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# Existence, uniqueness and continuous dependence of solutions of differential equations in Banach space

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**Abstract.** In this paper we will study differential equations of the form

$$x' = f(t, x) \quad \text{for a.e. } t \in [0, T], \tag{1}$$

by the assumption that $f\colon [0, T] \times X \to X$ satisfies the Carathéodory conditions, where $(X, |\cdot|)$ is a separable Banach space. The main result of this paper deals of the existence of solutions of (1) satisfying an initial condition $x(0) = x_0$, where $x_0 \in X$ ([8], [10]). Furthermore, the uniqueness and continuous dependence of solutions on the right-hand side of (1) will be here investigated.

1. Let $\mu$ denote the Lebesgue measure on the real line $R$. A $\mu$-measurable function of $[0, T]$ into $X$ will be called, simply, measurable. As usual, we will say that $f\colon [0, T] \times X \to X$ satisfies the Carathéodory conditions if: (i) $f(\cdot, x)\colon [0, T] \to X$ is measurable for fixed $x \in X$, (ii) $f(t, \cdot)\colon X \to X$ is continuous for fixed $t \in [0, T]$ and (iii) there exists a Lebesgue integrable function $m_f\colon [0, T] \to R$ such that $|f(t, x)| \leqslant m_f(t)$ for $x \in X$ and a.e. $t \in [0, T]$.

It is well known ([2], [5]) that the Carathéodory conditions are not sufficient to guarantee the local existence of solutions of an initial value problem for differential equations in Banach spaces. Therefore, we need extra conditions involving of the ball measure of noncompactness $\beta$. Here the ball measure of noncompactness $\beta(A)$ of a bounded subset $A$ of $X$ is defined by

$$\beta(A) = \inf\{r > 0\colon A \text{ can be covered by finitely many balls of radius} \leqslant r\}.$$

For the properties of $\beta$ we refer e.g. to [1]. We can directly obtain $\beta(A) \leqslant \operatorname{diam}(A)$ (the diameter of $A$), $\beta(A) = \beta(\bar{A})$, $\beta(A) = 0$ iff $\bar{A}$ is compact, and if $A \subset B$ then $\beta(A) \leqslant \beta(B)$. Furthermore, it was proved in [6] and [7] (see also [2]), that the ball measure of noncompactness has following properties.

LEMMA 1.1. *Let $(X, |\cdot|)$ be a separable Banach space and $(x_n)$ a sequence* *of continuously differentiable functions* $x_n$: $[0, T] \to X$ *such that* $|x_n(t)| \leqslant c$ *in* $[0, T]$ *for each* $n$.

*Then* $\psi(t) = \beta(\{x_n(t)\colon n \geqslant 1\})$ *is absolutely continuous and*

$$\dot{\psi}(t) \leqslant \beta(\{\dot{x}_n(t)\colon n \geqslant 1\}) \quad \textit{for almost every } t \in [0, T].$$

LEMMA 1.2. *Let* $(X, |\cdot|)$ *be a separable Banach space and* $(x_n)$ *a bounded sequence of continuous functions of* $[0, T]$ *into* $X$.

*Then* $\psi(t) = \beta(\{x_n(t)\colon n \geqslant 1\})$ *is measurable and*

$$\beta(\{\int_0^T x_n(t)\,dt\colon n \geqslant 1\}) \leqslant \int_0^T \psi(t)\,dt.$$

Using of the known extension of the Tieze's Theorem, we will show that the above properties of the ball measures of noncompactness can be extended on the case of absolutely continuous and Lebesgue integrable, respectively functions $x_n$: $[0, T] \to X$. Recall, that we say that a function $x$: $[0, T] \to X$ is absolutely continuous if there exists a Lebesgue integrable function $y$: $[0, T] \to X$ such that $x(t) = x(0) + \int_0^t y(s)\,ds$ for $t \in [0, T]$. It is easy to see that in this case $x$ has at almost every $t \in [0, T]$ the stong derivative $\dot{x}(t)$ satisfying $\dot{x}(t) = y(t)$ for a.e. $t \in [0, T]$.

LEMMA 1.3. *Let* $(X, |\cdot|)$ *be a separable Banach space and* $(x_n)$ *a sequence of absolutely continuous functions* $x_n$: $[0, T] \to X$ *such that* $|\dot{x}_n(t)| \leqslant m(t)$ *for almost every* $t \in [0, T]$, *where* $m$: $[0, T] \to R$ *is a Lebesgue integrable function. Then* $\psi(t) = \beta(\{x_n(t)\colon n \geqslant 1\})$ *is absolutely continuous and* $\dot{\psi}(t) \leqslant \beta(\{\dot{x}_n(t)\colon n \geqslant 1\})$ *for almost all* $t \in [0, T]$.

Proof. Let $I = [0, T]$, $A(t) = \{x_n(t)\colon n \geqslant 1\}$ and $\dot{A}(t) = \{\dot{x}_n(t)\colon n \geqslant 1\}$ for $t \in I$ and almost every $t \in I$, respectively. In virtue of the Lusin's Theorem, for every $\eta > 0$ and $n = 1, 2, \ldots$ there is a closed set $E_\eta^n \subset I$ with $\mu(I \setminus E_\eta^n) < \eta/2^n$ and such that $\dot{x}_n |\, E_\eta^n$ (the restriction of $\dot{x}_n$ to $E_\eta^n$) is continuous. It is easy to see that $E_\eta = \bigcap_{n=1}^{\infty} E_\eta^n \neq \emptyset$. Indeed, suppose $E_\eta = \emptyset$. Then

$$\mu(I) = \mu(I \setminus E_\eta) \leqslant \sum_{n=1}^{\infty} \mu(I \setminus E_\eta^n) < \sum_{n=1}^{\infty} (\eta/2^n) = \eta \qquad \text{for each } \eta > 0.$$

Therefore, $E_\eta \neq \emptyset$. Suppose furthermore, that $E_\eta$ is such that $m |\, E_\eta$ of $m$ to $E_\eta$ is continuous, too. Let $c_\eta = \sup\{m(t)\colon t \in E_\eta\}$. In virtue of the Tietze–Dugundji's Theorem, for every $n = 1, 2, \ldots$ and fixed $\eta > 0$ there exists a continuous extension $y_n^\eta$ of $\dot{x}_n |\, E_\eta$ on $I$, such that $y_n^\eta(t) \in \operatorname{conv}\{\dot{x}(s)\colon s \in E_\eta\}$ for $t \in I$. We can assume that $|\dot{x}_n(s)| \leqslant c_\eta$ for each $s \in E_\eta$. Therefore, for $t \in I$ we have $|y_n^\eta(t)| \leqslant c_\eta$, too. Let $z_n^\eta(t) = x_n(t) + \int_{t_\eta}^t y_n^\eta(s)\,ds$ for $t \in I$, where $t_\eta = \inf E_\eta$.

We can easy to see that $(z_n^\eta)$ is a sequence of continuously differentiable functions of $I$ into $X$ such that $|\dot{z}_n^\eta(t)| \leqslant c_\eta$ for $t \in I$ and fixed $\eta > 0$. Furthermore, $\dot{z}_n^\eta(t) = y_n^\eta(t) = \dot{x}_n(t)$ for $t \in E_\eta$, $n = 1, 2, \ldots$ and $\eta > 0$. Therefore, for each $n = 1, 2, \ldots$ and $\eta > 0$ there exists a constant $C_\eta^n$ such that $z_n^\eta(t) - x_n(t) = C_\eta^n$ for $t \in E_\eta$. But $z_n^\eta(t_\eta) = x_n(t_\eta)$. Then $C_\eta^n = 0$, and $x_n(t) = z_n^\eta(t)$ for $t \in E_\eta$; $n = 1, 2, \ldots$, $\eta > 0$. Let $B_\eta(t) = \{z_n^\eta(t)\colon n \geqslant 1\}$, $\dot{B}_\eta(t) = \{\dot{z}_n^\eta(z)\colon n \geqslant 1\}$, $\psi_\eta(t) = \beta(B_\eta(t))$ for fixed $t \in I$. We have $\psi(t) = \psi_\eta(t)$ for $t \in E_\eta$ for each $\eta > 0$. In virtue of Lemma 1.1, for every $\eta > 0$, $\psi_\eta$ is absolutely continuous and satisfies $\dot{\psi}_\eta(t) \leqslant \beta(\dot{B}_\eta(t))$ for almost every $t \in [0, T]$.

Let $\eta = 1/2^m$ and choose a sequence $E_1 \subset E_2 \subset \ldots$ of closed subsets of $I$ such that $\mu(I \setminus E_m) < 1/2^m$ and so that $\psi(t) = \psi_m(t)$ for $t \in E_m$, where $\psi_m$ is the above defined absolutely continuous function of $I$ into $R$, corresponding to $\eta = 1/2^m$. Let $E = \bigcup_{m=1}^{\infty} E_m$. We have $\mu(I \setminus E) = 0$. For every $t \in E$ there exists $m$ such that $\psi(t) = \psi_m(t)$ and $A(t) = B_m(t)$. Hence it follows that $\psi$ is absolutely continuous in $I$ and that $\dot{\psi}(t) = \dot{\psi}_m(t)$, $\dot{A}(t) = \dot{B}_m(t)$ for almost every $t \in I$. Therefore, for almost every $t \in I$ we have

$$\dot{\psi}(t) = \dot{\psi}_m(t) \leqslant \beta(\dot{B}_m(t)) = \beta(\dot{A}(t))$$

which completes the proof.

LEMMA 1.4. *Let $(X, |\cdot|)$ be a separable Banach space and $(x_n)$ a sequence of measurable functions of $[0, T]$ into $X$ such that $|x_n(t)| \leqslant m(t)$ for almost every $t \in [0, T]$ and $n = 1, 2, \ldots$, where $m$ is a Lebesgue integrable function of $[0, T]$ into $R$. Then $\Phi(t) = \beta(A(t))$, where $A(t) = \{x_n(t)\colon n \geqslant 1\}$, is measurable and $\beta(\{\int_E x_n(t)\,dt\colon n \geqslant 1\}) \leqslant \int_E \Phi(t)\,dt$ for each measurable set $E \subset [0, T]$.*

Proof. Let us consider a sequence $(y_n)$ satisfying the assumptions of Lemma 1.2. We shall show first, the for every measurable set $E \subset [0, T]$ we have

$$\beta(\{\int_E y(t)\,dt\colon n \geqslant 1\}) \leqslant \int_E \psi(t)\,dt,$$

where $\psi(t) = \beta(\{y_n(t)\colon n \geqslant 1\})$ for $t \in [0, T]$. It is clear that we can consider only measurable sets $E$ with $\mu(E) > 0$. It follows from Lemma 1.2. that for every open interval $(t_1, t_2) \subset [0, T]$ we have

$$\beta(\{\int_{t_1}^{t_2} y_n(s)\,ds\colon n \geqslant 1\}) \leqslant \int_{t_1}^{t_2} \psi(s)\,ds.$$

Let $\varepsilon > 0$ and let $\{(t_i, t_i')\colon 1 \leqslant i \leqslant N_\varepsilon\}$ be a family of disjoint open intervals such that

$$\int_{E \triangle H} c\,dt < \varepsilon/8 \quad \text{and} \quad \int_{E \triangle H} \psi(t)\,dt < \varepsilon/2,$$

where

$$H = \bigcup_{i=1}^{N_\varepsilon} (t_i, t_i') \cap [0, T],$$

and $c > 0$ is such that

$$|y_n(t)| \leqslant c \quad \text{for } t \in [0, T] \text{ and } n = 1, 2, \ldots$$

We have

$$\int_E y_n(s)\,ds = \int_{H \cup (E \setminus H)} y_n(s)\,ds - \int_{H \setminus E} y_n(s)\,ds$$
$$= \int_H y_n(s)\,ds + \int_{E \setminus H} y_n(s)\,ds - \int_{H \setminus E} y_n(s)\,ds.$$

Therefore,

$$\beta\Big(\Big\{\int_E y_n(s)\,ds:\ n \geqslant 1\Big\}\Big)$$
$$\leqslant \sum_{i=1}^{N_\varepsilon} \int_{(t_i, t_i') \cap [0,T]} \psi(s)\,ds + \beta\Big(\Big\{\int_{E \setminus H} y_n(s)\,ds:\ n \geqslant 1\Big\} + \Big(\Big\{\int_{H \setminus E} y_n(s)\,ds:\ n \geqslant 1\Big\}\Big)$$
$$\leqslant \int_H \psi(s)\,ds + 4 \int_{E \div H} c\,dt.$$

Furthermore, we have

$$\int_H \psi(s)\,ds \leqslant \int_E \psi(s)\,ds + \int_{E \div H} \psi(s)\,ds.$$

Then,

$$\beta\Big(\Big\{\int_E y_n(s)\,ds:\ n \geqslant 1\Big\}\Big) \leqslant \int_E \psi(s)\,ds + \varepsilon,$$

for each $\varepsilon > 0$. Therefore,

$$\beta\Big(\Big\{\int_E y_n(s)\,ds:\ n \geqslant 1\Big\}\Big) \leqslant \int_E \psi(s)\,ds.$$

Let $m$: $[0, T] \to R$ be a Lebesgue integrable function such that $|x_n(t)| \leqslant m(t)$, for almost every $t \in [0, T]$ and $n \geqslant 1$. Let us associate, to any $\varepsilon > 0$, a number $\delta_\varepsilon > 0$ such that, for each measurable set $G \subset [0, T]$ with $\mu(G) < \delta_\varepsilon$ we have

$$\int_G m(t)\,dt < \varepsilon/2.$$

Now, for each fixed $n \geqslant 1$ and $\varepsilon > 0$, let $E_\varepsilon^n$ be a closed subset of $[0, T]$, with

$$\mu([0, T] \setminus E_\varepsilon^n) < \delta_\varepsilon/2^n,$$

and such that the restrictions $x_n |\ E_\varepsilon^n$ and $m |\ E_\varepsilon^n$ of $x$ and $m$ are continuous.

It is easily seen that

$$E_\varepsilon = \bigcap_{n=1}^{\infty} E_\varepsilon^n \neq \emptyset \quad \text{for each } \varepsilon > 0.$$

Let

$$c_\varepsilon = \sup \{m(t)\colon t \in E_\varepsilon\},$$

and suppose that

$$|x_n(t)| \leqslant c_\varepsilon \quad \text{for } t \in E_\varepsilon \text{ and } n = 1, 2, \ldots$$

By virtue of the Tietze-Dugundji theorem ([3], [4]) for every $n \geqslant 1$ there exists a continuous extension $y_n^\varepsilon$ of $x_n|\, E_\varepsilon$ on $[0, T]$, such that

$$|y_n^\varepsilon(t)| \leqslant c_\varepsilon \quad \text{for } t \in [0, T].$$

Therefore,

$$\Phi_\varepsilon(t) = \beta(\{\int_{E_\varepsilon} y_n^\varepsilon(t)\, dt\colon n \geqslant 1\})$$

is measurable and satisfies

$$\beta(\{\int_{E_\varepsilon} y_n(t)\, dt\colon n \geqslant 1\}) \leqslant \int_{E_\varepsilon} \Phi_\varepsilon(t)\, dt$$

for each $\varepsilon > 0$. But, for $t \in E_\varepsilon$, we have

$$\Phi_\varepsilon(t) = \Phi(t).$$

Then $\Phi$ is measurable on $[0, T]$ and

$$\beta(\{\int_{E_\varepsilon} x_n(t)\, dt\colon n \geqslant 1\}) \leqslant \int_{E_\varepsilon} \Phi(t)\, dt.$$

Now, we have

$$\beta(\{\int_0^T x_n(t)\, dt\colon n \geqslant 1\}) \leqslant \int_{E_\varepsilon} \Phi(t)\, dt + \beta(\{\int_{G_\varepsilon} x_n(t)\, dt\colon n \geqslant 1\}),$$

where

$$G_\varepsilon = [0, T] \setminus E_\varepsilon.$$

It is not difficult to verify that

$$\operatorname{diam}(\{\int_{G_\varepsilon} x_n(t)\, dt\colon n \geqslant 1\}) \leqslant \varepsilon.$$

Therefore, for each $\varepsilon > 0$, we get

$$\beta(\{\int_0^T x_n(t)\, dt\colon n \geqslant 1\}) \leqslant \int_0^T \Phi(t)\, dt + \varepsilon,$$

which implies that

$$\beta(\{\int_0^T x_n(t)\, dt\colon n \geqslant 1\}) \leqslant \int_0^T \Phi(t)\, dt.$$

Hence, it follows, as in the first part of the proof, that

$$\beta\Big(\Big\{\int_E x_n(t)\,dt:\ n \geqslant 1\Big\}\Big) \leqslant \int_E \Phi(t)\,dt$$

for each measurable set $E \subset [0, T]$. The proof is complete.

We will need in this paper the following known result of the theory of differential inequalities ([9], Theorem III.16.2).

LEMMA 1.5. *Suppose* $\omega$: $[0, T]\times R \to R$ *satisfies the Carathéodory conditions and let* $y_0 \in R$. *Then there exists the right-hand maximum solution* $g$ *of*

$$\text{(1.2)}\qquad \begin{aligned} &\dot{y}(t) = \omega\big(t, y(t)\big) \quad \textit{for a.e. } t \in [0, T],\\ &y(0) = y_0. \end{aligned}$$

*Furthermore, for every absolutely continuous function* $\Phi$: $[0, T] \to R$ *satisfying* $\Phi(0) \leqslant y_0$ *and* $\dot{\Phi}(t) \leqslant \omega\big(t, \Phi(t)\big)$ *for a.e.* $t \in [0, T]$ *we have* $\Phi(t) \leqslant g(t)$ *for each* $t \in [0, T]$.

As a corollary of above lemma we get

LEMMA 1.6. *Suppose* $\omega$: $[0, T]\times R \to R$ *satisfies the Carathéodory conditions and* $\omega(t, x_1) \leqslant \omega(t, x_2)$ *for each* $x_1 \leqslant x_2$ *and almost all* $t \in [0, T]$ *and let* $g$ *be the right-hand maximum solution of* (1.2). *If* $\Phi$: $[0, T] \to R$ *is continuous and such that*

$$\Phi(t) \leqslant y_0 + \int_0^t \omega\big(s, \Phi(s)\big)\,ds \quad \textit{for } t \in [0, T],$$

*then*

$$\Phi(t) \leqslant g(t) \quad \textit{for each } t \in [0, T].$$

Proof. Let

$$h(t) = y_0 + \int_0^t \omega\big(s, \Phi(s)\big)\,ds \quad \text{for } t \in [0, T].$$

For $t \in [0, T]$ we have $\Phi(t) \leqslant h(t)$. Then $\omega\big(t, \Phi(t)\big) \leqslant \omega\big(t, h(t)\big)$ for a.e. $t \in [0, T]$. Therefore $\dot{h}(t) \leqslant \omega\big(t, h(t)\big)$ for a.e. $t \in [0, T]$. Furthermore, $h(0) = y_0$. Then, by Lemma 1.5, we get $\Phi(t) \leqslant h(t) \leqslant g(t)$ for each $t \in [0, T]$, which completes the proof.

Let $(X, |\cdot|)$ be separable Banach space and let us denote by $F$ the space of all functions $f$: $[0, T]\times X \to X$ satisfying the Carathéodory conditions. Let $G = \{f \in F:\ \int_0^T \sup[|f(t, x)|:\ x \in X]\,dt = 0\}$ and suppose $\mathscr{F} = F/G$ a quotient space defined by the equivalence relation $\sim$ defined by $f_1 \sim f_2$ iff $f_1 - f_2 \in G$. We will denote elements of $\mathscr{F}$ as the same as elements of $F$. Let $\varrho$ be a metric defined on $\mathscr{F}$ by

$$\varrho(f_1, f_2) = \int_0^T \sup\{|f_1(t, x) - f_2(t, x)|:\ x \in X\}\,dt.$$

**2.** Let us consider an initial-value problem

$$\text{(2.1)} \qquad \begin{aligned} &\dot{x}(t) = f(t, x(t)) \quad \text{for a.e. } t \in [0, T], \\ &x(0) = x_0, \end{aligned}$$

where $f \in \mathcal{F}$ and $x_0 \in X$ are given.

Let us denote by $\Omega$ the set of all functions $\omega$: $[0, T] \times R \to R$ satisfying the Carathéodory conditions and such that $\omega(t, \cdot)$ is nondecreasing for fixed $t \in [0, T]$ and with the property that an initial-value problem

$$\begin{aligned} &\dot{y}(t) = \omega(t, y(t)) \quad \text{for a.e. } t \in [0, T], \\ &y(t_0) = 0 \end{aligned}$$

has only the trivial solutions, for each fixed $t_0 \in [0, T]$.

We will prove now the following the existence theorem.

THEOREM 2.1. *Let $f \in \mathcal{F}$, $x_0 \in X$ and suppose for every $x \in X$ there exist a neighborhood $U_x$ of $x$ and a function $\omega_x \in \Omega$ such that $\beta(f(t, B)) \leqslant \omega_x(t, \beta(B))$ for each bounded $B \subset U_x$, $x \in X$ and a.e. $t \in [0, T]$. Then there exists at least one solution of* (2.1).

Proof. We shall show first that there exist $T_0 \in (0, T]$ and an absolutely continuous function $x^0$: $[0, T_0] \to X$ such that $\dot{x}^0(t) = f(t, x^0(t))$ for a.e. $t \in [0, T_0]$ and $x^0(0) = x_0$. Hence it will be follow the existence of solutions of (2.1) on the whole interval $[0, T]$.

Let $U_0$ and $\omega_0 \in \Omega$ be a neighborhood and a function defined, in virtue of our assumptions on $f$, corresponding to $x = x_0$.

Suppose for the simplicity that $U_0$ is an open ball of $X$ with the center $x_0$ and a radius $r_0 > 0$. Let $T_0 \leqslant T$ be such that

$$\int_0^{T_0} m_f(s)\,ds < r_0$$

and let us define a sequence $(x_n)$ of functions of $[0, T_0]$ into $X$ in the following way

$$\text{(2.2)} \qquad x_n(t) = x_0 + \int_0^{t_n} f(s, x_n(s))\,ds$$

for $t \in [0, T_0]$, where $t_n = \max(0, t - 1/n)$.

It is easy to see that (2.2) define each $x_n$ $(n = 1, 2, \ldots)$ on the whole interval $[0, T_0]$ and that $x_n(t) \in U_0$ for each $t \in [0, T_0]$. Let $A = \{x_n\colon n \geqslant 1\}$ and $A(t) = \{x_n(t)\colon n \geqslant 1\}$ and $\psi(t) = \beta(A(t))$. It is easy to see that $A$ is bounded and uniformly equicontinuous subset of the Banach space $C([0, T_0], X)$ containing all continuous functions of $[0, T_0]$ into $X$ with the supremum norm $\|\cdot\|$. Furthermore, by the definition of $x_n$ and a property of $m_f$, we can easy to see that $x_n$ is absolutely continuous for each $n = 1, 2, \ldots$

and satisfies

$$|\dot{x}_n(t)| = |f(t_n, x_n(t_n))| \leqslant m_f(t_n) \leqslant m_f(t) \quad \text{for a.e. } t \in [0, T_0].$$

Therefore, by virtue of Lemma 1.3, $\psi$ is an absolutely continuous function of $[0, T_0]$ into $R$. Furthermore, by the definition of $x_n$ we have

$$x_1(t) = x_0 + \int_{G_1^t} f(s, x_1(s))\,ds,$$

$$x_2(t) = x_0 + \int_{G_1^t} f(s, x_2(s))\,ds + \int_{G_2^t} f(s, x_2(s))\,ds,$$

$$x_n(t) = x_0 + \int_{G_1^t} f(s, x_n(s))\,ds + \sum_{k=2}^{n} \int_{G_k^t} f(s, x_n(s))\,ds,$$

where $G_1^t = E_1^t$, $G_n^t = E_{n+1}^t \setminus E_n^t$ for $n \geqslant 1$ and $E_n^t = [0, t_n]$ for $n = 1, 2, \ldots$ and fixed $t \in [0, T_0]$.

Let $\Gamma$ denote the family of all sequences $(n_k)$ of positive integers such that $n_k \geqslant k$ for $k = 1, 2, \ldots$ For each $(n_k) \in \Gamma$ we have

$$\sum_{k=1}^{\infty} \int_{G_k^t} f(s, x_{n_k}(s))\,ds \leqslant \int_{\bigcup_{k=1}^{\infty} G_k^t} m_f(s)\,ds \leqslant \int_0^T m_f(s)\,ds.$$

Therefore, for each $(n_k) \in \Gamma$, the series $\sum_{k=1}^{\infty} \int_{G_k^t} f(s, x_{n_k}(s))\,ds$ is converging. For fixed $p = 1, 2, \ldots$ and $t \in [0, T_0]$, let

$$H_p(t) = \Big\{\sum_{k=1}^{p} \int_{G_k^t} g(s, x_{n_k}(s))\,ds \colon (n_k) \in \Gamma\Big\},$$

$$K_p(t) = \Big\{\sum_{k=p+1}^{\infty} \int_{G_k^t} f(s, x_{n_k}(s))\,ds \colon (n_k) \in \Gamma\Big\}.$$

We have

$$A(t) = \{x_0\} + H_p(t) + K_p(t) \quad \text{and} \quad H_p(t) = \sum_{k=1}^{p} \Lambda_k(t),$$

where

$$\Lambda_k(t) = \Big\{0, \ldots, 0, \int_{G_k^t} f(s, x_n(s))\,ds \colon n \geqslant k\Big\} \quad \text{for } t \in [0, T_0],\ k \geqslant 1.$$

Furthermore, for each $t \in [0, T_0]$ and fixed $p = 1, 2, \ldots$ we have

$$\operatorname{diam}(K_p(t)) \leqslant 2 \sum_{k=p+1}^{\infty} \int_{G_k^t} m_f(s)\,ds.$$

Since $\sum_{k=1}^{\infty} \int_{G_k^t} m_f(s)\,ds$ is converging, then $\sum_{k=p+1}^{\infty} \int_{G_k^t} m_f(s)\,ds \to 0$ as $p \to \infty$. Therefore, for every $\varepsilon > 0$ there exists $p_\varepsilon \geqslant 1$ such that $\sum_{k=p+1}^{\infty} \int_{G_k^t} m_f(s)\,ds < \varepsilon/2$. Then diam $(K_{p_\varepsilon}(t)) < \varepsilon$ for each $t \in [0, T_0]$. On the other hand, for fixed $t \in [0, T_0]$ we have

$$\begin{aligned}
\beta(H_{p_\varepsilon}(t)) &\leqslant \sum_{k=1}^{p_\varepsilon} \beta(A_k(t)) \\
&\leqslant \sum_{k=1}^{p_\varepsilon} \beta\Big(\Big\{\int_{G_k^t} f(s, x_n(s))\,ds \colon n \geqslant k\Big\}\Big) \\
&\leqslant \int_{\bigcup_{k=1}^{p} G_k^t} \beta\big(\{f(s, x_n(s)) \colon n \geqslant k\}\big)\,ds \\
&\leqslant \int_0^t \beta\big(\{f(s, x_n(s)) \colon n \geqslant 1\}\big)\,ds \\
&\leqslant \int_0^t \omega_0\big(s, \beta(\{x_n(s) \colon n \geqslant 1\})\big)\,ds.
\end{aligned}$$

Therefore,

$$\beta(A(t)) \leqslant \int_0^t \omega_0\big(s, \beta(A(s))\big)\,ds + \varepsilon \qquad \text{for } t \in [0, T_0] \text{ and } \varepsilon > 0.$$

Hence it follows that

$$\psi(t) \leqslant \int_0^t \omega_0(s, \psi(s))\,ds \qquad \text{for } t \in [0, T_0].$$

Furthermore, $\psi(0) = 0$. Then, in virtue of Lemma 1.6, hence it follows that $\psi(t) \leqslant g(t)$ for $t \in [0, T_0]$, where $g$ is the right-hand maximum solution of $\dot{y}(t) = \omega_0(t, y(t))$ for a.e. $t \in [0, T_0]$ and $y(0) = 0$. By the properties of $\omega_0$, we have $g(t) = 0$ for $t \in [0, T_0]$. Therefore, $\psi(t) = 0$ for $t \in [0, T_0]$ and $\overline{A(t)}$ is compact for each $t \in [0, T_0]$. Hence it follows that $\bar{A}$ is compact; therefore there exists a subsequence, say $(x_k)$ of $(x_n)$ and $x^0 \in C([0, T_0], X)$ such that $\|x_k - x_0\| \to 0$ as $k \to \infty$. It is easy to verify that $x^0$ is a solution of (2.1) in $[0, T_0]$.

Now in a similar way as above we can show the existence of $T_1 \in (T_0, T]$ and absolutely continuous function $x^1\colon [T_0, T_1] \to X$ such that $\dot{x}^1(t) = f(t, x^1(t))$ for a.e. $t \in [T_0, T_1]$, $x^1(T_0) = x^0(T_0)$. Hence it follows at once that a function $z_1\colon [0, T_1] \to X$ of the form $z_1(t) = x^0(t)$ for $t = [0, T_0)$ and $z_1(t) = x^1(t)$ for $t \in [T_0, T_1]$ satisfies: $\dot{z}_1(t) = f(t, z_1(t))$ for a.e. $t \in [0, T_1]$ and $z_1(0) = x_0$. Continuing this process we can easy to show the existence of solution of (2.1) on the whole interval $[0, T]$. The proof is complete.

THEOREM 2.2. *Let $f \in \mathscr{F}$ and suppose $(f_n)$ is a sequence of $\mathscr{F}$ such that*
(i) $\lim_{n \to \infty} \varrho(f_n, f) = 0$,
(ii) *for each $x \in X$ there are a neighborhood $U_x$ of $x$ and $\omega_x \in \Omega$ such that $\beta(f_n(t, B)) \leqslant \omega_x(t, \beta(B^{1/n}))$ for each $n = 1, 2, \ldots$ a.e. $t \in [0, T]$ and bounded $B \subset U_x$; $x \in X$, where $B^{1/n} = \{x \in X\colon \operatorname{dist}(x, B) < 1/n\}$. Then for each $x_0 \in X$,* (2.1) *has at least one solution.*

Proof. By $\lim_{n \to \infty} \varrho(f_n, f) = 0$, there exists a subsequence, say again $(f_n)$ of $(f_n)$ such that $|f_n(t, x) - f(t, x)| \to 0$ as $n \to \infty$ for a.e. $t \in [0, T]$ and uniformly with respect to $x \in X$. Since $f(t, x) = [f(t, x) - f_n(t, x)] + f_n(t, x)$ for $t \in [0, T]$ and $x \in X$, then for each bounded set $B \subset U_x$ and each $n = 1, 2, \ldots$ we have $f(t, B) \subset H^t(B) + f_n(t, B)$, where $H_n^t(B) = \{[f(t, x) - f_n(t, x)]\colon x \in B\}$. Since $\sup\{|f(t, x) - f_n(t, x)|\colon x \in X\} \to 0$ as $n \to \infty$ for almost every fixed $t \in [0, T]$, then for each $n \geqslant 1$, there exists $N_n \geqslant 1$ such that $\operatorname{diam}(H_n^t(B)) \leqslant 1/n$ for $n \geqslant N_n$ and a.e. $t \in [0, T]$.

Therefore, for each bounded set $B \subset U_x$, $n \geqslant N_n$ and a.e. $t \in [0, T]$ we have $\beta(f(t, B)) \leqslant 1/n + \omega_x(t, \beta(B^{1/n}))$. Hence, by continuity of $\omega_x(t, \cdot)$, properties of the ball measure of noncompactness, we can easy to see that $\beta(f(t, B)) \leqslant \omega_x(t, \beta(B))$ for each $x \in X$, bounded $B \subset U_x$ and a.e. $t \in [0, T]$. Now the existence of solution of (2.1) follows immediately from Theorem 2.1. This completes the proof.

Using the classical method of successive approximations we can easy to prove the existence and uniqueness theorem, by the assumption that $f$ is locally Lipschitzean with respect to $x \in X$. Recall that $f \in \mathscr{F}$ is said to be *locally Lipschitzean* with respect to $x \in X$ if for every $x \in X$ there are an open set $U_x$ with $x \in U_x \subset X$ and a Lebesgue integrable function $k_x\colon [0, T] \to R$ such that $|f(t, x_1) - f(t, x_2)| \leqslant k_x(t)|x_1 - x_2|$ for each $x_1, x_2 \in U_x$ and almost every $t \in [0, T]$.

THEOREM 2.3. *Let $f \in \mathscr{F}$ be locally Lipschitzean with respect to $x \in X$. Then for every $x_0 \in X$ there exists exactly one solution of* (2.1).

Proof. There exists an open set $U_0 \subset X$ with $x_0 \in U_0$ and a Lebesgue integrable function $k_0$ of $[0, T]$ into $R$ such that $|f(t, x_1) - f(t, x_2)| \leqslant k_0(t)|x_1 - x_2|$ for a.e. $t \in [0, T]$ and $x_1, x_2 \in U$. Suppose, for the simplicity that $U_0$ is an open ball with the center $x_0$ and a redius $r_0 > 0$. Let $T_0 \leqslant T$ be such that $\int_0^{T_0} m_f(t)dt < r_0$ and let us define for each $t \in [0, T_0]$, the sequence $(x_n(t))$ of the form: $x_1(t) = x_0$ and $x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds$ for $n = 2, 3, \ldots$ It is easy to see that $x_n(t) \in U_0$ for each $t \in [0, T_0]$. Hence and the property of $f$ it follows that $(x_n)$ is uniformly converging on $[0, T_0]$ to the uniqueness solution $x^0$ of (2.1) on $[0, T_0]$.

Now, let $x_1 = x_0(T_0)$ and let $r_1 > 0$, an open set $U_1 \subset X$ containing $x_1$ and $k_1$: $[0, T] \to R$ be such that $\|f(t, y_1) - f(t, y_2)\| \leqslant k_1(t)|y_1 - y_2|$ for a.e. $t \in [0, T]$ and each $y_1, y_2 \in U_1$.

Similarly as above we can show the existence of the uniqueness solution $x^1$ of $\dot{x}(t) = f(t, x(t))$ for a.e. $t \in [T_0, T_1]$ and $x(T_0) = x_1$ on some interval $[T_0, T_1]$, where $T_1 \leqslant T$ is such that $\int_{T_0}^{T_1} m_f(t)dt < r_1$. It is easy to see that the function $z_1$ defined by $z_1(t) = x^0(t)$ for $t \in [0, T_0)$ and $z_1(t) = x^1(t)$ for $t \in [T_0, T_1]$ is a solution of (2.1) on $[0, T_1]$. Continuing this process we can show the existence of the unique solution of (2.1) on the whole interval $[0, T]$. This completes the proof.

THEOREM 2.4. *Let $f \in \mathscr{F}$ satisfy the assumption of Theorem* 2.1. *Suppose, for fixed $x_0 \in X$,* (2.1) *has exactly one solution $x(f)$. Then $x(f)$ continuously depends on $f \in \mathscr{F}$, i.e., for each sequence $(f_n)$ of $\mathscr{F}$ such that $\varrho(f_n, f) \to 0$ as $n \to \infty$, we have $\|x(f_n) - x(f)\| \to 0$ as $n \to \infty$, where $x(f_n)$ denotes the solution of* (2.1) *corresponding to $f_n$.*

Proof. Let $x = x(f)$ and $x_n = x(f_n)$. We have

$$x_n(t) = x_0 + \int_0^t f_n(s, x_n(s))ds \quad \text{for } t \in [0, T]$$

and

$$x(t) = x_0 + \int_0^t f(s, x(s))ds \quad \text{for } t \in [0, T].$$

Let us denote by $(x_k)$ an arbitrary subsequences of $(x_n)$. Since $f \in \mathscr{F}$ and $\varrho(f_k, f) \to 0$ as $k \to \infty$, then there exist a subsequence, say again $(f_k)$ of $(f_k)$ and Lebesgue integrable function $m$: $[0, T] \to R$ such that $\|f_k(t, x)\| \leqslant m(t)$ and $\|f(t, x)\| \leqslant m(t)$ for $x \in X$ and a.e. $t \in [0, T]$. Hence it follows that $(x_k)$ is bounded and uniformly equicontinuous sequence of $C([0, T], X)$. Let $U_0$, $\omega_0 \in \Omega$ and $T_0 \in (0, T]$ be such as in the proof of Theorem 2.1, corresponding to $x_0 \in X$ and $m$. In a similar way as in the proof of Theorem 2.2 we can prove that $\beta(\{\dot{x}_k(t): k \geqslant 1\}) \leqslant \omega_0(t, \beta(\{x_k(t): k \geqslant 1\}))$ for a.e. $t \in [0, T_0]$. In virtue of Lemma 1.3, the function $\psi(t) = \beta(\{x_k(t): k \geqslant 1\})$ is absolutely continuous in $[0, T_0]$ and satisfies

$$\dot{\psi}(t) \leqslant \beta(\{\dot{x}_k(t):\ k \geqslant 1\})$$

for a.e. $t \in [0, T_0]$. Furthermore, we have $\psi(0) = 0$. Therefore, for a.e. $t \in [0, T_0]$ we have $\dot{\psi}(t) \leqslant \omega_0(t, \psi(t))$. Hence and Lemma 1.5 it follows that $\psi(t) = 0$ for each $t \in [0, T_0]$. Similarly, now we can find $T_1 \in (T_0, T]$ and $\omega_1 \in \Omega$ such that $\dot{\psi}(t) \leqslant \omega_1(t, \psi(t))$ for a.e. $t \in [T_0, T_1]$. Since $\psi(T_0) = 0$, then hence it follows that $\psi(t) = 0$ for $t \in [T_0, T_1]$.

Continuing this process we can show that $\psi(t) = 0$ for $t \in [0, T]$. Therefore $A = \{x_k:\ k \geqslant 1\}$ is relatively compact in $C([0, T], X)$. Then, there exists a subsequence, say again $(x_k)$ of $(x_k)$ and $\bar{x} \in C([0, T], X)$ such that $\|x_k - \bar{x}\| \to 0$ as $k \to \infty$. Since

$$\Big|\bar{x}(t) - x_0 - \int_0^t f(s, \bar{x}(s))\,ds\Big| \leqslant |\bar{x}(t) - x_k(t)| + \int_0^t \big|f_k(s, x_k(s)) - f_k(s, \bar{x}(s))\big|\,ds + \int_0^t \big|f_k(s, \bar{x}(s)) - f(s, \bar{x}(s))\big|\,ds$$

for $t \in [0, T]$ and $k = 1, 2, \ldots$, then we can easy to see that $\bar{x} = x(f)$. By the unicity of solutions $x(f)$, hence it follows that every subsequence of $(x_n)$ has a subsequence converging to as the same limit $x(f)$. Therefore, $|x_n - x| \to 0$ as $n \to \infty$. This completes the proof.

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