(f, \Delta)$ changes only by ε (cf. Remark 1.2 and inequality (11)). Then

$$(H)\int_a^b f\,dt = \sum_{j=1}^{k} [F(y_j')-F(x_j')],$$

hence

$$\Big|(H)\int_a^b f\,dt - S(f, \Delta_h)\Big| \leqslant \sum_{t_j\notin C} |F(y_j')-F(x_j')-f(t_j)(y_j'-x_j')|$$
$$+ \sum_{t_j\in C} |F(y_j')-F(x_j')| + \sum_{t_j\in C} |f(t_j)|(y_j'-x_j').$$

Using (i), (ii) and estimating the last sum as in the proof of the preceding theorem we conclude

$$\Big|\int_a^b f\,dt - S(f, \Delta)\Big| \leqslant \mathrm{const}\cdot\varepsilon$$

which completes the proof.

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