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On d-characteristic and d_Ξ-characteristic of linear operators

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Let X be a linear space over complex or real scalars and let A be a linear operator transforming X into itself. Let

$$Z_A = \{x\colon Ax = 0\}, \quad R_A = \{y\colon y = Ax\}.$$

The pair of numbers (α_A, β_A) where $\alpha_A = \dim Z_A$, $\beta_A = \dim X/R_A$ (by X/R_A we denote the quotient space) is called a *d-characteristic*. If both numbers are finite, we say that the operator A has a *finite* d-characteristic. The number $\varkappa_A = \beta_A - \alpha_A$ (well determined if at least one of the numbers α_A and β_A is finite) is called an *index* of the operator A ([2], [3], [4]). There is also another concept of dimensional characteristic of linear operator. We consider simultaneously with the space X some space Ξ of linear functionals ξ defined on X. We shall assume that Ξ is total, i.e. if $\xi x = 0$ for all $\xi \in \Xi$, then $x = 0$. By $Z^\Xi_{A^*}$ we denote the set

$$Z^\Xi_{A^*} = \{\xi \in \Xi\colon \xi Ax = 0 \text{ for all } x \in X\}.$$

The pair of numbers (α_A, β^Ξ_A), where $\beta^\Xi_A = \dim Z^\Xi_{A^*}$ is called a *d_Ξ-characteristic*. The number $\varkappa^\Xi_A = \beta^\Xi_A - \alpha_A$ is called a *Ξ-index* of A. Obviously $\beta^\Xi_A \leqslant \beta_A$ and $\varkappa^\Xi_A \leqslant \varkappa_A$.

In this note connections between d-characteristic and d_Ξ-characteristic are considered.

If $\Xi = X$, i.e. Ξ is equal to the space of all linear functionals, then the d_Ξ-characteristic of A is equal to its d-characteristic. It is also true in the case where X is a linear topological locally convex space, $\Xi = X^+$, i.e. Ξ is equal to the space of all linear continuous functionals, and A is *normally solvable*, i.e. it is a closed operator [1] such that R_A is closed.

[1] Operator A is *closed* if its graph is closed.

Basing themselves on this fact I. C. Gochberg and M. G. Krein [3] have determined d-characteristic for normally solvable operators as d_{Ξ^+}-characteristic. Obviously, d-characteristic and d_{Ξ}-characteristic are not needed equal.

EXAMPLE 1. Let X be a space $C[0,1]$ of all continuous functions $x = x(t)$ determined on segment $[0,1]$. Let Ξ be a space of functionals ξ of type

$$\xi x = \int_0^1 x(t)\xi(t)\,dt$$

where $\xi(t)$ is a continuous function. It is easy to check that Ξ is a total space of functionals.

Let $y(t) = A[x(t)] = x(t) - x(0)$. This operator transforms X into itself. The set R_A is a set of functions belonging to $C[0,1]$ and such that $x(t) \in R_A$ if and only if $x(0) = 0$. This implies that the d-characteristic of A is equal to $(1,1)$. On the other hand, $\beta_A^{\Xi} = 0$, because if $\xi y = 0$ for all $y \in R_A$, then $\xi = 0$. Hence the d_{Ξ}-characteristic is not equal to the d-characteristic.

In this example, however, the space Ξ is not preserved by the conjugate operator A^*, i.e. by the operator $A^*(\xi)$ determined by the equality $A^*(\xi)x = \xi(Ax)$. In the following example the space Ξ is also preserved by the conjugate operator.

EXAMPLE 2. Let $X = C^{\infty}[0,1]$ be a space of all infinitely differentiable functions determined on the segment $[0,1]$. Let Ξ be a space of functionals ξ if the type

$$\xi x = \int_0^1 x(t)\xi(t)\,dt$$

where $\xi(t)$ is an infinitely differentiable function such that the function itself and all its derivates are equal to zero at the point zero. It is easy to prove that Ξ is a total space. Let A be the following operator:

$$y(t) = A[x(t)] = \int_t^1 x(\tau)\,d\tau\,.$$

It is easy to see that R_A is a space of all functions $y(t)$ infinitely differentiable and such that $y(1) = 0$. This implies that the d-characteristic of A is equal to $(0,1)$. Since, if $\xi \in \Xi$ and $\xi(Ax) = 0$ for all x, then $\xi = 0$, we have $\beta_A^{\Xi} = 0$ and the d_{Ξ}-characteristic is equal to $(0,0)$.

Further,

$$\xi(Ax) = \int_0^1 \xi(t)\Big(\int_t^1 x(\tau)\,dt\Big)\,dt = \int_0^1 \Big(\int_0^t \xi(t)\,dt\Big)\,x(\tau)\,d\tau\,;$$

therefore $(\xi A)\tau = \int_0^\tau \xi(t)\,dt$ and the conjugate operator A^* is invertible and transforms Ξ onto itself.

Let X be a linear space. Let Ξ be a total space of functionals. Let A transforming X into itself be such that the conjugate operator A^* transforms Ξ into Ξ. From the definition of the d_Ξ-characteristic it follows that $\alpha_A = \beta^X_{A^*}$ and $\beta^\Xi_A = \alpha_{A^*}$, i.e. that if a pair of numbers (a, b) is the d_Ξ-characteristic of operator A, then the paiı (b, a) is the d_X-characteristic of operator A^* considered in Ξ.

As it is shown by example 2, this is not true for d-characteristic, because in this case the d-characteristic of A is equal to $(0, 1)$ and the d-characteristic of A^* is equal to $(0, 0)$.

We do not know an example of this kind when X and Ξ are complete normed spaces and A is continuous.

As we see, for different total spaces Ξ and Ξ_0 the d_Ξ-characteristic is not needed equal to the d_{Ξ_0}-charactaristic. But the following theorem is true:

THEOREM 1. *Let X be a linear space and let Ξ be a total space of functionals determined on X. Let T be such an operator transforming X into X that the conjugate operator T^* transforms Ξ into Ξ. The d_Ξ-characteristic of operator $A = I+T$ determined on the space X is equal to the d_{Ξ_0}-characteristic of the operator A considered only on X_0, where X_0, Ξ_0 are arbitrary spaces such that $TX \subset X_0 \subset X$ and $\Xi T^* \subset \Xi_0 \subset \Xi$.*

The proof is trivial. It follows from the fact that each solution of equation $Ax = (I+T)x = 0$ belonging to X belongs to X_0, and respectively every solution of equation $\xi A = \xi(I+T) = 0$ belonging to Ξ belongs also to Ξ_0.

Now we will apply this theorem to the integral equation

$$x(s) + \int_0^1 T(s, t)x(t)\,dt = x_0(s)\,, \tag{1}$$

where $T(s, t)$ and $x_0(t)$ are continuous functions, and the given equation is considered in the space $C[0, 1]$ of all continuous functions on segment $[0, 1]$. The operator $Tx = \int_0^1 T(s, t)x(t)\,dt$ is compact ([2]) ([1], p. 98), whence the operator $A = I+T$ is normally solvable and the numbers α_A and $\beta^{X^+}_A$ are both finite and equal ([1], p. 151-161). In this case the space X^+ of all continuous linear functionals is the space of functionals ξ of type

$$\xi x = \int_0^1 x(t)\,d\xi(t)$$

([2]) An equivalent term is „complete continuous".

where $\xi(t)$ is a function with bounded variation ([1], p. 59). But

$$\xi Tx = \int_0^1\int_0^1 T(s,t)x(t)\,dt\,d\xi(s) = \int_0^1 x(t)\int_0^1 T(s,t)\,d\xi(s)\,dt\,.$$

Hence the conjugate operator T^* transforms the space of all continuous functionals X^+ into the space $\widetilde{C}$ of all functionals η of the type

$$\eta x = \int_0^1 x(t)\eta(t)\,dt$$

where $\eta(t)$ is a continuous function.

Therefore, applying Theorem 1, we find that the $d_{\widetilde{C}}$-characteristic of $A = I+T$ is equal to the d_{X^+}-characteristic and that the $\widetilde{C}$-index is equal to 0. If we remark that A is normally solvable, we obtain a classical formulation of Fredholm's Alternative ([6], ch. II).

In a similar way we find that if $T(s,t)$ satisfies the Hölder inequality or is k-times differentiable, infinitely differentiable or analytic, then equation (1) satisfies Fredholm's Alternative, if as space X we assume the corresponding space of functions of one variable and as Ξ we assume the family of functionals

$$\xi x = \int_0^1 \xi(t)x(t)\,dt$$

where $\xi(t)$ belongs to a corresponding class.

There are also operators T transforming X into X such that the d_{Ξ}-characteristic of operator $A = I+T$ is equal to the d-characteristic for each Ξ. Indeed, this occurs if T is a finite dimensional operator, i.e.

$$T = \sum_{i=1}^{n} x_i\xi_i(x), \qquad \text{where } x_i \in X \text{ and } \xi_i \in \Xi\,.$$

The proof is the same as the proof for an integral equation with a degenerate kernel ([6], p. 61-64). This implies

THEOREM 2. *Let X be a Banach space. Let T be a compact operator mapping X into X, approximable by finite dimensional operators. Let Ξ be an arbitrary total space of continuous functionals determined on X and preserved by a conjugate operator T^*. Then the d_{Ξ}-characteristic is equal to the d-characteristic for every Ξ.*

The proof is the same as the classical proof of Fredholm's Alternative based on the approximation of continuous kernels by degenerate kernels ([5], p. 33-38). These considerations are the same for all Ξ.

Unfortunately we do not know whether it is possible to approximate compact operators by finite dimensional operators in every Banach space X. It is possible if in the space X there is a basis. But we do not know whether there is a basis in every separable Banach space ([1], p. 111).

As an application of Theorem 2 we will consider the integral equation

$$x(t)+\int_0^1 K(t,s)x(s)ds = y(t) \tag{2}$$

where $x(t)$ and $y(t)$ are continuous functions on segment $[0,1]$ and $K(t,s) = k(t,s)K_0(t-s)$ where $k(t,s)$ is continuous and $K_0(u)$ is a non-negative, summable and even function. It is easy to check that the transformation

$$Kx = \int_0^1 K(t,s)x(s)ds$$

is compact in the space $C[0,1]$ of continuous functions defined on segment $[0,1]$.

Basing ourselves on the Theorem 2 we can formulate Fredholm's Alternative for equation (2), where the conjugate equation

$$x(t)+\int_0^1 K(s,t)x(s)ds = y(t)$$

is considered also in the space of continuous functions.

In the particular case $K_0(u) = 1/|u|^a$ $(0 < a < 1)$ we obtain Fredholm's Alternative for a weakly singular equation without using the method of iteration.

References

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