On solutions of Fourier's first problem for a system of non-linear parabolic equations in an unbounded domain

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Introduction. In the papers [5]-[9] M. Krzyżański has proved, among other theorems, some theorems on the existence and uniqueness of the solution of Fourier's first problem (as well as of the Cauchy problem) in an unbounded domain and in the class of functions E_2 for the linear equation of parabolic type of the normal form (for the definition of class E_δ, see § 1 of this paper)

$$(0.1) \quad \sum_{i,j=1}^{m} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(x, t) \frac{\partial u}{\partial x_j} + c(x, t) u - \frac{\partial u}{\partial t} = f(x, t) .$$

For this purpose M. Krzyżański has constructed certain auxiliary functions which permit to prove of the above-mentioned theorems for equation (0.1) with unbounded (continuous) coefficients fulfilling the inequalities

$$(0.2) \quad |a_{ij}(x, t)| \leqslant A , \quad |b_j(x, t)| \leqslant A_1 |x| + B_1 , \quad c(x, t) \leqslant A_2 |x|^2 + B_2 ,$$

A, A_1, A_2, B_1, B_2 being positive constants, $|x| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$.

In this paper we consider the problem posed by Professor Krzyżański. We deal with a system of non-linear equations with partial derivatives of the parabolic type of the form

$$(0.3) \quad \frac{\partial u_s}{\partial t} = F_s\left(x_1, ..., x_m, t, u_1, ..., u_n, \frac{\partial u_s}{\partial x_1}, ..., \frac{\partial u_s}{\partial x_m}, \frac{\partial^2 u_s}{\partial x_1^2}, \frac{\partial^2 u_s}{\partial x_1 \partial x_2}, ..., \frac{\partial^2 u_s}{\partial x_m^2}\right)$$

$$(s = 1, ..., n) ,$$

where in the equation with the index s the derivatives of functions $u_1, ..., u_{s-1}, u_{s+1}, ..., u_n$ do not appear.

Theorems I-III dealt with in this paper concern the solutions of Fourier's first boundary-value problem in an unbounded domain (they hold for the Cauchy problem as well). And thus in § 3 we prove the

theorem on the uniqueness of solutions of this problem in class E_2. Theorem II in § 4 concerns certain inequalities between the solutions of two systems of equations of (0.3) form in an unbounded domain and in class E_2. Theorem III (§ 5) speaks about the existence of a solution of Fourier's first problem in an unbounded domain if we assume the existence of a solution in a bounded domain. We assume the functions appearing in initial and boundary conditions to belong to class E_2, and we prove that in that case the solution also belongs to class E_2.

In all the theorems we assume that the functions F_s appearing on the right side of the system (0.3) fulfil a certain condition weaker than that of Lipschitz (the so-called ($\mathfrak{L}$)-condition; see § 2). In the statement on inequalities we assume additionaly that the F_s functions are not decreasing with regard to some variables.

The method of proving these statements is based on the method applied by M. Krzyżański. Using the above-mentioned auxiliary functions constructed by M. Krzyżański we show that the theorems mentioned previously remain true for the (0.3) system embracing equation (0.1) with unbounded coefficients fulfilling the (0.2)-conditions.

Fourier's problems for the system (0.3) have been treated by J. Szarski [17], [18] under weaker assumptions concerning the functions F_s, but nevertheless in domains whose intersections with hyperplanes $t =$ const are bounded.

A theorem similar to II has been established by W. Mlak [13] in a bounded domain.

The Cauchy problem for parabolic non-linear (quasi-linear) equations has been considered in papers [2], [19].

S. D. Eïdelman [2] has proved the correctness of this problem (in a sufficiently narrow strip) in class E_2 for a parabolic system of quasi-linear equations and under stronger assumptions than those in this paper, concerning, among other things, the character of increase of the non-linear term together with an increase of $|x|$ and $u_1, ..., u_n$.

In his paper [19] T. D. Ventcel has proved the existence and uniqueness of the solution of the Cauchy problem for a parabolic quasi-linear equation under strong assumptions concerning the coefficients and functions appearing in the initial condition.

I wish, here, to express my sincere thanks to Professor M. Krzyżański for posing the problem and for his valuable advice while I was working on that subject. I should also like to thank Professor J. Szarski for his helpful remarks.

§ 1. Notation and definitions. Let us denote by $x(x_1, x_2, ..., x_m)$ a point of the m-dimensional Euclidean space $\mathcal{E}^m$, and by $(x_1, x_2, ..., x_m, t)$, or shortly (x, t)—a point of the space $\mathcal{E}^{m+1}$.

Let D denote an open unbounded domain of $\mathcal{E}^{m+1}$, lying between the hyperplanes $t = 0$ and $t = T > 0$, whose boundary FD consists of m-dimensional unbounded domains S^0 and S^T of hyperplanes $t = 0$, $t = T$ and of surface σ which is not tangent to any hyperplane $t = $ const.

System (0.3) will be written shortly as

$$(1.1) \qquad \frac{\partial u_s}{\partial t} = F_s\left(x, t, u_i, \frac{\partial u_s}{\partial x_j}, \frac{\partial^2 u_s}{\partial x_j \partial x_k}\right) \qquad (s, i = 1, ..., n; \ j, k = 1, ..., m).$$

We assume that the functions $F_s(x, t, y_i, z_j, z_{jk})$ $(s = 1, ..., n)$ are defined in the domain Π defined as follows: $(x, t) \in D$ and y_i, z_j, z_{jk} arbitrary.

Let

$$(1.2) \qquad w_1(x, t), \ w_2(x, t), \ ..., \ w_n(x, t)$$

be a sequence of functions of class C^1 in D. We will make use of the following definition, due to J. Szarski.

DEFINITION I. An equation with the index s of (1.1) is called *parabolic with respect to sequence* (1.2) if for every system of numbers $z_{jk}, \bar{z}_{jk}$ $(j, k = 1, ..., m)$, $z_{jk} = z_{kj}$, $\bar{z}_{jk} = \bar{z}_{kj}$, such that the quadratic form

$$\sum_{j,k=1}^{m} (z_{jk} - \bar{z}_{jk}) \lambda_j \lambda_k$$

is non-positive for arbitrary vector $(\lambda_1, ..., \lambda_m)$, the following inequality holds:

$$F_s\left(x, t, w_i(x, t), \frac{\partial w_s(x, t)}{\partial x_j}, z_{jk}\right) - F_s\left(x, t, w_i(x, t), \frac{\partial w_s(x, t)}{\partial x_j}, \bar{z}_{jk}\right) \leqslant 0$$

$$\text{for} \quad (x, t) \in D.$$

If each equation of (1.1) is parabolic with respect to the same sequence, then we say that system (1.1) is *parabolic with respect to that sequence*. The solution of system (1.1) is called parabolic if this system is parabolic with respect to the solution.

We say that a solution $u_i(x, t)$ $(i = 1, ..., n)$ is *regular* in D if the functions $u_i(x, t)$ are continuous in the closure $\bar{D}$ of D and possess continuous derivatives appearing in the equations of (1.1) in the interior of D.

DEFINITION II. We will say that the function $F_s(x, t, y_i, z_j, z_{jk})$ satisfies the $(\mathfrak{L})$-*condition* if there exist positive constants $L_0, ..., L_4$ such that, for arbitrary $y_i, z_j, z_{jk}, \bar{y}_i, \bar{z}_j, \bar{z}_{jk}$ $(i = 1, ..., n; \ j, k = 1, ..., m)$, $y_s \geqslant \bar{y}_s$, we have the inequality

$$F_s(x, t, y_i, z_j, z_{jk}) - F_s(x, t, \bar{y}_i, \bar{z}_j, \bar{z}_{jk})$$

$$\leqslant L_0 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| + (L_1 |x| + L_2) \sum_{j=1}^{m} |z_j - \bar{z}_j| + (L_3 |x|^2 + L_4) \sum_{i=1}^{n} |y_i - \bar{y}_i|,$$

where $|x| = (\sum\limits_{i=1}^{m} x_i^2)^{1/2}$. It means that the function F_s satisfies the Lipschitz condition with respect to the variables $y_1, ..., y_{s-1}, y_{s+1}, ..., y_n, z_j, z_{jk}$ $(j, k = 1, ..., m)$ (at the same time some of the Lipschitz constants depend on x) and that the difference quotient of F_s for the variable y_s is bounded only from above by $L_3|x|^2 + L_4$.

By $E_\delta(M, K)$ or shortly E_δ we will denote the class of functions $\psi(x, t)$ for which positive constants M, K exist such that the inequality

$$|\psi(x, t)| \leqslant M\exp(K|x|^\delta)$$

is satisfied in D, and by $Z_\delta(A, B)$ or Z_δ, $\delta > 0$, we will denote the class of functions $\chi(x, t)$ for which positive constants A, B exist such that

$$|\chi(x, t)| \leqslant A|x|^\delta + B \quad \text{for} \quad (x, t) \,\epsilon\, D\,.$$

§ 2. Formulation of the problem.

Let $\varphi_i(x, t)$ $(i = 1, ..., n)$ be arbitrary given functions, defined and continuous in the set $\Sigma = S^0 + \sigma$.

In this paper we discuss the first boundary-value problem of Fourier, which we formulate as follows: to find a parabolic solution $u_i(x, t)$ $(i = 1, ..., n)$ of system (1.1), regular in D and such that

$$u_i(x, t) = \varphi_i(x, t) \quad (i = 1, ..., n) \quad \text{for} \quad (x, t) \,\epsilon\, \Sigma\,.$$

In what follows we will call this problem the (FI)-problem.

§ 3. The uniqueness of solution of the (FI)-problem.

THEOREM I. *If each function* $F_s(x, t, y_i, z_j, z_{jk})$ $(s = 1, ..., n)$ *fulfils the* (Ω)-*condition in* Π, *then the* (FI)-*problem possesses no more than one solution* $u_1(x, t), ..., u_n(x, t)$ *of class* E_2 [1] *in the domain* D.

Proof. Suppose there exist two such solutions, $u_i^{(1)}$ and $u_i^{(2)}$ $(i = 1, ..., n)$. Then the differences $u_i = u_i^{(1)} - u_i^{(2)}$ satisfy the system of equations

$$(3.1) \quad \frac{\partial u_s}{\partial t} = F_s\left(x, t, u_i^{(1)}, \frac{\partial u_s^{(1)}}{\partial x_j}, \frac{\partial^2 u_s^{(1)}}{\partial x_j \partial x_k}\right) - F_s\left(x, t, u_i^{(2)}, \frac{\partial u_s^{(2)}}{\partial x_j}, \frac{\partial^2 u_s^{(2)}}{\partial x_j \partial x_k}\right)$$
$$(s = 1, ..., n)\,.$$

We may assume that all the functions u_i $(i = 1, ..., n)$ belong to $E_2(M, K_0)$ with the same constants M and K_0.

Let D_R be an open bounded set separated from D by the cylindrical surface Γ_R with the equation $\sum\limits_{i=1}^{m} x_i^2 = R^2$.

We will denote by S_R^0, S_R^T and Σ_R the parts of the surfaces S^0, S^T and Σ, respectively, lying inside and on Γ_R, whereas by C_R we will de-

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[1] I.e. each function $u_i(x, t)$ $(i = 1, ..., n)$ belongs to E_2.

note the part of Γ_R contained in D. Further, let D^h, D_R^h, Σ^h, Σ_R^h, C_R^h denote the parts of the sets D, D_R, Σ, Σ_R, C_R, respectively, contained in the strip $0 \leqslant t < h$, $h \leqslant T$.

We first prove our theorem for a domain D^h, where h will be a conveniently chosen (sufficiently small) constant.

Now we introduce the functions $v_i^{(1)}$, $v_i^{(2)}$ according to the relations

$$u_i^{(1)} = v_i^{(1)} H(x, t; K), \qquad u_i^{(1)} = v_i^{(2)} H(x, t; K) \qquad (i = 1, \ldots, n),$$

where

$$(3.2) \qquad\qquad H(x, t; K) = \exp\left\{\frac{K|x|^2}{1-\mu t} + \nu t\right\}$$

is the auxiliary function constructed by M. Krzyżański in order to prove similar theorems for the linear parabolic equation (see, for instance, [6]). Here $\mu(K)$ and $\nu(K)$ are positive constants which will be conveniently chosen for $K > K_0$.

Writing $v_i = v_i^{(1)} - v_i^{(2)}$ we have $u_i = v_i H(x, t; K)$ in D^h where $h < 1/\mu(K)$. Introducing the functions v_i, $v_i^{(1)}$, $v_i^{(2)}$ into (3.1), subtracting and adding convenient expressions we get the system

$$(3.3) \qquad \frac{\partial v_s}{\partial t} H + v_s \frac{\partial H}{\partial t} =$$

$$F_s\left(x, t, v_i^{(1)} H, \frac{\partial v_s^{(1)}}{\partial x_j} H + v_s^{(1)} \frac{\partial H}{\partial x_j}, \frac{\partial^2 v_s^{(1)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(1)}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial v_s^{(1)}}{\partial x_j} \frac{\partial H}{\partial x_k} + v_s^{(1)} \frac{\partial^2 H}{\partial x_j \partial x_k}\right) -$$

$$- F_s\left(x, t, v_i^{(1)} H, \frac{\partial v_s^{(1)}}{\partial x_j} H + v_s^{(1)} \frac{\partial H}{\partial x_j}, \frac{\partial^2 v_s^{(2)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(1)}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial v_s^{(1)}}{\partial x_j} \frac{\partial H}{\partial x_k} + v_s^{(1)} \frac{\partial^2 H}{\partial x_j \partial x_k}\right) +$$

$$+ F_s\left(x, t, v_i^{(1)} H, \frac{\partial v_s^{(1)}}{\partial x_j} H + v_s^{(1)} \frac{\partial H}{\partial x_j}, \frac{\partial^2 v_s^{(2)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(1)}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial v_s^{(1)}}{\partial x_j} \frac{\partial H}{\partial x_k} + v_s^{(1)} \frac{\partial^2 H}{\partial x_j \partial x_k}\right) -$$

$$- F_s\left(x, t, v_i^{(2)} H, \frac{\partial v_s^{(2)}}{\partial x_j} H + v_s^{(2)} \frac{\partial H}{\partial x_j}, \frac{\partial^2 v_s^{(2)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(2)}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial v_s^{(2)}}{\partial x_j} \frac{\partial H}{\partial x_k} + v_s^{(2)} \frac{\partial^2 H}{\partial x_j \partial x_k}\right)$$

$$(s = 1, \ldots, n).$$

Let $\{R_a\}$ be an increasing sequence, $R_a \to \infty$ as $a \to \infty$. Let us consider (for a fixed a) the domain $D_{R_a}^h$. We write

$$A_a = \max_{(i)} \sup_{(x,t)\in \overline{D_{R_a}^h}} |v_i(x, t)|.$$

There exist an index i_a and a point $(x_a, t_a) \in \overline{D_{R_a}^h} = (D_{R_a}^h + S_{R_a}^h) + \Sigma_{R_a}^h + C_{R_a}^h$ such that $A_a = |v_{i_a}(x_a, t_a)|$. We will show that, the constants $\mu(K)$, $\nu(K)$ and $h(K)$ being conveniently chosen, the relation $(x_a, t_a) \in D_{R_a}^h + S_{R_a}^h$

implies $v_{i_a}(x_a, t_a) = 0$. Indeed, suppose the contrary; there are two cases to be considered:

   a) $v_{i_a}(x_a, t_a) > 0$. Then at (x_a, t_a) we have

$$\frac{\partial v_{i_a}}{\partial t} \geqslant 0, \quad \frac{\partial v_{i_a}}{\partial x_j} = 0 \quad (j = 1, \ldots, m) \quad \text{and} \quad \sum_{j,k=1}^{m} \frac{\partial^2 v_{i_a}}{\partial x_j \partial x_k} \lambda_j \lambda_k \leqslant 0 .$$

Substituting $x = x_a$, $t = t_a$ in the equation with the index i_a of (3.3) and taking into account the definition of parabolic equation (see § 1), we find that the difference of the first two terms on the right side of this equation is non-positive. Now we estimate the difference of the remaining two terms. According to the (Ω)-condition assumed we have

$$\frac{\partial v_{i_a}}{\partial t} H + v_{i_a} \frac{\partial H}{\partial t} \leqslant v_{i_a} L_0 \sum_{j,k=1}^{m} \left| \frac{\partial^2 H}{\partial x_j \partial x_k} \right| +$$

$$+ v_{i_a}(L_1|x| + L_2) \sum_{j=1}^{m} \left| \frac{\partial H}{\partial x_j} \right| + (L_3|x|^2 + L_4) \sum_{j=1}^{n} |v_i| H .$$

Since $\sum_{j=1}^{n} |v_i(x_a, t_a)| \leqslant n \cdot v_{i_a}(x_a, t_a)$, we have

$$(3.4) \quad \frac{\partial v_{i_a}}{\partial t} H \leqslant v_{i_a} \left[L_0 \sum_{j,k=1}^{m} \left| \frac{\partial^2 H}{\partial x_j \partial x_k} \right| + \right.$$

$$\left. + (L_1|x| + L_2) \sum_{j=1}^{m} \left| \frac{\partial H}{\partial x_j} \right| + (L_3|x|^2 + L_4) n H - \frac{\partial H}{\partial t} \right] \overset{\text{df}}{=} v_{i_a} F H .$$

We will show, as in paper [6], that the constants $\mu(K)$, $\nu(K)$ and $h(K)$ can be chosen (independently of a) in such a way that $FH \leqslant -NH$, N being an arbitrary positive number. Namely, we have

$$FH \leqslant H \left[\frac{4K^2 L_0}{(1 - \mu t)^2} \sum_{j,k=1}^{m} |x_j x_k| + \frac{2KL_0 m}{1 - \mu t} + \frac{2K}{1 - \mu t} (L_1|x| + L_2) \sum_{j=1}^{m} |x_j| + \right.$$

$$\left. + (L_3|x|^2 + L_4) n - \frac{\mu K |x|^2}{(1 - \mu t)^2} - \nu \right] .$$

Writing $L = \max(L_1, L_2)$ and taking into consideration the inequalities $|x_j| \leqslant |x|$, $|x| \leqslant |x|^2 + 1$, $1 - \mu t \leqslant 1$, we have

$$FH \leqslant \frac{H}{(1 - \mu t)^2} [4K^2 L_0 m^2 |x|^2 + 2KL_0 m + 4KLm|x|^2 +$$

$$+ 2KLm + nL_3|x|^2 + nL_4 - \mu K |x|^2 - \nu (1 - \mu t)^2] .$$

Choosing

$$\mu(K) = 4KL_0 m^2 + 4Lm + \frac{nL_3 + \lambda}{K} \quad (\lambda\text{-arbitrary positive constant}),$$

(3.5)

$$h(K) = \frac{1 - \gamma}{\mu(K)}, \quad 0 < \gamma < 1,$$

we obtain $\gamma \leqslant 1 - \mu t \leqslant 1$ in D^h, and hence

$$FH \leqslant \frac{H}{(1 - \mu t)^2}[-\lambda|x|^2 + 2KL_0 m + 2KLm + nL_4 - \nu\gamma^2].$$

Putting

$$(3.6) \quad \nu(K) = \frac{2KL_0 m + 2KLm + nL_4 + N}{\gamma^2} \quad (N\text{-arbitrary positive constant})$$

we have

$$FH \leqslant -\lambda H|x|^2 - NH \leqslant -NH < 0.$$

This is in contradiction with $\dfrac{\partial v_{t_a}(x_a, t_a)}{\partial t} \geqslant 0$ and (3.4).

b) $v_{t_a}(x_a, t_a) < 0$. Repeating the above reasoning for the function $\bar{v}_{t_a}(x, t) = -v_{t_a}(x, t)$, we also obtain a contradiction. Now, if $(x_a, t_a) \in \Sigma_{R_a}^h$, we have (by our assumption) $v_{t_a}(x_a, t_a) = 0$, and finally if $(x_a, t_a) \in C_{R_a}^h$, then

$$|v_{t_a}(x_a, t_a)| \leqslant \frac{M\exp(K_0|x_a|^2)}{\exp\left\{\dfrac{K|x_a|^2}{1 - \mu t_a} + \nu t_a\right\}} = \frac{M\exp(K_0 R_a^2)}{\exp\left\{\dfrac{KR_a^2}{1 - \mu t_a} + \nu t_a\right\}}.$$

Thus, since $|v_i(x, t)| \leqslant |v_{t_a}(x_a, t_a)|$ in $\overline{D_{R_a}^h}$, we have proved that, the constants $\mu(K), \nu(K)$ and $h(K)$ being chosen according to (3.5) and (3.6), we have in $\overline{D_{R_a}^h}$ the inequalities

$$|v_i(x, t)| \leqslant \frac{M\exp(K_0 R_a^2)}{\exp\left\{\dfrac{KR_a^2}{1 - \mu t_a} + \nu t_a\right\}} \quad (i = 1, \ldots, n).$$

Now let us take an arbitrary point $(\bar{x}, \bar{t}) \in D^h$. For an arbitrary number $\varepsilon > 0$ there exists an index $a_0(\varepsilon)$ such that for $a > a_0$ the point $(\bar{x}, \bar{t}) \in \overline{D_{R_a}^h}$ and

$$\frac{M\exp(K_0 R_a^2)}{\exp\left\{\dfrac{KR_a^2}{1 - \mu t_a} + \nu t_a\right\}} < \varepsilon.$$

Consequently, $v_i(\bar{x}, \tilde{t}) = 0$, whence $u_i(\bar{x}, \tilde{t}) = 0$ and thus $u_i(x, t) \equiv 0$ $(i = 1, ..., n)$ in the whole domain $\overline{D^h}$.

Similarly, the theorem can be proved for the parts of the domain D contained in the strips: $jh \leqslant t \leqslant (j+1)h$ $(j = 1, 2, ...)$. For this purpose it is sufficient to put $t = \tilde{t} + j \cdot h$ $(j = 1, 2, ...)$ (see, for instance, [9]).

§ 4. Centrain inequalities between solutions of the (FI)-problem for two systems of equations.
Under a supplementary assumption we will prove a stronger theorem than theorem I.

Let us take a system of functions $\psi_1(y_1, ..., y_n, \tau), ..., \psi_n(y_1, ..., y_n, \tau)$, where τ denotes a sequence of variables different from $y_1, ..., y_n$. The following definition is introduced: the function $\psi_s(y_1, ..., y_n, \tau)$ satisfies the (W)-condition with regard to variables $y_1, ..., y_n$ if for $y_i \leqslant \bar{y}_1$, $i \neq s$, $y_s = \bar{y}_s$ we have the inequality

$$\psi_s(y_1, ..., y_n, \tau) \leqslant \psi_s(\bar{y}_1, ..., \bar{y}_n, \tau) .$$

THEOREM II. $1°$ *Let* $u_i^{(1)}(x, t)$, $u_i^{(2)}(x, t)$ $(i = 1, ..., n)$ *be regular solutions, belonging to class* E_2, *of the systems of equations*

$$(4.1) \qquad \frac{\partial u_s^{(1)}}{\partial t} = F_s^{(1)}\left(x, t, u_i^{(1)}, \frac{\partial u_s^{(1)}}{\partial x_j}, \frac{\partial^2 u_s^{(1)}}{\partial x_j \partial x_k}\right) \qquad (s = 1, ..., n) ,$$

$$(4.2) \qquad \frac{\partial u_s^{(2)}}{\partial t} = F_s^{(2)}\left(x, t, u_i^{(2)}, \frac{\partial u_s^{(2)}}{\partial x_j}, \frac{\partial^2 u_s^{(2)}}{\partial x_j \partial x_k}\right) \qquad (s = 1, ..., n) .$$

$2°$ *For each* s $(s = 1, ..., n)$ *the equation with the index* s *of system* (4.1) *or* (4.2) *is parabolic (see* § 1) *with regard to the solution* $u_s^{(1)}(x, t)$ *and* $u_s^{(2)}(x, t)$, *respectively.*

$3°$ *For each* s $(s = 1, ..., n)$ *the function* $F_s^{(1)}(x, t, y_i, z_j, z_{jk})$ *or* $F_s^{(2)}(x, t, y_i, z_j, z_{jk})$ *satisfies the* (W)-*condition with respect to the variables* $y_1, ..., y_n$ *and the* (Ω)-*condition (independently of the respective equation of the system* (4.1) *or* (4.2) *being parabolic).*

$4°$ $F_s^{(1)}(x, t, y_i, z_j, z_{jk}) \leqslant F_s^{(2)}(x, t, y_i, z_j, z_{jk})$ $(s = 1, ..., n)$ *for* $(x, t, y_i, z_j, z_{jk}) \in \Pi$.

$5°$ $u_i^{(1)}(x, t) \leqslant u_i^{(2)}(x, t)$ $(i = 1, ..., n)$ *for* $(x, t) \in \Sigma$.

Under the assumptions $1°$-$5°$ *the inequalities*

$$u_i^{(1)}(x, t) \leqslant u_i^{(2)}(x, t) \qquad (i = 1, ..., n)$$

hold for $(x, t) \in \overline{D}$.

Proof. For simplification we may assume that all the equations of system (4.2) are parabolic with respect to the solution $u_i^{(2)}(x, t)$ and that all the functions $F_s^{(2)}$ $(s = 1, ..., n)$ satisfy the (Ω)- and (W)-conditions. In the remaining cases the proof is similar.

The differences $u_i = u_i^{(1)} - u_i^{(2)}$ belong to the class $E_2(M, K_0)$. Let us introduce the functions $v_i^{(1)}, v_i^{(2)}$ according to the relations $u_i^{(1)} = v_i^{(1)}H$, $u_i^{(2)} = v_i^{(2)}H$ $(i = 1, ..., n)$, where $H = H(x, t; K)$, $K > K_0$, is defined by (3.2). Writing $v_i = v_i^{(1)} - v_i^{(2)}$, we obtain $u_i = v_i H$.

Let us put

$$\tau_j^{(1)} = \frac{\partial v_s^{(1)}}{\partial x_j} H + v_s^{(1)} \frac{\partial H}{\partial x_j}, \qquad \tau_j^{(2)} = \frac{\partial v_s^{(2)}}{\partial x_j} H + v_s^{(2)} \frac{\partial H}{\partial x_j},$$

$$\tau_{jk}^{(1,1)} = \frac{\partial^2 v_s^{(1)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(1)}}{\partial x_j} \frac{\partial H}{\partial x_k} + \frac{\partial v_s^{(1)}}{\partial x_k} \frac{\partial H}{\partial x_j} + v_s^{(1)} \frac{\partial^2 H}{\partial x_j \partial x_k},$$

(4.3)

$$\tau_{jk}^{(1,2)} = \frac{\partial^2 v_s^{(1)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(2)}}{\partial x_j} \frac{\partial H}{\partial x_k} + \frac{\partial v_s^{(2)}}{\partial x_k} \frac{\partial H}{\partial x_j} + v_s^{(2)} \frac{\partial^2 H}{\partial x_j \partial x_k},$$

$$\tau_{jk}^{(2,2)} = \frac{\partial^2 v_s^{(2)}}{\partial x_j \partial x_k} H + \frac{\partial v_s^{(2)}}{\partial x_j} \frac{\partial H}{\partial x_k} + \frac{\partial v_s^{(2)}}{\partial x_k} \frac{\partial H}{\partial x_j} + v_s^{(2)} \frac{\partial^2 H}{\partial x_j \partial x_k}.$$

Subtracting the adequate equations of (4.2) from (4.1), introducing the functions $v_i^{(1)}, v_i^{(2)}, v_i$, and subtracting and adding convenient expressions, we receive:

$$(4.4) \quad \frac{\partial v_s}{\partial t} H + v_s \frac{\partial H}{\partial t} = F_s^{(1)}(x, t, v_i^{(1)}H, \tau_j^{(1)}, \tau_{jk}^{(1,1)}) - F_s^{(2)}(x, t, v_i^{(1)}H, \tau_j^{(1)}, \tau_{jk}^{(1,1)}) +$$

$$+ F_s^{(2)}(x, t, v_i^{(2)}H, \tau_j^{(2)}, \tau_{jk}^{(1,2)}) - F_s^{(2)}(x, t, v_i^{(2)}H, \tau_j^{(2)}, \tau_{jk}^{(2,2)}) +$$

$$+ F_s^{(2)}(x, t, v_i^{(1)}H, \tau_j^{(1)}, \tau_{jk}^{(1,1)}) - F_s^{(2)}(x, t, v_i^{(2)}H, \tau_j^{(2)}, \tau_{jk}^{(1,2)})$$

$$(s = 1, ..., n) .$$

Let us take an increasing sequence $\{R_a\}$, $R_a \to \infty$ as $a \to \infty$. We retain the meaning of the symbols $D^h, D_R^h, ..., C_R^h$ introduced in the proof of theorem I. Let us denote

$$\bar{A}_a = \max_{(i)} \sup_{(x,t) \in D_{R_a}^h} v_i(x, t) .$$

There exist an index i_a and a point $(x_a, t_a) \in \overline{D_{R_a}^h} = (D_{R_a}^h + S_{R_a}^h) + \Sigma_{R_a}^h + C_{R_a}^h$ such that $\bar{A}_a = v_{i_a}(x_a, t_a)$. We will show that, the constants $\mu(K), \nu(K)$ and $h(K)$ being conveniently chosen, the relation $(x_a, t_a) \in D_{R_a}^h + S_{R_a}^h$ implies the inequality $v_{i_a}(x_a, t_a) \leqslant 0$. Indeed, suppose that $(x_a, t_a) \in D_{R_a}^h + S_{R_a}^h$ and $v_{i_a}(x_a, t_a) > 0$. Then at (x_a, t_a) we have

$$\frac{\partial v_{i_a}}{\partial t} \geqslant 0, \qquad \frac{\partial v_{i_a}}{\partial x_j} = 0 \quad \text{as well as} \quad \sum_{j,k=1}^{m} \frac{\partial^2 v_{i_a}}{\partial x_j \partial x_k} \lambda_j \lambda_k \leqslant 0 .$$

Substituting $x = x_a$, $t = t_a$ in the equation with the index i_a of (4.4) we find, by virtue of assumption $4°$, that the difference of the first two terms on the right side of this equation is non-positive, and from the definition of the parabolic equations it follows that the difference of the next two terms is also non-positive.

Suppose that a certain number (e.g. q, $1 \leqslant q \leqslant n$) of the functions among $v_1, ..., v_n$ at point (x_a, t_a) admit a positive value the others having non-positive values. Without loss of generality we may assume that they are the functions $v_1, ..., v_q$ (and that $i_a \leqslant q$).

The difference of the last two terms in the equation with the index i_a of (4.4) at (x_a, t_a) can be written as

$$(4.5) \qquad F_{i_a}^{(2)}(x_a, t_a, v_1^{(1)}H, ..., v_q^{(1)}H, v_{q+1}^{(1)}H, ..., v_n^{(1)}H, \tau_j^{(1)}, \tau_{jk}^{(1,1)}) -$$

$$- F_{i_a}^{(2)}(x_a, t_a, v_1^{(2)}H, ..., v_q^{(2)}H, v_{q+1}^{(1)}H, ..., v_n^{(1)}H, \tau_j^{(2)}, \tau_{jk}^{(1,2)}) +$$

$$+ F_{i_a}^{(2)}(x_a, t_a, v_1^{(2)}H, ..., v_q^{(2)}H, v_{q+1}^{(1)}H, ..., v_n^{(1)}H, \tau_j^{(2)}, \tau_{jk}^{(1,2)}) -$$

$$- F_{i_a}^{(2)}(x_a, t_a, v_1^{(2)}H, ..., v_q^{(2)}H, v_{q+1}^{(2)}H, ..., v_n^{(2)}H, \tau_j^{(2)}, \tau_{jk}^{(1,2)}) .$$

By the (W)-condition the difference of the last two terms in (4.5) is non-positive; however, the difference of the first two terms is estimated by virtue of the (Ω)-condition. Taking at the same time into consideration the equalities

$$\frac{\partial v_{i_a}^{(1)}(x_a, t_a)}{\partial x_j} = \frac{\partial v_{i_a}^{(2)}(x_a, t_a)}{\partial x_j} \qquad (j = 1, ..., m) ,$$

we have

$$0 \leqslant \frac{\partial v_{i_a}}{\partial t} H \leqslant v_{i_a} \left[L_0 \sum_{j,k=1}^{m} \left| \frac{\partial^2 H}{\partial x_j \partial x_k} \right| + (L_1|x| + L_2) \sum_{j=1}^{m} \left| \frac{\partial H}{\partial x_j} \right| + \right.$$

$$\left. + (L_3|x|^2 + L_4) qH - \frac{\partial H}{\partial t} \right] \overset{\mathrm{df}}{=} v_{i_a} F_1 H .$$

Choosing conveniently $\mu(K), \nu(K)$ and $h(K)$, as in the proof of theorem I, we find $F_1 H \leqslant -NH < 0$. Thus we obtain a contradiction. Now, if $(x_a, t_a) \in \Sigma_{R_a}^h$, then (by our assumption) $v_{i_a}(x_a, t_a) \leqslant 0$ and finally, if $(x_a, t_a) \in C_{R_a}^h$, then we have

$$v_{i_a}(x_a, t_a) \leqslant \frac{M \exp(K_0 R_a^2)}{\exp\left\{ \dfrac{K R_a^2}{1 - \mu t_a} + \nu t_a \right\}} .$$

Thus we have proved that, the constants $\mu(K), \nu(K)$ and $h(K)$ being conveniently chosen, the last inequality always holds.

Let $(\bar{x}, \bar{t})$ be an arbitrary fixed point of $\overline{D^h}$. For every $\varepsilon > 0$ there exists such an $a_0(\varepsilon)$ that for $a > a_0$ the point $(\bar{x}, \bar{t}) \in \overline{D^h_{R_a}}$ and

$$\frac{M \exp(K_0 R_a^2)}{\exp\left\{\dfrac{KR_a^2}{1-\mu t_a} + \nu t_a\right\}} < \varepsilon .$$

Since

$$v_i(\bar{x}, \bar{t}) \leqslant v_{t_a}(x_a, t_a) \leqslant \frac{M \exp(K_0 R_a^2)}{\exp\left\{\dfrac{KR_a^2}{1-\mu t_a} + \nu t_a\right\}} \qquad (i = 1, ..., n) ,$$

we have $v_i(\bar{x}, \bar{t}) \leqslant 0$ $(i = 1, ..., n)$, whence $v_i(x, t) \leqslant 0$ and thus $u_i(x, t) \leqslant 0$ $(i = 1, ..., n)$ everywhere in $\overline{D^h}$.

As in the proof of theorem I, the change of variable t permits to prove this theorem in the whole domain D.

Remark. It is easy to observe that the change of direction of the inequalities in assumptions 4° and 5° changes the direction of the inequalities in our theorem. If, therefore, $F_s^{(1)} \equiv F_s^{(2)}$ in Π and $u_i^{(1)}(x, t) = u_i^{(2)}(x, t)$ in Σ, then $u_i^{(1)}(x, t) = u_i^{(2)}(x, t)$ in $\bar{D}$. Hence we obtain the uniqueness of the solution of the (FI)-problem, under stronger assumption, however, than in theorem I. In the case of one equation, theorem II is more general than theorem I because then the (W)-condition does not interfere.

We have proved theorem II in the case where all the equations of (4.2) are parabolic and all the functions $F_s^{(2)}$ $(s = 1, ..., n)$ satisfy the (Ω)-condition and the (W)-condition. In this case the equations of (4.1) are not necessarily parabolic, just as the adequate functions do not necessarily satisfy either the (Ω)- or the (W)-conditions. For example, the functions $u_1^{(1)} = \pi/2 + e^{-(x_1+x_2)}$, $u_2^{(1)} = \pi/2$ are the solution of the system

$$(4.6) \quad \begin{aligned} \frac{\partial u_1^{(1)}}{\partial t} &= \frac{\partial^2 u_2^{(1)}}{\partial x_1^2} - \left|\frac{\partial^2 u_1^{(1)}}{\partial x_2^2}\right| + \sqrt{\left|\sin u_2^{(1)}\right|} - 1 , \\[2mm] \frac{\partial u_2^{(1)}}{\partial t} &= -\left|\frac{\partial^2 u_2^{(1)}}{\partial x_1^2}\right| + \frac{\partial^2 u_2^{(1)}}{\partial x_2^2} - \sqrt{\left|u_1^{(1)} - u_2^{(1)}\right|} + u_1^{(1)} - u_2^{(1)} + e^{-\frac{x_1+x_2}{2}} - e^{-(x_1+x_2)} , \end{aligned}$$

whereas the functions $u_1^{(2)} = 3$, $u_2^{(2)} = 5$ constitute the solution of the system

$$(4.7) \quad \begin{aligned} \frac{\partial u_1^{(2)}}{\partial t} &= \frac{\partial^2 u_1^{(2)}}{\partial x_1^2} + \frac{\partial^2 u_1^{(2)}}{\partial x_2^2} , \\[2mm] \frac{\partial u_2^{(2)}}{\partial t} &= \frac{\partial^2 u_2^{(2)}}{\partial x_1^2} + \frac{\partial^2 u_2^{(2)}}{\partial x_2^2} + u_1^{(2)} - u_2^{(2)} + 2 . \end{aligned}$$

The inequalities $u_1^{(1)} < u_2^{(2)}$, $u_2^{(1)} < u_2^{(2)}$ are fulfilled in the part of space $x_1 \geqslant 0$, $x_2 \geqslant 0$, $t \geqslant 0$. All the assumptions of theorem II are satisfied. The equations of (4.6) are not parabolic and, moreover, the suitable functions do not satisfy either the (Ω)- or the (W)-condition.

COROLLARY. *In the class* Z_δ *(see* § 1*) of functions* $u_i(x, t)$ *(i* $= 1, ..., n$), δ *being an arbitrary positive constant, we can prove theorems* I *and* II *if the* (Ω)-*condition is replaced by a weaker one. Namely, it is sufficient to assume that there exist positive constants* $L_0, L_1, ..., L_8$ *such that for* $y_s \geqslant \bar{y}_s$ *(s* $= 1, ..., n$) *the following inequalities hold:*

$$F_s(x, t, y_i, z_j, z_{jk}) - F_s(x, t, \bar{y}_i, \bar{z}_j, \bar{z}_{jk})$$

$$\leqslant L_0 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| + (L_1|x| + L_2|y|^{1/\delta} + L_3|\bar{y}|^{1/\delta} + L_4) \sum_{j=1}^{m} |z_j - \bar{z}_j| +$$

$$+ (L_5|x|^2 + L_6|y|^{2/\delta} + L_7|\bar{y}|^{2/\delta} + L_8) \sum_{i=1}^{n} |y_i - \bar{y}_i| \,,$$

where

$$|y| = \Big(\sum_{i=1}^{n} y_i^2\Big)^{1/2}, \quad |\bar{y}| = \Big(\sum_{i=1}^{n} \bar{y}_i^2\Big)^{1/2}.$$

Proof. If there exist positive constants A, B such that $|y_i| \leqslant A|x|^\delta + B$, then

$$|y|^{1/\delta} = \Big(\sum_{i=1}^{n} |y_i|^2\Big)^{1/2\delta} \leqslant n^{1/2\delta}(A|x|^\delta + B)^{1/\delta}.$$

Put $A_1 \overset{\text{df}}{=} n^{1/2\delta}(A + B)^{1/\delta}$. In the case $|x| > 1$ we have

$$n^{1/2\delta}(A|x|^\delta + B)^{1/\delta} = n^{1/2\delta}|x|\Big(A + \frac{B}{|x|^\delta}\Big)^{1/\delta} \leqslant A_1|x|.$$

In the case $|x| \leqslant 1$,

$$n^{1/2\delta}(A|x|^\delta + B) \leqslant A_1.$$

Therefore in both cases $|y|^{1/\delta} \leqslant A_1(|x|+1)$. Since $2|x| \leqslant |x|^2+1$, we have $|y|^{2/\delta} \leqslant A_1^2(|x|^2 + 2|x| + 1) \leqslant 2A_1^2(|x|^2 + 1)$. Similarly $|\bar{y}|^{1/\delta} \leqslant A_1(|x|+1)$, $|\bar{y}|^{2/\delta} \leqslant 2A_1^2(|x|^2 + 1)$. By virtue of the above inequalities we receive

$$L_1|x| + L_2|y|^{1/\delta} + L_3|\bar{y}|^{1/\delta} + L_4 \leqslant \bar{L}_1|x| + \bar{L}_2,$$

$$L_5|x|^2 + L_6|y|^{2/\delta} + L_7|\bar{y}|^{2/\delta} + L_8 \leqslant \bar{L}_3|x|^2 + \bar{L}_4,$$

$\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4$ being certain positive constants. It means that for functions $u_i^{(1)}, u_i^{(2)}$ of class Z_δ inequalities of form (3.4) are satisfied. This completes the proof.

§ 5. A certain theorem on the existence of a solution of the (FI)-problem. As in paper [6], we will prove the following

LEMMA. *For* $K > \dfrac{1}{2m}\sqrt{\dfrac{nL_3+\lambda}{L_0}}$ *and* $k_0 > 0$, *the inequality*

$$\frac{H(x, t; K)}{H(x, t; K + k_0)} \leqslant \exp(-k_0|x|^2)$$

holds for $(x, t) \in \overline{D^{h_0}}$, *where* $h_0 = (1-\gamma)/\mu(K+k_0)$, $0 < \gamma < 1$, $\mu(K)$ *is defined by* (3.5), *and* $\nu(K)$ *by* (3.6).

Proof. By virtue of (3.6), $\nu(K+k_0) > \nu(K)$. From (3.5) it follows that for $K > \dfrac{1}{2m}\sqrt{\dfrac{nL_2+\lambda}{L_0}}$, $k_0 > 0$, we have $\mu(K+k_0) > \mu(K)$. Hence

$$\frac{H(x, t; K)}{H(x, t; K + k_0)} = \exp\Big\{\frac{K|x|^2}{1-\mu(K)t} - \frac{K|x|^2}{1-\mu(K+k_0)t} - \frac{k_0|x|^2}{1-\mu(K+k_0)t} +$$

$$+ [\nu(K) - \nu(K+k_0)]t\Big\} \leqslant \exp\Big\{-\frac{k_0|x|^2}{1-\mu(K+k_0)t}\Big\} \leqslant \exp(-k_0|x|^2).$$

The lemma is thus proved.

Let us put $\Omega_R = \Sigma_R + C_R$, $\Omega_R^h = \Sigma_R^h + C_R^h$.

HYPOTHESIS (H). *Let* $\Phi_i(x, t)$ $(i = 1, ..., n)$ *be arbitrary continuous functions belonging to class* E_2 *in the domain* $\overline{D}$. *For every* R *there exists a parabolic solution* $u_i(x, t)$ $(i = 1, ..., n)$ *of* (1.1) *regular in* D_R *and such that* $u_i(x, t) = \Phi_i(x, t)$ $(i = 1, ..., n)$ *in the set* Ω_R.

We write $f_s(x, t) = F_s(x, t, 0, 0, ..., 0)$. We will prove

THEOREM III. *If*

1° *the* (H)-*hypothesis is satisfied,*

2° *system* (1.1) *is parabolic with regard to all the solutions mentioned in hypothesis* (H),

3° *the functions* $f_s(x, t)$ $(s = 1, ..., n)$ *belong to* E_2,

4° *the functions* F_s $(s = 1, ..., n)$ *satisfy the* (Ω)-*condition*,

5° $\varphi_i(x, t)$ $(i = 1, ..., n)$ *are arbitrary given functions, defined, continuous, and belonging to* E_2 *in* Σ;

then

1. *there exists a solution of the* (FI)-*problem in* D^{h_0} *(see* § 2*), where* h_0 *is sufficiently small (defined as in the lemma),*

2. *this solution also belongs to class* E_2 *in* D^{h_0}.

Proof. Let us choose n functions $\Phi_i(x, t)$ continuous and of class $E_2(M, K_0)$ in $\overline{D}$, so that $\Phi_i(x, t) = \varphi_i(x, t)$ $(i = 1, ..., n)$ for $(x, t) \in \Sigma$. We may assume that all the functions $\varphi_i(x, t), \Phi_i(x, t)$ and $f_i(x, t)$ $(i = 1, ..., n)$ belong to $E_2(M, K_0)$ with the same constants M and K_0.

Let $\{R_a\}$ be an increasing sequence, $R_a \to \infty$ as $a \to \infty$. Let us fix natural numbers a and β, $\beta > a$. By $u_i^a(x, t)$ $(i = 1, ..., n)$ we denote a solution of (1.1), determined and regular in D_{R_a}, which satisfies the condition

$$u_i^a(x, t) = \Phi_i(x, t) \qquad (i = 1, ..., n) \qquad \text{for} \qquad (x, t) \in \Omega_{R_a} \ .$$

According to the (H)-hypothesis such a solution exists.

Similarly we define $u_i^\beta(x, t)$ $(i = 1, ..., n)$ in D_{R_β}.

For the functions $u_i^a(x, t)$ $(i = 1, ..., n)$ we have in D_{R_a}

$$(5.1) \qquad \frac{\partial u_s^a}{\partial t} = F_s\left(x, t, u_i^a, \frac{\partial u_s^a}{\partial x_j}, \frac{\partial^2 u_s^a}{\partial x_j \partial x_k}\right) \qquad (s = 1, ..., n) \ .$$

Next, we introduce the functions v_i^a, v_i^β determined by the relations

$$(5.2) \qquad u_i^a = v_i^a H(x, t; K), \qquad u_i^\beta = v_i^\beta H(x, t; K) \qquad (i = 1, ..., n) \ ,$$

where $H(x, t; K)$ is defined by (3.2); at the same time we choose

$$(5.3) \qquad K > \max\left(K_0, \frac{1}{2m} \sqrt{\frac{nL_3 + \lambda}{L_0}}\right) \ .$$

The functions v_i^a satisfy the system

$$(5.4) \qquad \frac{\partial v_s^a}{\partial t} H + v_s^a \frac{\partial H}{\partial t} =$$

$$F_s\left(x, t, v_i^a H, \frac{\partial v_s^a}{\partial x_j} H + v_s^a \frac{\partial H}{\partial x_j}, \frac{\partial^2 v_s^a}{\partial x_j \partial x_k} H + \frac{\partial v_s^a}{\partial x_k}\frac{\partial H}{\partial x_j} + \frac{\partial v_s^a}{\partial x_j}\frac{\partial H}{\partial x_k} + v_s^a \frac{\partial^2 H}{\partial x_j \partial x_k}\right) -$$

$$- F_s\left(x, t, v_i^a H, \frac{\partial v_s^a}{\partial x_j} H + v_s^a \frac{\partial H}{\partial x_j}, 0 + \frac{\partial v_s^a}{\partial x_k}\frac{\partial H}{\partial x_j} + \frac{\partial v_s^a}{\partial x_j}\frac{\partial H}{\partial x_k} + v_s^a \frac{\partial^2 H}{\partial x_j \partial x_k}\right) +$$

$$+ F_s\left(x, t, v_i^a H, \frac{\partial v_s^a}{\partial x_j} H + v_s^a \frac{\partial H}{\partial x_j}, 0 + \frac{\partial v_s^a}{\partial x_k}\frac{\partial H}{\partial x_j} + \frac{\partial v_s^a}{\partial x_j}\frac{\partial H}{\partial x_k} + v_s^a \frac{\partial^2 H}{\partial x_j \partial x_k}\right) -$$

$$- F_s(x, t, 0, 0, ..., 0) + F_s(x, t, 0, 0, ..., 0) \ .$$

We consider the part D^h of D contained in the strip $0 < t < h$, where $h \leqslant T$ will be determined later. We retain the notation of the domains and their boundaries introduced in § 3.

Since the functions $\Phi_i(x, t)$ belong to $E_2(M, K_0)$, in $\Omega_{R_a}^h$ each function v_i^a satisfies the inequality $|v_i^a(x, t)| \leqslant M$ $(i = 1, ..., n)$. We put

$$A_a = \max_{(i)} \sup_{(x,t) \in D_{R_a}^h} |v_i^a(x, t)| \ .$$

First of all we will show that, for every a, $|v_i^a(x, t)| \leqslant M$ in the domain $\overline{D_{R_a}^h}$.
There exist an index i_a and a point $(x_a, t_a) \in \overline{D_{R_a}^h} = (D_{R_a}^h + S_{R_a}^h) + \Omega_{R_a}^h$ such

that $A_a = |v_{ia}^a(x_a, t_a)|$. If $(x_a, t_a) \in \Omega_{R_a}^h$, then our assertion follows from the preceding remark. Suppose, therefore, that $(x_a, t_a) \in D_{R_a}^h + S_{R_a}^h$. If $v_{ia}^a(x_a, t_a) \geqslant 0$, then at the point (x_a, t_a) we have

$$\frac{\partial v_{ia}^a}{\partial t} \geqslant 0, \qquad \frac{\partial v_{ia}^a}{\partial x_j} = 0 \qquad (j = 1, ..., m) \qquad \text{as well as} \qquad \sum_{j,k=1}^{m} \frac{\partial^2 v_{ia}^a}{\partial x_j \partial x_k} \lambda_j \lambda_k \leqslant 0$$

for any real vector $(\lambda_1, ..., \lambda_m)$. Substituting $x = x_a$, $t = t_a$ in the equation with the index i_a of (5.4) and taking advantage of the definition of parabolic equations, we find that the difference of the first two terms in this equation is non-positive. The difference of the next two terms is estimated according to the (Ω)-condition and we obtain

$$(5.5) \qquad \frac{\partial v_{ia}^a}{\partial t} \leqslant \frac{v_{ia}^a}{H} \left[L_0 \sum_{j,k=1}^{m} \left| \frac{\partial^2 H}{\partial x_j \partial x_k} \right| + (L_1|x| + L_2) \sum_{j=1}^{m} \left| \frac{\partial H}{\partial x_j} \right| + \right.$$
$$\left. + (L_3|x|^2 + L_4) nH - \frac{\partial H}{\partial t} \right] + f_{ia}^{(1)}(x_a, t_a) ,$$

where $f_{ia}^{(1)}(x, t) = \frac{1}{H} f_{ia}(x, t)$ Since $f_{ia}(x, t) \in E_2(M, K_0)$, it follows from (3.2) and (5.3) that $|f_{ia}^{(1)}(x_a, t_a)| \leqslant M$.

The constants $\mu(K)$, $\nu(K)$ in the function $H(x, t; K)$ and the number $h(K)$ can be chosen—in the same way as in the proof of theorem I—in such a way that the expression between the brackets in (5.5) is smaller than $-1 \cdot H$. Therefore at (x_a, t_a) we have

$$0 \leqslant \frac{\partial v_{ia}^a}{\partial t} \leqslant - v_{ia}^a + M ,$$

whence $v_{ia}^a(x_a, t_a) \leqslant M$. If $v_{ia}^a(x_a, t_a) < 0$, then similarly it can be shown that $v_{ia}^a(x_a, t_a) \geqslant -M$, and hence, finally, $|v_i^a(x, t)| \leqslant M$ $(i = 1, ..., n)$ in the domain $\overline{D_{R_a}^h}$. M does not depend on a; thus we have also

$$|v_i^\beta(x, t)| \leqslant M \qquad (i = 1, ..., n) \qquad \text{in } \overline{D_{R_\beta}^h} .$$

We put

$$(5.6) \qquad u_i^{\alpha\beta} = u_i^\alpha - u_i^\beta, \qquad v_i^{\alpha\beta} = v_i^\alpha - v_i^\beta .$$

Consequently,

$$(5.7) \qquad u_i^{\alpha\beta} = v_i^{\alpha\beta} H(x, t; K) \qquad \text{and} \qquad |v_i^{\alpha\beta}| \leqslant 2M \qquad \text{in the domain } \overline{D_{R_a}^h} .$$

For the functions $u_i^\alpha, u_i^\beta, u_i^{\alpha\beta}$ $(i = 1, ..., n)$ we receive the system of the equations

$$(5.8) \qquad \frac{\partial u_s^{\alpha\beta}}{\partial t} = F_s\left(x, t, u_i^\alpha, \frac{\partial u_s^\alpha}{\partial x_j}, \frac{\partial^2 u_s^\alpha}{\partial x_j \partial x_k}\right) - F_s\left(x, t, u_i^\beta, \frac{\partial u_s^\beta}{\partial x_j}, \frac{\partial^2 u_s^\beta}{\partial x_j \partial x_k}\right) .$$

Further, we introduce the functions $\overset{*\alpha}{v_i}, \overset{*\beta}{v_i}$ according to the relations

$$(5.9) \quad u_i^\alpha = \overset{*\alpha}{v_i} H(x,t; K + k_0), \quad u_i^\beta = \overset{*\beta}{v_i} H(x,t; K + k_0) \quad (i = 1, \ldots, n),$$

k_0 being a positive constant. Writing $\overset{*\alpha\beta}{v_i} = \overset{*\alpha}{v_i} - \overset{*\beta}{v_i}$, we get

$$(5.10) \quad u_i^{\alpha\beta} = \overset{*\alpha\beta}{v_i} H(x,t; K + k_0) \quad (i = 1, \ldots, n).$$

In view of (5.8), (5.9) and (5.10) we obtain for the functions $\overset{*\alpha}{v_i}, \overset{*\beta}{v_i}, \overset{*\alpha\beta}{v_i}$ the system

$$(5.11) \quad \frac{\partial \overset{*\alpha\beta}{v_s}}{\partial t} H + \overset{*\alpha\beta}{v_s} \frac{\partial H}{\partial t} =$$

$$F_s\left(x, t, \overset{*\alpha}{v_i} H, \frac{\partial \overset{*\alpha}{v_s}}{\partial x_j} H + \overset{*\alpha}{v_s} \frac{\partial H}{\partial x_j}, \frac{\partial^2 \overset{*\alpha}{v_s}}{\partial x_j \partial x_k} H + \frac{\partial \overset{*\alpha}{v_s}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial \overset{*\alpha}{v_s}}{\partial x_j} \frac{\partial H}{\partial x_k} + \overset{*\alpha}{v_s} \frac{\partial^2 H}{\partial x_j \partial x_k}\right) -$$

$$- F_s\left(x, t, \overset{*\alpha}{v_i} H, \frac{\partial \overset{*\alpha}{v_s}}{\partial x_j} H + \overset{*\alpha}{v_s} \frac{\partial H}{\partial x_j}, \frac{\partial^2 \overset{*\beta}{v_s}}{\partial x_j \partial x_k} H + \frac{\partial \overset{*\alpha}{v_s}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial \overset{*\alpha}{v_s}}{\partial x_j} \frac{\partial H}{\partial x_k} + \overset{*\alpha}{v_s} \frac{\partial^2 H}{\partial x_j \partial x_k}\right) +$$

$$+ F_s\left(x, t, \overset{*\alpha}{v_i} H, \frac{\partial \overset{*\alpha}{v_s}}{\partial x_j} H + \overset{*\alpha}{v_s} \frac{\partial H}{\partial x_j}, \frac{\partial^2 \overset{*\beta}{v_s}}{\partial x_j \partial x_k} H + \frac{\partial \overset{*\alpha}{v_s}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial \overset{*\alpha}{v_s}}{\partial x_j} \frac{\partial H}{\partial x_k} + \overset{*\alpha}{v_s} \frac{\partial^2 H}{\partial x_j \partial x_k}\right) -$$

$$- F_s\left(x, t, \overset{*\beta}{v_i} H, \frac{\partial \overset{*\beta}{v_s}}{\partial x_j} H + \overset{*\beta}{v_i} \frac{\partial H}{\partial x_j}, \frac{\partial^2 \overset{*\beta}{v_s}}{\partial x_j \partial x_k} H + \frac{\partial \overset{*\beta}{v_s}}{\partial x_k} \frac{\partial H}{\partial x_j} + \frac{\partial \overset{*\beta}{v_s}}{\partial x_j} \frac{\partial H}{\partial x_k} + \overset{*\beta}{v_s} \frac{\partial^2 H}{\partial x_j \partial x_k}\right).$$

Let us write

$$A_{\alpha\beta} = \max_{(i)} \sup_{(x,t) \in \overline{D_{R_\alpha}^{h_0}}} |\overset{*\alpha\beta}{v_i}(x,t)|,$$

where h_0, $0 < h_0 < h$, will be determined later. There exist an index $i_{\alpha\beta}$ and a point $(x_{\alpha\beta}, t_{\alpha\beta}) \in \overline{D_{R_\alpha}^{h_0}} = (D_{R_\alpha}^{h_0} + S_{R_\alpha}^{h_0}) + \Sigma_{R_\alpha}^{h_0} + C_{R_\alpha}^{h_0}$ such that $A_{\alpha\beta} = |\overset{*\alpha\beta}{v_{i_{\alpha\beta}}}(x_{\alpha\beta}, t_{\alpha\beta})|$. By an argument similar to that in the proof of theorem I, we show that, for a convenient choice of $\mu(K)$, $\nu(K)$ and h_0, if $(x_{\alpha\beta}, t_{\alpha\beta}) \in D_{R_\alpha}^{h_0} + S_{R_\alpha}^{h_0}$, then $\overset{*\alpha\beta}{v_{i_{\alpha\beta}}}(x_{\alpha\beta}, t_{\alpha\beta}) = 0$. If $(x_{\alpha\beta}, t_{\alpha\beta}) \in \Sigma_{R_\alpha}^{h_0}$, then by the definition of u_i^α we have also $\overset{*\alpha\beta}{v_{i_{\alpha\beta}}}(x_{\alpha\beta}, t_{\alpha\beta}) = 0$. If finally $(x_{\alpha\beta}, t_{\alpha\beta}) \in C_{R_\alpha}^{h_0}$, then by (5.7), (5.10) and by the lemma we have at $(x_{\alpha\beta}, t_{\alpha\beta})$

$$|\overset{*\alpha\beta}{v_i}(x,t)| \leq \frac{2MH(x,t; K)}{H(x,t; K + k_0)} \leq 2M \exp(-k_0|x|^2) \quad (i = 1, \ldots, n).$$

Thus we have proved that the inequality

$$(5.12) \quad |\overset{*\alpha\beta}{v_i}(x,t)| \leq 2M \exp(-k_0 R_\alpha^2) \quad (i = 1, \ldots, n)$$

holds true in $\overline{D_{R_\alpha}^{h_0}}$.

Now let r be an arbitrary positive number and let us consider the domain $D_r^{h_0}$. We choose a so that $R_a > r$. By virtue of (5.10) and (5.12) we find

$$|u_i^{\alpha\beta}(x, t)| \leqslant 2MN_0 \exp(-k_0 R_a^2) \quad (i = 1, ..., n) \quad \text{for} \quad (x, t) \, \epsilon \, \overline{D_r^{h_0}},$$

where $N_0 = \max\limits_{D_r^{h_0}} H(x, t; K + k_0)$. Hence it follows that for every $\varepsilon > 0$ there exists an index $a_0(\varepsilon)$, $R_{a_0} > r$, such that for $\beta > a > a_0$ the inequalities

$$|u_i^{\alpha} - u_i^{\beta}| < \varepsilon \quad (i = 1, ..., n)$$

hold for $(x, t) \, \epsilon \, \overline{D_r^{h_0}}$. It means that the sequences

$$(5.13) \qquad \{u_1^{\alpha}(x, t)\}, \, \{u_2^{\alpha}(x, t)\}, \, ..., \, \{u_n^{\alpha}(x, t)\}$$

are almost uniformly convergent in $\overline{D^{h_0}}$ as $a \to \infty$. Observe that the limit functions $U_1(x, t), U_2(x, t), ..., U_n(x, t)$ of these sequences belong to class E_2. In fact,

$$U_i = \lim_{a \to \infty} u_i^{\alpha} = \lim_{a \to \infty} v_i^{\alpha} H(x, t; K) \quad (i = 1, ..., n)$$

and $|v_i^{\alpha}| \leqslant M$ for every i and a, whence

$$|U_i(x, t)| \leqslant M \exp\left\{\frac{K|x|^2}{1 - \mu t} + vt\right\} \quad (i = 1, ..., n), \quad \text{q.e.d.}$$

Now we will show that the functions $U_i(x, t)$ $(i = 1, ..., n)$ constitute the solution of the (FI)-problem in D^{h_0}. Obviously

$$U_i(x, t) = \varphi_i(x, t) \quad (i = 1, ..., n) \quad \text{for} \quad (x, t) \, \epsilon \, \Sigma^{h_0}.$$

It remains to prove that $U_i(x, t)$ $(i = 1, ..., n)$ is a (parabolic) solution of (1.1). It is sufficient to prove this in the domain $D_r^{h_0}$ for an arbitrary r. Let $u_i^r(x, t)$ $(i = 1, ..., n)$ be a parabolic solution of (1.1), regular in $D_r^{h_0}$, satisfying the condition $u_i^r(x, t) = \varphi_i(x, t)$ $(i = 1, ..., n)$ for $(x, t) \, \epsilon \, \Sigma_r^{h_0}$ and identical with $U_i(x, t)$ for $(x, t) \, \epsilon \, C_r^{h_0}$. From the almost uniform convergence of sequences (5.13) it follows that for every $\varepsilon > 0$ there is an index $a_1(\varepsilon)$, $R_{a_1} > r$, such that for $a > a_1$ the inequalities

$$(5.14) \qquad |U_i - u_i^{\alpha}| < \frac{\varepsilon}{N_1 + 1} \quad (i = 1, ..., n),$$

where $N_1 = \max\limits_{D_r^{h_0}} H(x, t, K + k_0)$, hold for $(x, t) \, \epsilon \, \overline{D_r^{h_0}}$. In the set $\Omega_r^{h_0}$ we have $U_i = u_i^r$; therefore

$$(5.15) \quad |u_i^r - u_i^{\alpha}| < \frac{\varepsilon}{N_1 + 1} \quad (i = 1, ..., n) \quad \text{for} \quad a > a_1, \quad (x, t) \, \epsilon \, \Omega_r^{h_0}.$$

The functions

$$(5.16) \qquad \overset{*r}{v_i} = \frac{u_i^r}{H(x, t; K + k_0)}, \qquad \overset{*a}{v_i} = \frac{u_i^a}{H(x, t; K + k_0)}, \qquad \overset{*ra}{v_i} = \overset{*r}{v_i} - \overset{*a}{v_i}$$

$$(i = 1, ..., n)$$

fulfil a system similar to (5.11). Let us put

$$A_{ra} = \max_{(i)} \sup_{(x,t) \in \overline{D}_r^{h_0}} |\overset{*ra}{v_i}(x, t)|.$$

There exist an index i_{ra} and a point $(x_{ra}, t_{ra}) \in \overline{D}_r^{h_0} = (D_r^{h_0} + S_r^{h_0}) + \Omega_r^{h_0}$ such that $A_{ra} = |\overset{*ra}{v_{i_{ra}}}(x_{ra}, t_{ra})|$. Reasoning as before, we find that if $(x_{ra}, t_{ra}) \in D_r^{h_0} + S_r^{h_0}$, then $\overset{*ra}{v_{i_{ra}}}(x_{ra}, t_{ra}) = 0$. If $(x_{ra}, t_{ra}) \in \Omega_r^{h_0}$, then from (5.15) and (5.16) it follows that

$$|\overset{*ra}{v_{i_{r\beta}}}(x_{ra}, t_{ra})| < \frac{\varepsilon}{N_1 + 1}.$$

Consequently,

$$|\overset{*ra}{v_i}(x, t)| < \frac{\varepsilon}{N_1 + 1} \qquad (i = 1, ..., n)$$

everywhere in $\overline{D}_r^{h_0}$. Taking into consideration (5.16), we have

$$|u_i^r - u_i^a| < \frac{N_1 \varepsilon}{N_1 + 1}.$$

Hence and from (5.14) we receive the inequalities

$$|U_i - u_i^r| \leqslant |U_i - u_i^a| + |u_i^a - u_i^r| < \varepsilon \qquad (i = 1, ..., n) \qquad \text{for} \qquad (x, t) \in \overline{D}_r^{h_0},$$

which means that $U_i(x, t) = u_i^r(x, t)$ $(i = 1, ..., n)$ in the domain $\overline{D}_r^{h_0}$. Therefore $U_i(x, t)$ $(i = 1, ..., n)$ is the solution of the (FI)-problem in D^{h_0}. As we have proved before, this solution belongs to class E_2.

According to theorem I the functions $U_i(x, t)$ $(i = 1, ..., n)$ constitute the unique solution of the problem in question.

If the domain D is the whole strip

$$D\{-\infty < x_i < +\infty \ (i = 1, ..., m), \ 0 < t < T\},$$

then theorems I-III remain true for the Cauchy problem, i.e. the problem of the existence and uniqueness of the solution $u_i(x, t)$ $(i = 1, ..., n)$ of system (1.1), satisfying the initial condition

$$u_i(x, 0) = \varphi_i(x) \qquad (i = 1, ..., n),$$

where the initial data $\varphi_i(x)$ are continuous functions, defined and belonging to class E_2 on the hyperplane $t = 0$.

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