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On interpolation of completely monotonic sequences

by F. H. SZAFRANIEC (Kraków)

Dedicated to the memory of Jacek Szarski

Abstract. We propose a characterization of these completely monotonic sequences which can be interpolated by completely monotonic functions.

1. Recall that a *sequence* $\{a_n\}_{n=0}^{\infty}$ of real numbers is said to be *completely monotonic* if

$$(1)\qquad (-1)^k \Delta^k a_n \geqslant 0, \quad n, k = 0, 1, 2, \ldots$$

Here $\Delta^k a_n = \Delta^1(\Delta^{k-1} a_n)$, $\Delta^1 a_n = a_n - a_{n+1}$, $\Delta^0 a_n = a_n$. A *completely monotonic function* is a function f which is continuous on $[0, +\infty)$ and such that

$$(-1)^k f^{(k)}(x) \geqslant 0, \quad k = 0, 1, 2, \ldots, \ 0 < x < +\infty.$$

The *interpolation* problem in question is as follows: given $\{a_n\}_{n=0}^{\infty}$ which is completely monotonic, to find a completely monotonic function f such that

$$f(n) = a_n, \quad n = 0, 1, 2, \ldots$$

The answer, given in [4], p. 163, says that a sequence $\{a_n\}_{n=0}^{\infty}$ can be interpolated if and only if it ceases to be completely monotonic where a_0 is decreased. In the present note we wish to *reformulate* this in terms of *positive definiteness*.

It is known that a sequence $\{a_n\}_{n=0}^{\infty}$ is completely monotonic if and only if

$$(2)\qquad 0 \leqslant \sum_{m,n=0}^{p} a_{m+n+1} c_m \bar{c}_n \leqslant \sum_{m,n=0}^{p} a_{m+n} c_m \bar{c}_n,$$

where $c_0, c_1, c_2, \ldots, c_p$ is an arbitrary finite sequence of numbers (no matter whether c's are allowed to be real or complex). The proof of this equivalence is via the integral representation

$$(3)\qquad a_n = \int_0^1 t^n \mu(dt), \quad n = 0, 1, 2, \ldots$$

(Let us mention here that, as we know, there is still no direct, without any appeal to (3), proof of (1) $\Leftrightarrow$ (2).)

2. We look separately over each of three inequalities in (2). In the inequality

$$0 \leqslant \sum_{m,n=0}^{p} a_{m+n+1} c_m \bar{c}_n, \tag{4}$$

a_0 does not appear. The inequality

$$\sum_{m,n=0}^{p} a_{m+n+1} c_m \bar{c}_n \leqslant \sum_{m,n=0}^{p} a_{m+n} c_m \bar{c}_n$$

is equivalent to

$$\sup_n |a_n| < +\infty. \tag{5}$$

This fact can be proved in a simple way ([2], and also [3]) using an appropriate Schwarz inequality for the sequence $\{a_n\}$. The same Schwarz inequality is involved in the proof [1] that the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies

$$0 \leqslant \sum_{m,n=0}^{p} a_{m+n} c_m \bar{c}_n$$

if and only if

$$\Big|\sum_{m=1}^{p} a_m c_m\Big|^2 \leqslant a_0 \sum_{m,n=1}^{p} a_{m+n} c_m \bar{c}_n. \tag{6}$$

Thus (2) is equivalent to (4), (5) and (6). Since no change of a_0 destroyes (5), the only condition where a_0 is essentially involved is just (6). In other words, a_0 demolishes complete monotonicity of $\{a_n\}$ if and only if so does it with (6). All this enables us to state the following

THEOREM. *A completely monotonic sequence $\{a_n\}_{n=0}^{\infty}$ can be interpolated by a completely monotonic function if and only if*

$$a_0 = \sup \frac{\Big|\sum_{m=1}^{p} a_m c_m\Big|^2}{\sum_{m,n=1}^{p} a_{m+n} c_m \bar{c}_n},$$

where the supremum is taken over all finite sequences $c_1, \ldots, c_p$ of either real or complex numbers.

Note added in proof (May, 1983). The question mentioned on the very top of this page has been recently answered in the paper *Equivalent definitions of positive definiteness* (to appear in Pac. J. Math.) by P. H. Maserick and the present author.

References

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UNIWERSYTET JAGIELLOŃSKI, INSTYTUT MATEMATYKI, KRAKÓW

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