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On the oscillation of $ty''+p(t)y = 0$ with $\int p$ almost periodic*

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Abstract. The equation of the title is oscillatory on $[a, \infty)$, $a > 0$, if $M\{P^2\} - (M\{P\})^2 > \frac{1}{4}$ and is non-oscillatory on $[a, \infty)$ if $M\{P^2\}-(M\{P\})^2 < \frac{1}{4}$, where $P(t) = \int^t P$ and $M\{Q\}$ denotes in general the mean value of Q as an almost periodic function. The consequence of this result when $p(t) = \sum_{n=1}^{\infty} a_n \sin(\beta_n t + \delta_n)$ is described, and a generalization to $y''+[a(t)p(t)+\beta(t)]y = 0$ is indicated.

A second order differential equation

$$y''+q(t)y = 0 \tag{1}$$

with $q(t)$ a continuous function is said to be *oscillatory on an interval I* if every solution not identically zero has an infinite number of zeros in I. Sobol [2] proved that (1) is oscillatory on the real line R provided that there exists a primitive $\int q$ which is an almost periodic non-constant function. Markus and Moore [1] proved that (1) is oscillatory on R provided that $q(t)$ is an almost periodic function with mean value

$$M\{q\} = \lim_{T\to\infty} \frac{1}{T} \int_0^T q(s)\,ds$$

non-negative.

The point of this note is to discuss the oscillation problem for equations of the form

$$ty''+p(t)y = 0 \tag{2}$$

with $\int p$ almost periodic on R. Perhaps the simplest non-trivial equation of the form (2) is

$$ty''+ya\sin\beta t = 0, \tag{3}$$

which we proved in [3] to be oscillatory on $[a, \infty)$, $a > 0$, if $(a/\beta)^2 > \frac{1}{2}$ and non-oscillatory on $[a, \infty)$ if $(a/\beta)^2 < \frac{1}{2}$. Wong [6] later showed that

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(3) is non-oscillatory on $[a, \infty)$ if $(\alpha/\beta)^2 = \frac{1}{2}$. The following results show that the oscillatory behavior for general equations of the form (2) is typlified by the oscillatory behavior just described for (3).

THEOREM 1. *If* $P(t) = \int^t p$ *is an almost periodic function on* R *and* $a > 0$, *then* (2) *is oscillatory on* $[a, \infty)$ *provided*

$$M\{P^2\} - (M\{P\})^2 > \tfrac{1}{4} \tag{4}$$

and non-oscillatory on $[a, \infty)$ *provided*

$$M\{P^2\} - (M(P\})^2 < \tfrac{1}{4}. \tag{5}$$

COROLLARY 1. *If* $p(t)$ *is an almost periodic function on* R *with Fourier series* $\sum_{n=1}^{\infty} A_n e^{i\nu_n t}$ $(A_n = M\{p(t)e^{-i\nu_n t}\})$ *and* $|\nu_n| \geqslant \nu > 0$ $(n = 1, \ldots)$, *then* (2) *is oscillatory on* $[a, \infty)$ $(a > 0)$ *provided*

$$\sum_{n=1}^{\infty} |A_n/\nu_n|^2 > \tfrac{1}{4}$$

and non-oscillatory on $[a, \infty)$ $(a > 0)$ *provided*

$$\sum_{n=1}^{\infty} |A_n/\nu_n|^2 < \tfrac{1}{4}.$$

COROLLARY 2. *If* $\sum_{n=1}^{\infty} \alpha_n$ *and* $\sum_{n=1}^{\infty} \alpha_n/\beta_n$ *are absolutely convergent series and* $a > 0$, *then* (2) *with*

$$p(t) = \sum_{n=1}^{\infty} \alpha_n \sin(\beta_n t + \delta_n)$$

is oscillatory on $[a, \infty)$ *if*

$$\sum_{n=1}^{\infty} \left|\frac{\alpha_n}{\beta_n}\right|^2 > \tfrac{1}{2}$$

and non-oscillatory on $[a, \infty)$ *if*

$$\sum_{n=1}^{\infty} \left|\frac{\alpha_n}{\beta_n}\right|^2 < \tfrac{1}{2}.$$

The oscillation problem for equation (2) in the case of equality in (4) and (5) is in general an open question. However, those equations in this case for which there exists a constant $\varepsilon > 0$ such that

$$\text{(6)}\qquad \begin{aligned} &\lim_{t\to\infty} t^{\varepsilon}\Big[M\{P\} - \frac{1}{t}\int_0^t P(s)\,ds\Big] = 0,\\ &\lim_{t\to\infty} t^{\varepsilon}\Big[M\{P^2\} - \frac{1}{t}\int_0^t P^2(s)\,ds\Big] = 0, \end{aligned}$$

can be shown to be non-oscillatory by using a theorem of J. S. W. Wong ([6], Theorem 5, p. 206).

Theorem 1 has applications to the oscillation problem for (2) when $p(t)$ is an almost periodic function. In this case, Theorem 1 is applicable provided $p(t)$ has an almost periodic primitive $P(t)$, which is the case if and only if $P(t)$ is bounded. In addition, if $M\{p\} > 0$, then (2) is oscillatory by the well-known Fite–Wintner theorem ([4], Corollary 3.1, p. 603). Thus, in this case there occurs the additional open problem when $M\{p\} \leqslant 0$ and $\int^t p$ is unbounded.

Proof of Theorem 1. Let $O(1)$ and $o(1)$ be generic symbols denoting a continuous function $f(t)$ with the property that $|f(t)| \leqslant C$ on $[a, \infty)$ for some constant C and $f(t) \to 0$, as $t \to \infty$, respectively. Let $\mu = M\{P^2\}$, $\nu = M\{P\}$ and

$$r(t) = \frac{1}{t}\int_0^t P(s)\,ds - \nu.$$

Then

$$Q(t) \equiv \int_t^\infty \frac{p(s)}{s}\,ds = \frac{\nu - P(t) - r(t)}{t} + 2\int_t^\infty \frac{r(s)}{s^2}\,ds = \frac{\nu - P(t) + o(1)}{t},$$

$$\int^t Q(s)\,ds = -r(t) + 2t\int_t^\infty \frac{r(s)}{s^2}\,ds = o(1).$$

Theorem 1 is a consequence of Theorems 1.4 and 1.5 of [3], p. 180 (Corollary 5.3 of [4], p. 615, or Corollary 3.2 of [5], p. 362), which state that (2) is oscillatory provided there exists an $\varepsilon > 0$ such that

$$\text{(6)}\qquad 4\int_t^\infty \Big(\int_s^\infty Q^2(\tau)\,d\tau\Big)^2 ds \geqslant (1+\varepsilon)\int_t^\infty Q^2(s)\,ds,$$

and (2) is non-oscillatory provided

$$\text{(7)}\qquad 4\int_t^\infty \Big(\int_s^\infty Q^2(\tau)\,d\tau\Big)^2 ds \leqslant (1-\varepsilon)\int_t^\infty Q^2(s)\,ds.$$

In our case here,

$$\int_t^\infty Q^2(s)\,ds = \frac{\mu-\nu^2+o(1)}{t}, \qquad \int_t^\infty \Big(\int_s^\infty Q^2(\tau)\,d\tau\Big)^2 ds = \frac{(\mu-\nu^2)^2+o(1)}{t},$$

and so (4) implies (6) and (5) implies (7).

Proof of Corollary 1. It has been established in the theory of almost periodic functions that the primitives $P(t)$ of an almost periodic function is an almost periodic function provided that the Fourier exponents γ_k satisfy $|\gamma_k| \geqslant \gamma > 0$. Thus, $p(t)$ has an almost periodic primitive $P(t)$ with Fourier series $\sum_{n=1}^{\infty} \frac{A_n}{i\nu_n} e^{i\nu_n t}$, which implies that $M\{P\} = 0$ and

$$M\{P^2\} = \sum_{n=1}^{\infty} |A_n/\nu_n|^2$$

by Parseval's equality.

Proof of Corollary 2. The function

$$P(t) = -\sum_{n=1}^{\infty} \frac{a_n}{\beta_n}\cos(\beta_n t+\delta_n)$$

is bounded by $\sum_{n=1}^{\infty} |a_n/\beta_n|$, hence, $P(t)$ is an almost periodic function on R. Furthermore, $P(t)$ has Fourier coefficients

$$M\{P\},\ \frac{a_n}{2\beta_n}e^{-\delta_n i},\ -\frac{a_n}{2\beta_n}e^{\delta_n i} \qquad (n = 1, \ldots);$$

thus, Parseval's equality implies

$$M\{P^2\} = (M\{P\})^2 + \tfrac{1}{2}\sum |a_n/\beta_n|^2.$$

Inspection of the proof of Theorem 1 indicates that the almost periodicity of $P(t)$ is quite incidental, all that was required was the existence of $M\{P\}$ and $M\{P^2\}$. We reflect this fact as well as some further generalizations in the following theorem.

THEOREM 2. *Assume that* β, $p \in C[a, \infty)$ $(a > 0)$, $P(t) = \int^t p$, $\nu \equiv M\{P\}$ *and* $\mu \equiv M\{P^2\}$ *exist,* $t^2\beta(t) \to \lambda$ *as* $t \to \infty$, $a \in C^2[a, \infty)$, $t^3 a''(t) = O(1)$, $a'(t) \to 0$ *and* $ta(t) \to 1$ *as* $t \to \infty$. *If*

$$\mu+\lambda-\nu^2 > \tfrac{1}{4},$$

then

$$y'' + [a(t)p(t)+\beta(t)]y = 0 \tag{8}$$

is oscillatory on $[a, \infty)$, *and if*

$$\mu+\lambda-\nu^2 < \tfrac{1}{4},$$

then (8) *is non-oscillatory on* $[a, \infty)$.

Proof. The situation here differs from the situation described in Theorem 1 primarily in the fact that here

$$\int^{t} Q(s)\,ds \equiv \int^{t}\Big[\int_{s}^{\infty}\big(a(\tau)p(\tau)+\beta(\tau)\big)d\tau\Big]\,ds = \lambda\log t+o(1);$$

hence, Theorems 1.1 and 1.2 of [3], p. 178 (Corollary 5.3 of [4], p. 615) can be applied.

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