ANNALES
POLONICI MATHEMATICI
XLII (1983)

Some results in the theory of a third-order linear differential equation

by Michal Greguš and Milan Gera (Bratislava)

Dedicated to the memory of Jacek Szarski

Abstract. This paper investigates the existence of oscillatory and non-oscillatory solutions of a third-order linear differential equation. Besides, conditions are derived for the asymptotic behavior of its non-trivial non-oscillatory solutions.

1. Consider the differential equation

$$y'''+p(x)y''+q(x)y'+r(x)y = 0, \tag{a}$$

where p, r are continuous functions and q is continuously differentiable function on the interval $I = (a, \infty)$, $-\infty \leqslant a < \infty$.

Suppose that the coefficients p, q, r satisfy one of the following conditions on the interval I:

$$p(x) \geqslant 0, \quad 2r(x)-q'(x)-p(x)q(x) \geqslant 0, \tag{P}$$

moreover, the sign of the equality in the second inequality does not hold on any subinterval of I;

$$p(x) \leqslant 0, \quad 2r(x)-q'(x)-p(x)q(x) \leqslant 0, \tag{$\bar{\mathrm{P}}$}$$

moreover, the sign of the equality in the second inequality does not hold on any subinterval of I.

The adjoint differential equation of equation (a) is

$$\big[\big(z'-p(x)z\big)'+q(x)z\big]'-r(x)z = 0. \tag{b}$$

Write

$$lu = u''+p(x)u'+q(x)u, \quad u \in C^2(I); \quad E(x, x_0) = \exp \int_{x_0}^{x} p(t)\,dt$$

for $x, x_0 \in I$ and transform equation (b) by the substitution $z = uE(x, x_0)$.

Then we get the equation

$$[luE(x, x_0)]' - r(x)E(x, x_0)u = 0. \tag{$\bar{\text{b}}$}$$

A non-trivial solution of a linear differential equation will be called *oscillatory on I* if its set of zeros is not bounded above. Other solutions will be called *non-oscillatory on I*.

A linear differential equation is termed *oscillatory on I* if it has at least one oscillatory solution, and *non-oscillatory on I* in the opposite case.

We say that a linear differential equation of order n is *disconjugated* on an interval $J \subset I$ if none of its non-trivial solutions has more than $n-1$ zeros (including multiplicity) on J.

We shall derive some conditions for the existence of oscillatory and non-oscillatory solutions of the differential equations (a) and ($\bar{\text{b}}$).

2. First we introduce the notion of bands of solutions of the differential equation (a) [6]. It will enable us to look into the structure of solutions of the differential equation (a).

Let y_1, y_2 be solutions of the differential equation (a) with the properties $y_1(x_0) = y_1'(x_0) = 0$, $y_1''(x_0) \neq 0$; $y_2(x_0) = y_2''(x_0) = 0$, $y_2'(x_0) \neq 0$, $x_0 \in I$.

We call the set of solutions $y = c_1y_1 + c_2y_2$ of the differential equation (a) (c_1, c_2 are arbitrary constants) the band of solutions of equation (a) at the point x_0.

The band of solutions at x_0 fulfils the second order differential equation

$$uy'' - u'y' + lu\,y = 0, \tag{c}$$

where $u = y_1y_2' - y_1'y_2$ is a solution of the differential equation ($\bar{\text{b}}$) with the properties $u(x_0) = u'(x_0) = 0$, $u''(x_0) \neq 0$.

If $u(x) \neq 0$ for $x > x_0$, then the band of solutions at x_0 of equation (a) is called ***regular*** on the interval (x_0, ∞) and equation (a) is oscillatory on I iff equation (c) is oscillatory on (a, ∞). Moreover, two linearly independent solutions of the band at x_0 separate the null-points of each other to the right from the point x_0.

The solutions of the differential equation (a) fulfil the following integral identity

$$\begin{aligned} Fx_0[y(x)] &\equiv [y'^2(x) - q(x)y^2(x) - 2y(x)y''(x)]E(x, x_0) \\ &= Fx_0[y(x_0)] + \int_{x_0}^{x} p(t)E(t, x_0)y'^2(t)\,dt + \\ &\quad + \int_{x_0}^{x} [2r(t) - q'(t) - p(t)q(t)]E(t, x_0)y^2(t)\,dt. \end{aligned} \tag{1}$$

The solutions of the differential equation $(\bar{\mathrm{b}})$ fulfil the integral identity

$$(\bar{1}) \qquad \bar{F}x_0[u(x)] \equiv [2u(x)u''(x)+2p(x)u(x)u'(x)+ + q(x)u^2(x)-u'^2(x)]E(x,x_0) = \bar{F}x_0[u(x_0)]+\int_{x_0}^{x} p(t)E(t,x_0)u'^2(t)\,dt + \int_{x_0}^{x}[2r(t)-q'(t)-p(t)q(t)]E(t,x_0)u^2(t)\,dt.$$

The following lemma is an immediate consequence of the integral identities.

Lemma 1. *Let $x_0 \in I$ and y (u) be a solution of the differential equation* (a) $((\bar{\mathrm{b}}))$ *with the properties $y(x_0) = y'(x_0) = 0$, $y''(x_0) \neq 0$ $(u(x_0) = u'(x_0) = 0$, $u''(x_0) \neq 0)$.*

(a) *Let further assumption* (P) *hold. Then the solution y (u) does not have any null-point on the interval I to the left* (*to the right*) *of x_0. Moreover, if $q(x) \leqslant 0$ on I, then it holds for y (u), that $y(x)y'(x) < 0$, $y(x)y''(x) > 0$ for $x \in (a, x_0)$ $(u(x)u'(x) > 0$, $u(x)[u'(x)E(x,x_0)]' > 0$ for $x > x_0)$.*

(b) *Let further assumption* $(\bar{\mathrm{P}})$ *hold. Then the solution y (u) does not have any null-point on I to the right* (*to the left*) *of x_0. If, besides, $q(x) \leqslant 0$ on I, then it holds for y (u), that $y(x)y'(x) > 0$, $y(x)y''(x) > 0$ for $x > x_0$ $(u(x)u'(x) < 0$, $u(x)[u'(x)E(x,x_0)]' > 0$ for $x \in (a, x_0))$.*

Using Lemma 1 we can prove the following theorem in the similar way as in paper [7], Theorem 14.

Theorem 1. *Let assumption* (P) *hold. Then the differential equation* (a) $((\bar{\mathrm{b}}))$ *has at least one solution y (u) without null-points on I for which $Fx_0[y(x)] < 0$ $(\bar{F}x_0[u(x)] > 0)$ holds on the interval I. If $q(x) \leqslant 0$ on I, then $y(x)y'(x) < 0$, $y(x)y''(x) > 0$ $(u(x)u'(x) > 0$, $u(x)[u'(x)E(x,x_0)]' > 0)$ for $x \in I$.*

Remark 1. The assertion of Theorem 1 holds also under assumption $(\bar{\mathrm{P}})$, but the inequalities $y(x)y'(x) > 0$, $y(x)y''(x) > 0$ $(u(x)u'(x) < 0$, $u(x)[u'(x)E(x,x_0)]' > 0)$ hold on the interval I for the solution y (u) in the case $q(x) \leqslant 0$.

Lemma 2. *Let the differential equation $lu = 0$ be disconjugated on I and let $r(x) \geqslant 0$ $(\leqslant 0)$ for $x \in I$. Then the solution y of the differential equation* (a) *with the properties $y(x_0) = y'(x_0) = 0$, $y''(x_0) \neq 0$, $x_0 \in I$ does not have further null-point and further null-point of the derivative to the left* (*right*) *of x_0 on I.*

This lemma follows from Corollary 1 of Lemma 1.1 in paper [2].

It follows from Lemma 1 resp. Lemma 2 and from the properties of bands [6] (or from Corollary 1 of Theorem 1 in paper [3], too).

COROLLARY 1. *Let the differential equation* $lu = 0$ *be disconjugated on* I *and let* $r(x) \leqslant 0$ ($\geqslant 0$) *for* $x \in I$. *If* (P) ((P̄)) *holds, then the differential equation* (a) *is disconjugated on* I.

COROLLARY 1′. *Let* $q(x) \leqslant 0$, $r(x) \leqslant 0$ ($r(x) \geqslant 0$) *for* $x \in I$ *and, moreover, let* (P) ((P̄)) *hold. Then the differential equation* (a) *is disconjugated on* I.

The following theorem follows from Theorem 1 and from Theorem 1 in paper [1].

THEOREM 2. *Let* (P) *or* (P̄) *hold. Then the differential equation* (a) *is oscillatory on* I *iff the differential equation* (b̄) *is oscillatory.*

THEOREM 3. *Let* (P) ((P̄)) *hold and let the differential equation* (a) *be oscillatory on* I. *Then a non-trivial solution* y (u) *of* (a) ((b̄)) *is non-oscillatory on* I *iff* $Fx_0[y(x)] < 0$ ($\bar{F}x_0[u(x)] > 0$) *for* $x \in I$.

The proof of Theorem 3 is similar to the proof of Theorem 3.2 in paper [10].

COROLLARY 2. *Let* (P) ((P̄)) *hold. A necessary and sufficient condition for equation* (a) ((b̄)) *to be oscillatory on* I *is that every its non-trivial solution* y (u) *satisfying the condition* $Fx_0[y(x_0)] \geqslant 0$ ($\bar{F}x_0[u(x_0)] \leqslant 0$) *at some point* $x_0 \in I$, *be oscillatory on* I.

COROLLARY 2′. *Let* (P) ((P̄)) *hold. Then the following three conditions are equivalent:*

(a) *The differential equation* (a) ((b̄)) *is oscillatory on* I.

(b) *Every non-trivial solution of* (a) ((b̄)) *with at least one zero in* I *is oscillatory on* I.

(c) *Any non-trivial non-oscillatory solution of* (a) ((b̄)) *has not any null-points on* I.

COROLLARY 3. *Let* (P) ((P̄)) *hold. Then equation* (a) ((b̄)) *is non-oscillatory on* I *if there exists such a non-trivial, non-oscillatory solution* y (u) *of* (a) ((b̄)) *and such* $x_0 \in I$ *that* $Fx_0[y(x_0)] \geqslant 0$ ($\bar{F}x_0[u(x_0)] \leqslant 0$).

THEOREM 4. *Let* $q(x) \leqslant 0$ *on* I *and let* (P) *hold. If the differential equation* (a) *is oscillatory on* I, *then for any non-trivial non-oscillatory solution* y *of* (a) *on* I *we have* $y(x)y'(x) < 0$, $y(x)y''(x) > 0$ *for all* $x \in I$.

Proof. Let y be non-trivial non-oscillatory solution of the oscillatory equation (a). It follows from Theorem 3 that y fulfils the inequality $Fx_0[y(x)] < 0$ for $x \in I$. The integral identity (1) implies $y(x)y''(x) > 0$ on I, because $q(x) \leqslant 0$ for $x \in I$. We can suppose, without loss of generality, that $y(x) > 0$ for $x \in I$. Then $y''(x) > 0$ on I. It is necessary to prove that $y'(x) < 0$ for $x \in I$. Let, on the contrary, there exists such a number $\tau \in I$ that $y'(\tau) \geqslant 0$. Since $y''(x) > 0$ on I, therefore $y'(x) > 0$ for $x > \tau$. It follows from Theorem 1 that there exists a solution v of (a) with the properties $v(x) > 0$, $v'(x) < 0$ for $x \in I$.

Therefore

$$\begin{vmatrix} v(x), & y(x) \\ v'(x), & y'(x) \end{vmatrix} > 0$$

for $x \in I$. Thus every non-trivial linear combination of solutions y and v is non-oscillatory on I. This is in a contradiction with the assumption that (a) is oscillatory on I. The solution $\bar{y}(x) = y(x_0)v(x) - v(x_0)y(x)$ of (a) with a null-point at $x_0 \in (\tau, \infty)$ fulfils at x_0 the relation $Fx_0[\bar{y}(x_0)] = \bar{y}'^2(x_0) > 0$, that is, it oscillates on I by Corollary 2 and the theorem is proved.

THEOREM 5. *Let* (P) *hold and let the integral* $\int_{x_0}^{\infty} p(t)dt$, $x_0 \in I$, *be convergent. Let further* $q(x) \leqslant 0$ *and let equation* (a) *be oscillatory on* I. *Then there exists one and only one non-oscillatory solution* $v(x) \not\equiv 0$ (*up to the linear dependence*) *of the differential equation* (a), *for which, on the interval* I, $v(x)v'(x) < 0$, $v(x)v''(x) > 0$ *hold.*

Proof. Theorem 1 implies the existence of at least one solution v with the desired property. Assume that there exists one more non-oscillatory solution $\bar{v}$ *of* (a), independent of v. It follows from Theorem 4, that $\bar{v}$ has on I the same properties as v. Clearly $y = \bar{v}(x_0)v(x) - v(x_0)\bar{v}(x)$ is a solution of (a), and $y(x_0) = 0$. By Corollary $2'$, y is an oscillatory solution on I. Since $\lim_{x\to\infty} v'(x) = \lim_{x\to\infty} \bar{v}'(x) = 0$, we have

$$\lim_{x\to\infty} y'(x) = 0. \tag{2}$$

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence of null-points of y on (x_0, ∞) such that $\lim_{n\to\infty} x_n = \infty$. The supposition (P) and the integral identity (1) for y imply $Fx_1[y(x)] \geqslant Fx_1[y(x_1)]$ for all $x \geqslant x_1$ and therefore

$$y'^2(x_n) \geqslant y'^2(x_1)E(x_1, x_n), \quad n = 1, 2, \ldots$$

It follows from Lemma 1, that $y'(x_1) \neq 0$.

From these facts and from the convergency of $\int_{x_0}^{\infty} p(t)dt$ we get that $\lim_{n\to\infty} y'^2(x_n)$ is not equal to zero. This is a contradiction with (2) and the theorem is proved.

The proofs of the following theorems are similar to the proofs of Theorems 4 and 5.

THEOREM 4$'$. *Let* $q(x) \leqslant 0$ *on* I *and let* $(\bar{\mathrm{P}})$ *hold. If the differential equation* (a) *is oscillatory, then* $u(x)u'(x) < 0$, $u(x)[u'(x)E(x, x_0)]' > 0$ *hold for every non-trivial non-oscillatory solution* u *of the differential equation* $(\bar{\mathrm{b}})$ *on* I.

THEOREM 5$'$. *Let* $(\bar{\mathrm{P}})$ *hold and let the integral* $\int_{x_0}^{\infty} p(t)dt$, $x_0 \in I$, *be convergent. Let further* $q(x) \leqslant 0$ *and* (a) *be oscillatory on* I. *Then there exists* *one and only one non-trivial, non-oscillatory solution* u *of* $(\bar{\mathrm{b}})$ *(up to the linear dependence) for which* $u(x)u'(x) < 0$, $u(x)[u'(x)E(x, x_0)]' > 0$, $x \in I$, *hold.*

From Theorems 4, 4′ it follows

COROLLARY 4. *Let* $q(x) \leqslant 0$ *and let* (P) $((\bar{\mathrm{P}}))$ *hold. If the differential equation* (a) *is oscillatory on* I, *then every non-trivial solution* y (u) *of* (a) $((\bar{\mathrm{b}}))$ *is oscillatory on* I, *if it fulfils one of the following conditions at some point* $x_0 \in I$:

$$y(x_0)y'(x_0) \geqslant 0; \quad y(x_0)y''(x_0) \leqslant 0 \quad (u(x_0)u'(x_0) \geqslant 0;$$
$$u(x_0)\big(u'(x)E(x, x_0)\big)'_{x=x_0} \leqslant 0).$$

3. We shall derive some sufficient conditions for the differential equation (a) in order to be oscillatory on I.

LEMMA 3. *Let* w *be such a positive, twice differentiable function on* $J \subset I$ *that* $lw \leqslant 0$ *for* $x \in J$. *Then the differential equation* $ly = 0$ *is disconjugated on* J.

This lemma is proved in paper [9] (see [4], too).

THEOREM 6. *Let* (P) *hold and let the second order differential equation*

$$v'' + \tfrac{1}{4}q(x)v = 0 \tag{3}$$

be oscillatory on I. *Then the differential equation* (a) *is oscillatory on* I, *too.*

Proof. Let y be a non-trivial solution of (a) with the property $Fx_0[y(x_0)] \geqslant 0$ for some $x_0 \in I$ and be non-oscillatory on I. Then there exists such $x_1 \geqslant x_0$ that $y(x) \neq 0$ for $x > x_1$. Assume e.g. $y(x) > 0$ for $x > x_1$. We can suppose it since $Fx_0[y(x)] = Fx_0[-y(x)]$. It follows from (P) that $Fx_0[y(x)] > 0$ for $x > x_0$ and therefore $Fx_1[y(x)] > 0$ for $x > x_1$. Denote $Y(x) = \sqrt{y(x)}$ for $x > x_1$. Then

$$y(x) = Y^2(x), \quad y'(x) = 2Y(x)Y'(x), \quad y''(x) = 2Y(x)Y''(x) + 2Y'^2(x)$$

and

$$0 < Fx_1[Y^2(x)] = -4Y^3(x)[Y''(x) + \tfrac{1}{4}q(x)Y(x)]$$

for $x > x_1$. Hence we get

$$Y''(x) + \tfrac{1}{4}q(x)Y(x) < 0, \quad x \in (x_1, \infty). \tag{4}$$

By Lemma 3 the differential equation (3) is therefore disconjugated on (x_1, ∞). This is applied to obtain a contradiction with the assumption on equation (3) to be oscillatory on I and Theorem 6 is proved.

COROLLARY 5. *Let* (P) *hold and let* $q(x) \geqslant 0$ *on* I. *If the differential equation* (a) *is non-oscillatory on* I *and if* y *is an arbitrary non-trivial solution of* (a) *with the property* $Fx_0[y(x_0)] \geqslant 0$ *at some number* $x_0 \in I$, *then there exists such a number* $\tau \in I$ *that* $y(x)y'(x) > 0$ *holds for* $x > \tau$.

Proof. Let (a) be non-oscillatory on I and let y be a positive solution of (a), mentioned in the proof of Theorem 6. Then by (4) regarding the assumption $q(x) \geqslant 0$ on I for $Y''(x)$ we get that $Y''(x) < 0$ for $x > x_1$. Since $Y(x) > 0$ for $x \in (x_1, \infty)$ there exists such a number $\tau \geqslant x_1$, that $Y'(x) > 0$ for $x > \tau$ and thus $y'(x) > 0$ on (τ, ∞), too.

Remark 2. (a) In the case $p(x) \equiv 0$ on I from Theorem 6 and Corollary 2 we get Theorem 4 in paper [11].

(b) If (a) is non-oscillatory on I and (P) holds, then the differential equation (3) is non-oscillatory on I.

LEMMA 4. *Let $x_0 \in I$ and let $r(x) \geqslant 0$ for $x \geqslant x_0$. Then the differential equation* (a) *is disconjugated on $[x_0, \infty)$ if there exists a function f with continuous third derivative on (x_0, ∞) such that $f(x) > 0$, $f'(x) > 0$ and $f'''(x) + p(x)f''(x) + q(x)f'(x) + r(x)f(x) \leqslant 0$ for $x > x_0$.*

The proof of Lemma 4 is given in [4] as the proof of Corollary 1′ of Theorem 2.

The following lemma follows from Lemma 4 and Corollary 5.

LEMMA 5. *Let $q(x) \geqslant 0$, $r(x) \geqslant 0$ for $x \in I$ and let* (P) *hold. Then* (a) *is non-oscillatory on I iff there exists such a solution y of* (a) *and such a number $\tau \in I$ that $y(x)y'(x) > 0$ for $x > \tau$.*

If, moreover,

$$\text{(E)} \qquad \int_{x_0}^{\infty} E(x_0, s)\,ds = \infty, \qquad x_0 \in I,$$

then $y(x)y''(x) > 0$ holds on (τ, ∞), too.

From Lemma 5 and Lemma 4 follows

COROLLARY 6. *Let $q(x) \geqslant 0$, $r(x) \geqslant 0$ for $x \in I$ and let* (P) *hold. Then* (a) *is non-oscillatory on I iff there exists such $\tau \in I$ and such a function $w \in C^3\big((\tau, \infty)\big)$ with the property $w(x) > 0$, $w'(x) > 0$, $w'''(x) + p(x)w''(x) + q(x)w'(x) + r(x)w(x) \leqslant 0$ for $x > \tau$.*

COROLLARY 7. *Let $q(x) \geqslant 0$, $r(x) \geqslant 0$ for $x \in I$ and let* (P) *and* (E) *hold. Besides let the second order differential equation*

$$\text{(5)} \qquad v'' + q(x)v = 0$$

be oscillatory on I. Then the differential equation (a) *is oscillatory on I, too.*

Proof. Assume the contrary, that is, let (a) be non-oscillatory on I. Then by Lemma 5 there exists such a number $\tau \in I$ and such a solution y of (a), that $y(x) > 0$, $y'(x) > 0$, $y''(x) > 0$ for $x > \tau$. Denote $z = y'(x)$. Then from (a) we get

$$z'' + q(x)z = -p(x)z' - r(x)y \leqslant 0 \qquad \text{for } x > \tau.$$

By Lemma 3 the differential equation (5) is therefore disconjugated on

(τ, ∞). This is in a contradiction with the supposition and Corollary 7 is proved.

LEMMA 6. *Let be $\tau \in I$. Let $g(x)$ be such that $g(x) > 0$, $g'(x) > 0$ for $x \geqslant \tau$ and let $g''(x)E(x, \tau)$ have a continuous and non-positive derivative on $[\tau, \infty)$ if $p(x) \geqslant 0$ on $[\tau, \infty)$, then*

$$\liminf_{x\to\infty} \frac{g(x)}{xg'(x)} \geqslant \tfrac{1}{2}.$$

The proof of this lemma follows from Corollary 1 of Lemma 1 of paper [5].

THEOREM 7. *Let $q(x) \geqslant 0$, $r(x) \geqslant 0$ for $x \in I$ and let* (P) *hold. Let further the second order differential equation*

$$v'' + p(x)v' + [q(x) + \tfrac{1}{2}x\theta r(x)]v = 0 \tag{6}$$

be oscillatory on I for some $\theta \in (0, 1)$. Then the differential equation (a) *is oscillatory on I.*

Proof. Suppose the assertion is false. Let namely equation (a) be non-oscillatory on I. Then by Lemma 5 there exists such a solution y of (a) and such a number $\tau \in I$ that $y(x) > 0$, $y'(x) > 0$ hold for $x > \tau$. From equation (a) we get then

$$\big(y''(x)E(x, \tau)\big)' = -[q(x)y'(x) + r(x)y(x)]E(x, \tau) \leqslant 0$$

for $x \geqslant \tau$. Taking it into consideration and regarding $p(x) \geqslant 0$ on I from Lemma 6 we get

$$\liminf_{x\to\infty} \frac{y(x)}{xy'(x)} \geqslant \tfrac{1}{2}.$$

Thus there exists for the number $\theta \in (0, 1)$ such a $T > \tau$ $(T > 0)$, that

$$\frac{y(x)}{xy'(x)} > \frac{\theta}{2} \quad \text{for } x > T,$$

hence

$$\frac{y(x)}{y'(x)} > \frac{x}{2}\theta \quad \text{for } x > T. \tag{7}$$

Consider now the equation

$$v'' + p(x)v' + \left[q(x) + \frac{y(x)}{y'(x)} r(x)\right]v = 0 \quad (x > T). \tag{8}$$

Since the function $v = y'(x) > 0$ is a solution of this equation on (T, ∞), equation (8) is disconjugated on (T, ∞). On account of it regarding inequality (7) by Sturm Comparison Theorem the differential equation (6) is also disconjugated on (T, ∞). But this is in a contradiction with the assumption, which completes the proof.

COROLLARY 8. *Let* $q(x) \geqslant 0$, $r(x) \geqslant 0$ *for* $x \in I$ *and let* (P) *hold. If the second order differential equation*

$$v'' + p(x)v' + q(x)v = 0$$

is oscillatory on I, *then the differential equation* (a) *is oscillatory on* I, *too.*

Remark 3. By Theorem 7 it follows Theorem 3.1 in [10]. Besides it Theorem 7 completes Theorem 2 in [5].

The proof of the following theorem and corollary is similar to the proof of Theorem 6 and Corollary 5 (using Theorem 2, Corollary 3, respectively).

THEOREM 8. *Let* $(\overline{\mathrm{P}})$ *hold and let the second order differential equation*

$$v'' + p(x)v' + \tfrac{1}{4}q(x)v = 0$$

be oscillatory on I. *Then equation* (a) *is oscillatory on* I.

COROLLARY 9. *Let* $q(x) \geqslant 0$ *on* I *and let* $(\overline{\mathrm{P}})$ *hold. If the differential equation* (a) *is non-oscillatory on* I, *then for an arbitrary non-trivial solution* u *of* $(\overline{\mathrm{b}})$ *with the property* $\overline{F}x_0[u(x_0)] \leqslant 0$ *at some number* $x_0 \in I$ *there exists such a number* $\tau \in I$ *that* $u(x)u'(x) > 0$ *for* $x > \tau$.

Remark 4. From Theorem 6 and Theorem 8 follows Theorem 1 of [8].

References

[1] I. M. Dolan, *On the relationship between the oscillatory behavior of a linear third-order differential equation and its adjoint,* J. Diff. Eqs. 7 (1970), p. 367–388.

[2] M. Gera, *Einige Integralbedingungen für das Nichtoszillieren der linearen Differentialgleichung dritter Ordnung,* Acta F.R.N. Univ. Commen. Math. 31 (1975), p. 73–102.

[3] —, *Allgemeine Bedingungen der Nichtoszillationsfähigkeit und der Oscillationsfähigkeit für die lineare Differentialgleichung dritter Ordnung* $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$, Mat. Čas. 20 (1970), p. 49–61.

[4] —, *Nichtoszillatorische und oszillatorische Differentialgleichungen dritter Ordnung,* Čas. pěstov. mat. 96 (1971), p. 278–293.

[5] —, *Bedingungen für die Existenz oszillatorischer Lösungen der Gleichung* $x''' + a(t)x'' + b(t)x' + c(t)x = 0$, $c(t) \geqslant 0$, Mat. Čas. 25 (1975), p. 23–40.

[6] M. Greguš, *Bands of solutions of a differential equation of the third order with continuous coefficients and their application,* Acta Math. Univ. Comen. 39 (1980), p. 67–73.

[7] —, *Über die lineare homogene Differentialgleichung dritter Ordnung,* Wiss. Z. Univ. Halle, Math. Nat. 12 (1963), p. 265–268.

[8] G. D. Jones, *Oscillation criteria for third order differential equations,* SIAM J. Math. Anal. 7 (1976), p. 13–15.

[9] V. A. Kondrat'ev, *An elementary deduction of the necessary and sufficient condition of unoscillability of solution of the second order linear differential equations*, Usp. Mat. Nauk 12, 3 (75) (1957), p. 159–160 (in Russian).
[10] A. C. Lazer, *The behavior of solutions of the differential equation* $y''' + p(x)y' + q(x)y = 0$, Pacific J. Math. 17 (1966), p. 435–466.
[11] M. Ráb, *Oscilační vlastnosti integrálu diferenciální rovnice 3 řádu*, Práce Brněnské základny Českoslov. akademie věd. 27 (1955), p. 349–360.

DEPARTMENT OF MATHEMATICAL ANALYSIS MFFUK
BRATISLAVA, CZECHOSLOVAKIA

Reçu par la Rédaction le 9.02.1981