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Solutions of two types of differential equations

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A general solution of the equations

$$(1)\qquad (a_1x^2+b_1x+c_1)^n\frac{\partial^n z}{\partial x^n}+(a_2y^2+b_2y+c_2)^k\frac{\partial^k z}{\partial y^k}+Dz=0\,,$$

$$(2)\qquad (a_1x^2+b_1x+c_1)^n(a_2y^2+b_2y+c_2)^k\frac{\partial^{n+k} z}{\partial x^n\partial y^k}+Dz=0$$

is given in this article. The coefficients $a_i\neq 0$, b_i, c_i $(i=1,2)$, D are real numbers, and the orders n, k are positive integers.

Remark 1. By means of (1) and (2) it is possible to find the solutions of the equations

$$(1\mathrm{a})\qquad f(x)\frac{\partial^n z}{\partial x^n}+g(y)\frac{\partial^k z}{\partial y^k}+Dz=0\,,$$

$$(2\mathrm{a})\qquad f(x)g(y)\frac{\partial^{n+k} z}{\partial x^n\partial y^k}+Dz=0$$

in a neighbourhood of the point $P(x,y)$ if the functions $\sqrt[n]{f(x)}$, $\sqrt[k]{g(y)}$ are real at this point and if they can be expanded into Taylor series which we reduce to the first three terms $\sqrt[n]{f(x)}=c_1+b_1x+a_1x^2$, $\sqrt[k]{g(y)}=c_2+ +b_2y+a_2y^2$; at least a_1, a_2 must be distinct from zero.

Remark 2. From the following can be seen that our problem may be extended immediately to analogous equations with any number of independent variables.

(A) The preliminary problem. First let us consider the problem: to give the form of the nth derivative of the function

$$(3)\qquad y=\exp\left\{\int\frac{\alpha x+\beta}{ax^2+bx+c}\,dx\right\},$$

where a, b, c, α, β are constants.

To this purpose, let us define the "deformed nth power" of the binomial $ax+\beta$ and let us denote this power by a bold dot placed in front of the binomial: $^{\bullet}(ax+\beta)^n$.

DEFINITION. Let $i \leqslant k$ be non-negative integers; let n be a natural number and put

$$(4)\qquad \begin{aligned} a_i^k &= (a-ia)(a-\overline{i+1}\cdot a)(a-\overline{i+2}\cdot a)\dots(a-\overline{k-1}\cdot a)\,, \\ a_i^i &= 1; \end{aligned}$$

$$(5)\qquad \begin{aligned} \beta_0^k &= \beta^{k-1}(\beta_0-\overline{k-1}\cdot b)+(k-1)c\beta_0^{k-2}a_{k-2}^{k-1}\,, \\ \beta_0^0 &= 1; \end{aligned}$$

$$(6)\qquad {}^{\bullet}(a^i\beta^k) = \beta_0^k a_k^{i+k}\,,$$

$$(7)\qquad {}^{\bullet}(ax+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} {}^{\bullet}(a^{n-k}\beta^k)x^{n-k}\,.$$

Remark 3. An especially simple form is assumed by the "deformed power" for $c=0$; e.g.,

$$\begin{aligned} {}^{\bullet}(ax+\beta)^3 = a(a-a)(a-2a)x^3+3\beta(a-a)(a-2a)x^2+ \\ +3\beta(\beta-b)(a-2a)x+\beta(\beta-b)(\beta-2b)\,. \end{aligned}$$

THEOREM. *For the n-th derivative of the exponential function* (3) *we have*

$$(8)\qquad y^{(n)} = \frac{{}^{\bullet}(ax+\beta)^n}{(ax^2+bx+c)^n}\,y\,.$$

Proof of the theorem.

LEMMA. *Let $i \leqslant k$ be non-negative integers. Then we get*

$$(9)\qquad a_i^k = a_{i+1}^{k+1}+(k-i)\,a\,a_{i+1}^k\,.$$

Proof. According to the relation (4) the formula holds

$$\begin{aligned} a_{i+1}^{k+1}+(k-i)aa_{i+1}^k &= a_{i+1}^k(a-ka)+(k-i)aa_{i+1}^k \\ &= a_{i+1}^k(a-ia) = a_i^{i+1}a_{i+1}^k = a_i^k\,. \end{aligned}$$

Let us suppose now, using the method of complete induction, that (8) holds for orders $1, 2, \dots, n$. Then, by differentiating the equation (8), we obtain after a modification:

$$y^{(n)} = \frac{{}^{\bullet}(ax+\beta)^n[(a-2an)x+\beta-bn]+[{}^{\bullet}(ax+\beta)^n]'(ax^2+bx+c)}{(ax^2+bx+c)^{n+1}}\,y\,.$$

Thus we have prove that the following identity is valid

$$\cdot(ax+\beta)^{n+1} = \cdot(ax+\beta)^n[(a-2an)x+\beta-bn]+[\cdot(ax+\beta)^n]'(ax^2+bx+c)\,.$$

It is, however, valid if and only if on both sides of the equation the coefficients of the power x^{n-k} are equal; $k=-1, 0, \dots, n$.

On the left side there is the member $\binom{n+1}{k+1}\cdot(a^{n-k}\beta^{k+1})x^{n-k}$ — according to (7). The coefficient of the power x^{n-k} on the right side of the equation is of the form

$$\Big[\dots+\binom{n}{k+1}\cdot(a^{n-k-1}\beta^{k+1})x^{n-k-1}+\binom{n}{k}\cdot(a^{n-k}\beta^k)x^{n-k}+\dots\Big]\times$$
$$\times[(a-2an)x+\beta-bn]+\Big[\dots+(n-k-1)\binom{n}{k+1}\cdot(a^{n-k-1}\beta^{k+1})x^{n-k-2}+$$
$$+(n-k)\binom{n}{k}\cdot(a^{n-k}\beta^k)x^{n-k-1}+(n-k+1)\binom{n}{k-1}\cdot(a^{n-k+1}\beta^{k-1})x^{n-k}+\dots\Big]\times$$
$$\times(ax^2+bx+c)\ (^1)\,.$$

Thus it is to be proved that the following equation is satisfied

$$\binom{n+1}{k+1}\cdot(a^{n-k}\beta^{k+1})x^{n-k}=\Big\{\binom{n}{k+1}\cdot(a^{n-k-1}\beta^{k+1})[a-2an+(n-k-1)a]+$$
$$+\binom{n}{k}\cdot(a^{n-k}\beta^k)[\beta-bn+(n-k)b]+\binom{n}{k-1}\cdot(a^{n-k+1}\beta^{k-1})c(n-k+1)\Big\}x^{n-k}\,.$$

We modify the coefficient of the right side

$$\binom{n}{k+1}\beta_0^{k+1}a_{k-1}^n[a-(n+k+1)a]+\binom{n}{k}\beta_0^k a_k^n(\beta-kb)+\binom{n}{k-1}\beta_0^{k-1}a_{k-1}^n c(n-k+1)$$
$$=\binom{n}{k+1}\beta_0^{k+1}\underbrace{a_{k+1}^n(a-na)}_{=a_{k+1}^{n+1},\ \text{see (4)}}+\binom{n}{k+1}\beta_0^{k+1}a_{k+1}^n[-(k+1)a]+$$
$$+\binom{n}{k}\underbrace{[\beta_0^k(\beta-kb)+kc\beta_0^{k-1}a_{k-1}^k]}_{=\beta_0^{k+1},\ \text{see (5)}}a_k^n$$
$$=\binom{n}{k+1}\beta_0^{k+1}a_{k+1}^{n+1}-\frac{n!}{k!(n-k-1)!}\beta_0^{k+1}a_{k+1}^n a+\binom{n}{k}\beta_0^{k+1}\underbrace{[a_{k+1}^{n+1}+(n-k)aa_{k+1}^n]}_{=a_n^k,\ \text{see (9)}}$$
$$=\binom{n+1}{k+1}\beta_0^{k+1}a_{k+1}^{n+1}\,,$$

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(1) Evidently, if the indices are $k=-1,\ 0$, we write $\binom{n}{-2}=\binom{n}{-1}=0$.

which, according to (6), is actually the coefficient of the left side of the above equation.

We have proved: If the theorem holds for the order n, it also holds for the order $n+1$. It remains to be shown that the theorem holds for the order 1.

By differentiating the function (3) we obtain

$$y' = \frac{\alpha x + \beta}{ax^2 + bx + c} y .$$

Since, according to definition, the formula ${}^{\bullet}(\alpha x + \beta)^1 = \alpha x + \beta$ holds, it is evident that for $n = 1$ the theorem is true.

Thus the proof is completed.

(B) Solutions of the equations (1), (2). Let us suppose that the solution of the equations (1), (2) has the form

$$(10) \qquad z = \exp\left\{\int \frac{\alpha_1 x + \beta_1}{a_1 x^2 + b_1 x + c_1} dx\right\} \exp\left\{\int \frac{\alpha_2 y + \beta_2}{a_2 y^2 + b_2 y + c_2} dy\right\},$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$, are constants not yet determined. For computing any partial derivative $\frac{\partial^{n+k} z}{\partial x^n \partial y^k}$ it is, naturally, possible to use the theorem from paragraph (A). Therefore, by substituting (10) into the equations (1), (2), we obtain after a small modification

$$(11.1) \qquad {}^{\bullet}(\alpha_1 x + \beta_1)^n + {}^{\bullet}(\alpha_2 y + \beta_2)^k + D = 0 ,$$

$$(11.2) \qquad {}^{\bullet}(\alpha_1 x + \beta_1)^n \cdot {}^{\bullet}(\alpha_2 y + \beta_2)^k + D = 0 .$$

Now, for α_1, α_2, we substitute these values

$$(12) \qquad \alpha_1 = (n-1) a_1 , \qquad \alpha_2 = (k-1) a_2 .$$

In this way we eliminate the variables x, y from the equations (11) — which we can see from (4), (6) and (7) — and consequently, we obtain characteristic equations, i.e., algebraic equations for the not yet determined constants β_1, β_2

$$(13.1) \qquad (\beta_1)_0^n + (\beta_2)_0^k + D = 0 ,$$

$$(13.2) \qquad (\beta_1)_0^n \cdot (\beta_2)_0^k + D = 0 .$$

These equations can be written down according to (5).

Further operations are familiar to every reader.

Remark 4. If $b_i^2-4a_ic_i>0$, $i=1,2$, and if there are tables of the values of Euler's function beta at our disposal, then, particularly in the case $n\geqslant 3$, $k\geqslant 3$, it is possible to proceed in the following way.

We introduce a substitution for the independent variables

$$x=\xi+\lambda\,,\qquad y=\eta+\varkappa\,, \tag{14}$$

where the constants λ, $\varkappa$ satisfy:

$$a_1\lambda^2+b_1\lambda+c_1=0\,,\qquad a_2\varkappa^2+b_2\varkappa+c_2=0\,.$$

Let us denote the given equations (1), (2), after the substitution (14)

$$(A_1\xi^2+B_1\xi)^n\frac{\partial^n z}{\partial\xi^n}+(A_2\eta^2+B_2\eta)^k\frac{\partial^k z}{\partial\eta^k}+Dz=0\,, \tag{1*}$$

$$(A_1\xi^2+B_1\xi)^n(A_2\eta^2+B_2\eta)^k\frac{\partial^{n+k}z}{\partial\xi^n\partial^k\eta}+Dz=0\,. \tag{2*}$$

Then the characteristic equations (13) can be, according to (5) and with regard to the denotation of the constants in (1*), (2*), written as follows

$$\beta_1(\beta_1-B_2)\dots(\beta_1-\overline{n-1}\cdot B_1)+\beta_2(\beta_2-B_2)\dots(\beta_2-\overline{k-1}\cdot B_2)+D=0\,, \tag{15.1}$$

$$\beta_1(\beta_1-B_1)\dots(\beta_1-\overline{n-1}\cdot B_1)\beta_2(\beta_2-B_2)\dots(\beta_2-\overline{k-1}\cdot B_2)+D=0\,. \tag{15.2}$$

If we, afterwards, write

$$\beta_1\colon B_1=p\,,\qquad \beta_2\colon B_2=q\,, \tag{16}$$

it is possible to express, after a modification, the equations (15) by means of the function gamma

$$\frac{1}{(B_2)^k}\cdot\frac{\Gamma(p+1)}{\Gamma(p-n+1)}+\frac{1}{(B_1)^n}\cdot\frac{\Gamma(q+1)}{\Gamma(q-k+1)}+\frac{D}{(B_1)^n(B_2)^k}=0\,,$$

$$\frac{\Gamma(p+1)}{\Gamma(p-n+1)}\cdot\frac{\Gamma(q+1)}{\Gamma(q-k+1)}+\frac{D}{(B_1)^n(B_2)^k}=0\,,$$

and, finally, after a further modification, by means of the function beta

$$\frac{1}{(B_2)^k\Gamma(k)B(p-n+1,n)}+\frac{1}{(B_1)^n\Gamma(n)B(q-k+1,k)}+\frac{D}{(B_1)^n(B_2)^k\Gamma(n)\Gamma(k)}=0\,, \tag{17.1}$$

$$\frac{1}{B(p-n+1,n)}\cdot\frac{1}{B(q-k+1,k)}+\frac{D}{(B_1)^n(B_2)^k\Gamma(n)\Gamma(k)}=0\,. \tag{17.2}$$

If we now choose a certain value for q, for instance, we easily determine, using the tables, the corresponding value for p. Hence, through (16) we obtain the pair of values β_1, β_2 to be determined.

Thus it is evident that the tables of the values of the function beta make it possible for us to solve the characteristic equation without having to compute the roots of an algebraic equation of a degree higher than two.

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