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An example of a continuous function without the usual, the approximative and the distributional derivative

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Abstract. A real continuous function defined on the whole real line is constructed. Then it is proved that the function does not possess a finite usual, approximative or distributional derivative (in the sense of Definition 1 and 2 below) at any point of its domain.

1. Introduction. The classical examples of a continuous function without a finite derivative at any point of its domain, given by Weierstrass, van der Waerden and others, can be found in many text-books on mathematical analysis or theory of real functions (cf. for instance Sikorski [2], X, § 8).

It is natural to ask whether it is possible to construct an example of such a continuous function which, moreover, does not possess a finite approximative or distributional derivative at any point of its domain in the sense of the following definitions.

Definition 1 (cf. R. Sikorski [2], X, § 9). We say that a function $g\colon I \to \boldsymbol{R}$, where I is an interval in $\boldsymbol{R} = (-\infty, +\infty)$, has an approximative derivative $k \in \boldsymbol{R}$ at a point $x_0 \in I$ if and only if there is a set $F \subset I$ such that

(i) x_0 is a point of density of F, i.e.,

$$\lim_{\substack{x_0 \in Q \\ |Q| \to 0}} \frac{m(Q \cap F)}{|Q|} = 1,$$

where $|Q|$ denotes the length of an interval Q, m denotes the Lebesgue measure,

(ii) $\dfrac{g(x_0+h)-g(x_0)}{h} \to k$ as $h \to 0$ and $x_0+h \in F$.

Definition 2. We say that a function *g has the distributional derivative $k \in \boldsymbol{R}$* at a point x_0 if and only if the distribution g' has the value

k at the point x_0; i.e. (cf. S. Łojasiewicz [1])

$$\lim_{a\to 0} g'(x_0+ax) = k.$$

The first example of a continuous function without the distributional derivative at any point of its domain was given by Z. Zieleźny [4]. In [3] H. Świątak has constructed a class of continuous functions defined on $\boldsymbol{R}$ without the usual, the approximative and the distributional derivative at any point $x_0 \in \boldsymbol{R}$.

In this paper we also give (in Section 2) a construction of a continuous function defined on $\boldsymbol{R}$ which does not possess the usual, the approximative and the distributional derivative at any point $x_0 \in \boldsymbol{R}$. The idea of this construction is very geometrical and it is quite different from that used in [3]. Moreover, our function does not enter into the classes considered in [3] and in [4].

We start with the geometrical interpretation of the behaviour of the difference quotient. Namely, the necessary and sufficient condition for the existence of a finite derivative $g'(x_0) = k$ is that

$$\frac{g(x_0+ax)-g(x_0)}{a} \to kx$$

almost uniformly (i.e. uniformly on each compact set) when $a\to 0$. Hence we obtain the following

LEMMA 1. *If $g'(x_0) = k \in \boldsymbol{R}$, then the convergence*

$$\lambda\left[g\left(x_0+\frac{x}{\lambda}\right)-g(x_0)\right]\to kx \quad \textit{when } \lambda\to+\infty \tag{1.1}$$

is almost uniform (a.u.).

It also appears (see Lemma 4 and Lemma 5 in Section 3) that convergence (1.1) in measure and in the distributional sense is a necessary condition for the existence, respectively, of the approximative and the distributional derivative $g'(x_0) = k \in \boldsymbol{R}$ (we adopt the same notation for the derivative in all cases).

Thus, in order to obtain a function f with the desired property it suffices to construct it in such a way that the graph of its homothetic function $x\to\lambda[f(x_0+x/\lambda)-f(x_0)]$ (where x_0 is any fixed point of the domain of f) should pass through two "disjoint" (in the sense of Lemma 3) parallelograms $\boldsymbol{R}_{1\lambda}, \boldsymbol{R}_{2\lambda}$ for $\lambda = \lambda_\nu$, $\nu = 1, 2, \ldots$, where $\{\lambda_\nu\}$ is some sequence increasing to infinity ($\lambda_\nu \nearrow +\infty$).

This will imply that convergence (1.1) (with g replaced by f) is impossible in each sense under consideration (neither a.u. nor in measure nor in the distributional sense).

The construction of the function f is given in Section 2. Using some auxiliary facts obtained in Section 3, we prove in Section 4 that the constructed function f has the required property.

This result was essentially obtained by the author in the diplome paper. The author thanks Professor S. Łojasiewicz for indicating the problem and suggesting the idea of construction.

2. Construction. Let ε be a fixed positive number and let Δ denote the interval $[0, 1]$. We fix positive numbers a and λ_1 and suppose that they fulfil the inequalities

$$a \geqslant 4\varepsilon, \quad \lambda_1 > 1. \tag{2.1}$$

Now we define by induction a sequence of functions $\{f_\mu(\cdot)\}_{\mu=1,2,\ldots}$: we put

$$f_1(x) = \frac{a}{\lambda_1}\sin\lambda_1 2\pi x, \quad x \in \boldsymbol{R},$$

and supposing that the function

$$f_{\mu-1}(x) = \sum_{\nu=1}^{\mu-1} \frac{a}{\lambda_\nu}\sin\lambda_\nu 2\pi x, \quad x \in \boldsymbol{R}$$

with some $\lambda_\nu > 0$ is already defined, we construct f_μ as follows: We fix a point $x_0 \in \boldsymbol{R}$. Notice that $f_{\mu-1}$ has a finite derivative at the point x_0:

$$f'_{\mu-1}(x_0) = \sum_{\nu=1}^{\mu-1} 2\pi a\cos\lambda_\nu 2\pi x_0 =: k_{\mu-1}.$$

Hence and from Lemma 1 it follows that for our ε and Δ there is $\tilde{\lambda}_\mu$ such that

$$\left|\lambda\left[f_{\mu-1}\left(x_0+\frac{x}{\lambda}\right) - f_{\mu-1}(x_0)\right] - k_{\mu-1}\cdot x\right| < \varepsilon \quad \text{for } \lambda \geqslant \tilde{\lambda}_\mu,\ x \in \Delta. \tag{2.2}$$

We may find such $\tilde{\lambda}_\mu$ by means of the mean-value theorem:

$$\begin{aligned}
&\left|\lambda\left[f_{\mu-1}\left(x_0+\frac{x}{\lambda}\right) - f_{\mu-1}(x_0)\right] - k_{\mu-1}\cdot x\right| \\
&= \left|\lambda\cdot 2\pi\frac{x}{\lambda}\sum_{\nu=1}^{\mu-1}\frac{a}{\lambda_\nu}\cdot\lambda_\nu\cos\left(\lambda_\nu 2\pi\left(x_0+\frac{\Theta_{\mu-1}\cdot x}{\lambda}\right)\right) - 2\pi a x\sum_{\nu=1}^{\mu-1}\cos\lambda_\nu 2\pi x_0\right| \\
&= \left|2\pi a x\cdot 2\pi\frac{\Theta_{\mu-1}\cdot x}{\lambda}\sum_{\nu=1}^{\mu}\lambda_\nu\sin\left(\lambda_\nu\cdot 2\pi\left(x_0+\frac{\Theta'_{\mu-1}\Theta_{\mu-1}\cdot x}{\lambda}\right)\right)\right| \\
&\leqslant 4\pi^2 a\frac{\Theta_{\mu-1}\cdot x^2}{\lambda}\sum_{\nu=1}^{\mu-1}\lambda_\nu,
\end{aligned}$$

where $\Theta_{\mu-1}$, $\Theta'_{\mu-1}$ are suitable constants from the interval $(0, 1)$.

Thus, in order that inequality (2.2) be fulfilled, it sufficies to put

$$\tilde{\lambda}_\mu = \frac{4\pi^2 a}{\varepsilon}\sum_{\nu=1}^{\mu-1}\lambda_\nu. \tag{2.3}$$

Then, setting

$$\lambda_\mu = \max\left(\tilde{\lambda}_\mu,\, 2^\mu\lambda_{\mu-1}\cdot\frac{a}{\varepsilon}\right) \tag{2.4}$$

we find that

$$\lambda_\mu > \lambda_{\mu-1} > 1, \tag{2.5}$$

since $a \geqslant 4\varepsilon$ and $\lambda_1 > 1$; also, we have

$$\frac{a}{\lambda_\mu} \leqslant \frac{\varepsilon}{2^\mu\cdot\lambda_{\mu-1}}. \tag{2.6}$$

Now we put

$$f_\mu(x) = f_{\mu-1}(x) + \frac{a}{\lambda_\mu}\cdot\sin\lambda_\mu 2\pi x = \sum_{\nu=1}^{\mu}\frac{a}{\lambda_\mu}\sin\lambda_\nu 2\pi x; \qquad x\in\boldsymbol{R}, \tag{2.7}$$

and the process of construction can be continued. We thus obtain a sequence $\{f_\mu\}_{\mu=1,2,\ldots}$, which is uniformly convergent on $\boldsymbol{R}$, because its majorant $\sum_{\nu=1}^{\infty}\varepsilon/2^\nu$ is convergent in view of (2.5) and (2.6). Hence the function

$$f(x) = \sum_{\nu=1}^{\infty}\frac{a}{\lambda_\nu}\sin\lambda_\nu 2\pi x; \qquad x\in\boldsymbol{R}, \tag{2.8}$$

is continuous as the limit of the uniformly convergent sequence of continuous functions.

Remark 2.1. Though the point x_0 was used in the construction of the function f_μ, it follows from the construction that neither f_μ nor f depends on x_0; thus x_0 could have been chosen arbitrarily in $\boldsymbol{R}$.

3. Auxiliary results. To show that the continuous function f given by (2.8) has the required properties we need some lemmas.

LEMMA 2. *If l is an integer, then the following inequalities are fulfilled:*

$$\sin(\gamma+2\pi x)-\sin\gamma \leqslant -1 \quad \text{for } \gamma\in[2\pi l, 2\pi l+\tfrac{1}{2}\pi),\ x=\tfrac{3}{4} \quad \text{or for } \gamma\in[2\pi l+\tfrac{1}{2}\pi, 2\pi l+\pi),\ x=\tfrac{1}{4}, \tag{3.1}$$

$$\sin(\gamma+2\pi x)-\sin\gamma \geqslant 1 \quad \text{for } \gamma\in[2\pi l+\pi, 2\pi l+\tfrac{3}{2}\pi),\ x=\tfrac{3}{4} \quad \text{or for } \gamma\in[2\pi l+\tfrac{3}{2}\pi, 2\pi l+2\pi),\ x=\tfrac{1}{4}. \tag{3.2}$$

The proof is straightforward and elementary, and therefore, omitted.

Before we formulate the next lemma, we give the following

DEFINITION 3. We say that a real function g defined on a set containing an interval $[p, q]$ passes through the parallelogram

$$\mathscr{R} = \{(x, y) \in R^2; p \leqslant x \leqslant q, kx + A \leqslant y \leqslant kx + B\}$$

(with some constants $k, A, B \in R$, $A < B$) if the inequality

$$kx + A \leqslant g(x) \leqslant kx + B$$

is satisfied for $x \in [p, q]$.

The lemma which follows is very simple, but it is crucial for the proof of the theorem below.

LEMMA 3. *If a function g passes through two parallelograms*

$$\mathscr{R}_1 = \{(x, y) \in R^2; p \leqslant x \leqslant q, kx - A \leqslant y \leqslant kx + A\},$$

$$\mathscr{R}_2 = \{(x, y) \in R^2; r \leqslant x \leqslant s, kx - C \leqslant y \leqslant kx - B\}$$

$$(or\ \mathscr{R}_2' = \{(x, y) \in R^2; r' \leqslant x \leqslant s', kx + B' \leqslant y \leqslant kx + C'\}),$$

where the constants k, A, B, C, p, q, r, s (B', C', r', s') satisfy the inequalities

$$0 < A < B < C \quad (0 < A < B' < C'),$$

$$\varepsilon \leqslant B - A \quad (\varepsilon \leqslant B' - A),$$

$$0 < r < s < p < q \quad (0 < r' < s' < p < q),$$

then for any linear function $y = mx$ we have

$$(3.3)\quad |mx - g(x)| \geqslant \frac{\varepsilon}{2} \quad \textit{either for } x \in [p, q] \textit{ or for } x \in [r, s] \ (x \in [r', s']).$$

Proof. We prove the lemma only for the pair $\mathscr{R}_1, \mathscr{R}_2$ of parallelograms, since for the pair $\mathscr{R}_1, \mathscr{R}_2'$ the proof is quite analogous.

We set $M = (p, kp - A)$, $N = (s, ks - B)$. It is easily seen that the graph of the linear function

$$y = \left(k - \frac{A + B}{p + s}\right)x$$

passes through the origin and through the point

$$\tfrac{1}{2}(M + N) = \left(\frac{p + s}{2}, \frac{kp - A + ks - B}{2}\right).$$

Now we observe that

$$\left(k-\frac{A+B}{p+s}\right)x-(kx-B)\geqslant B-\frac{A+B}{2s}\cdot s=\frac{B-A}{2}\geqslant\frac{\varepsilon}{2}\quad \text{for } x\in[r,s],$$

$$(kx-A)-\left(k-\frac{A+B}{p+s}\right)x\geqslant \frac{A+B}{2p}\cdot p-A=\frac{B-A}{2}\geqslant\frac{\varepsilon}{2}\quad \text{for } x\in[p,q].$$

Hence and from the assumption that

$$g(x)\leqslant kx-B \quad \text{if } x\in[r,s]\ (g \text{ passing through } \mathscr{R}_2),$$

$$g(x)\geqslant kx-A \quad \text{if } x\in[p,q]\ (g \text{ passing through } \mathscr{R}_1)$$

we get

$$\left(k-\frac{A+B}{p+s}\right)x-g(x)\geqslant\frac{\varepsilon}{2}\quad \text{when } x\in[r,s],$$

$$g(x)-\left(k-\frac{A+B}{p+s}\right)x\geqslant\frac{\varepsilon}{2}\quad \text{when } x\in[p,q],$$

which proves the lemma in the case $m=k-\dfrac{A+B}{p+s}$. To complete the proof it suffices to observe that for m different from $\left(k-\dfrac{A+B}{p+s}\right)$ inequality (3.3) has to be satisfied either for $x\in[r,s]$ or for $x\in[p,q]$.

Now, we give two lemmas which provide a necessary condition for the existence of the finite approximative and the distributional derivative, respectively, at a point x_0.

LEMMA 4. *If a real function g defined on $\boldsymbol{R}$ possesses the approximative derivative equal to $k\in\boldsymbol{R}$ at a point x_0, then*

$$\lambda\left[g\left(x_0+\frac{x}{\lambda}\right)-g(x_0)\right]\xrightarrow{(m)}kx \quad \textit{as } \lambda\to\infty, \tag{3.4}$$

where the symbol $\xrightarrow{(m)}$ denotes the convergence in measure in any finite interval Δ.

Proof. For the proof we need to show that

$$\lim_{\lambda\to\infty} m\big(\{x\in\Delta;\ |\lambda[g(x_0+x/\lambda)-g(x_0)]-kx|\geqslant\varepsilon\}\big)=0 \tag{3.5}$$

for any $\varepsilon>0$ and any compact interval Δ (m denotes the Lebesgue measure).

According to Definition 1, the assumption of the lemma implies that there is a set $F\subset\boldsymbol{R}$ such that x_0 is the point of density of F, so we have

$$\lim_{\substack{|Q|\to 0\\ x_0\in Q}}\frac{m(Q\setminus F)}{|Q|}=0 \quad (Q \text{ is an interval}). \tag{3.6}$$

Similarly, by condition (ii) of Definition 1 we get

$$\frac{g(x_0+x/\lambda)-g(x_0)}{x/\lambda}\to k, \qquad \text{as } \frac{x}{\lambda}\to 0,\ x_0+\frac{x}{\lambda}\in F.$$

Hence, for any $\varepsilon>0$ and any finite interval Δ there is $\tilde{\lambda}$ such that

(3.7) $|\lambda[g(x_0+x/\lambda)-g(x_0)]-kx|<\varepsilon$ if $\lambda\geqslant\tilde{\lambda}$, $x\in\Delta$ and $x_0+x/\lambda\in F$.

Let us fix $\varepsilon>0$ and $\Delta=[u,w]$. Suppose that $\tilde{\lambda}$ in (3.7) is found just for these ε and Δ. We set

$$v=\max(|u|,|w|).$$

From (3.6) it follows that for $\eta/2v$ there is $\delta>0$ such that

(3.8) $$m(Q\setminus F)<\frac{\eta}{2v}\cdot|Q| \quad \text{if only } |Q|<\delta \text{ and } x_0\in Q,$$

where η is a positive constant.

Let us consider the family of intervals

$$Q_\lambda=[x_0-v/\lambda,\ x_0+v/\lambda] \quad (\lambda>0).$$

It is easily seen that

(3.9) $$x_0\in Q_\lambda, \quad x_0+x/\lambda\in Q_\lambda \quad \text{for } x\in\Delta,\ |Q_\lambda|=2v/\lambda.$$

Setting

$$\lambda_0=\max(\tilde{\lambda},2v/\delta),$$

we obtain by (3.7)

$$|\lambda[g(x_0+x/\lambda)-g(x_0)]-kx|<\varepsilon \quad \text{for } \lambda\geqslant\lambda_0,$$

if $x\in\Delta$, $x_0+x/\lambda\in F$. Hence, the condition

$$|\lambda[g(x_0+x/\lambda)-g(x_0)]-kx|\geqslant\varepsilon \quad \text{for } \lambda\geqslant\lambda_0,\ x\in\Delta$$

mplies that

$$x_0+x/\lambda\in Q_\lambda\setminus F$$

and, consequently,

(3.10) $$m(\{x\in\Delta;|\lambda[g(x_0+x/\lambda)-g(x_0)]-kx|\geqslant\varepsilon\})\leqslant\lambda\cdot m(Q_\lambda\setminus F) \quad \text{for } \lambda\geqslant\lambda_0.$$

In turn, by (3.9) we get

$$|Q_\lambda|<\delta \quad \text{for } \lambda\geqslant\lambda_0,$$

which, owing to (3.8) and (3.9), yields the estimation

$$\lambda\cdot m(Q_\lambda\setminus F)\leqslant\lambda\cdot\frac{\eta}{2v}|Q_\lambda|=\eta.$$

The last inequality together with (3.10) completes the proof of the lemma.

LEMMA 5. *If a continuous real function g defined on $\boldsymbol{R}$ possesses the distributional derivative equal to $k \in \boldsymbol{R}$ at a point x_0, then*

$$\lambda[g(x_0+x/\lambda)-g(x_0)] \quad \text{converges to } kx \quad \text{as } \lambda\to\infty$$

in the distributional sense, i.e.,

$$\int_{-\infty}^{+\infty} \lambda[g(x_0+x/\lambda)-g(x_0)]\cdot\varphi(x)\,dx \to \int_{-\infty}^{+\infty} kx\cdot\varphi(x)\,dx, \tag{3.11}$$

as $\lambda\to\infty$, $\varphi\in\mathcal{D}$, where $\mathcal{D}$ denotes the set of all infinitely derivable functions on $\boldsymbol{R}$ with compact supports.

This lemma is a version of Lemma II proved by H. Świątak in paper [3].

4. The main result. Now we are in a position to prove the following

THEOREM. *The function f given by* (2.8), *with a and λ_ν satisfying conditions* (2.1) *and* (2.4) (*with μ replaced by ν*), *is continuous on $\boldsymbol{R}$ and does not possess a finite usual, approximative or distributional derivative at any point $x_0\in\boldsymbol{R}$.*

Proof. The continuity of f was already proved in Section 2. For the second part of the assertion let us consider the expression

$$\lambda[f(x_0+x/\lambda)-f(x_0)] = \lambda F_1(x;\lambda)+\lambda F_2(x;\lambda)+\lambda F_3(x;\lambda), \tag{4.1}$$

where x_0 is a fixed point of $\boldsymbol{R}$, $\lambda>0$, and

$$F_1(x;\lambda) = \sum_{\nu=1}^{\mu-1}\frac{a}{\lambda_\nu}\sin\left(\lambda_\nu 2\pi\left(x_0+\frac{x}{\lambda}\right)\right)-\sum_{\nu=1}^{\mu-1}\frac{a}{\lambda_\nu}\sin\lambda_\nu 2\pi x_0,$$

$$F_2(x;\lambda) = \frac{a}{\lambda_\mu}\sin\left(\lambda_\mu 2\pi\left(x_0+\frac{x}{\lambda}\right)\right)-\frac{a}{\lambda_\mu}\sin\lambda_\mu 2\pi x_0,$$

$$F_3(x;\lambda) = \sum_{\nu=\mu+1}^{\infty}\frac{a}{\lambda_\nu}\sin\left(\lambda_\nu 2\pi\left(x_0+\frac{x}{\lambda}\right)\right)-\sum_{\nu=\mu+1}^{\infty}\frac{a}{\lambda_\nu}\sin\lambda_\nu 2\pi x_0.$$

In view of (2.6) we obtain the estimation:

$$\lambda|F_3(x;\lambda)|\leqslant 2\frac{\lambda}{\lambda_\mu}\sum_{\nu=\mu+1}^{\infty}\frac{\varepsilon}{2^\nu} = \frac{\lambda}{\lambda_\mu}\,\frac{\varepsilon}{2^{\mu-1}},\quad x\in\boldsymbol{R},$$

and so we have

$$\lambda|F_3(x;\lambda)|\leqslant\frac{\lambda}{\lambda_\mu}\cdot\frac{\varepsilon}{4}\quad \text{for } \mu\geqslant 3,\ x\in\boldsymbol{R}. \tag{4.2}$$

Owing to (2.2) (we may consider the point x_0 in Section 2 as that fixed above — compare Remark 2.1) and (2.4) we get

$$|\lambda F_1(x;\lambda)-k_{\mu-1}x|<\varepsilon \quad \text{for } \lambda\geqslant\lambda_\mu,\ x\in\Delta=[0,1]. \tag{4.3}$$

Directly from the definition of $F_2(x,\lambda)$ we have

$$\lambda_\mu F_2(1;\ \lambda_\mu)=0. \tag{4.4}$$

Similarly, from the definition of $F_2(x,\lambda)$ and from (2.1) we obtain, by Lemma 2, the inequalities:

$$\begin{aligned}
&\lambda_\mu F_2(\tfrac{1}{4};\lambda_\mu)\leqslant -4\varepsilon && \text{if } \lambda_\mu 2\pi x_0\in[2\pi l+\tfrac{1}{2}\pi,\ 2\pi l+\pi),\\
&\lambda_\mu F_2(\tfrac{1}{4};\lambda_\mu)\geqslant 4\varepsilon && \text{if } \lambda_\mu 2\pi x_0\in[2\pi l+\tfrac{3}{2}\pi,\ 2\pi l+2\pi),\\
&\lambda_\mu F_2(\tfrac{3}{4};\lambda_\mu)\leqslant -4\varepsilon && \text{if } \lambda_\mu 2\pi x_0\in[2\pi l,\ 2\pi l+\tfrac{1}{2}\pi),\\
&\lambda_\mu F_2(\tfrac{3}{4};\lambda_\mu)\geqslant 4\varepsilon && \text{if } \lambda_\mu 2\pi x_0\in[2\pi l+\pi,\ 2\pi l+\tfrac{3}{2}\pi),
\end{aligned} \tag{4.5}$$

where l is an integer.

Since the function $x\to a\sin 2\pi x$ has a bounded derivative, it is uniformly continuous on $\boldsymbol{R}$ and we have the implication

$$|x-x'|<\delta=\frac{\varepsilon}{4\cdot 2\pi a}\ |\Rightarrow\ |a\sin 2\pi x-a\sin 2\pi x'|<\tfrac{1}{4}\varepsilon. \tag{4.6}$$

Hence and from (4.4) we get

$$\lambda_\mu|F_2(x;\lambda_\mu)|\leqslant\tfrac{1}{4}\varepsilon \quad \text{for } x\in[1-\varepsilon/8a\pi,\ 1]. \tag{4.7}$$

Similarly, from (4.6) and from (4.5) we obtain

$$\begin{aligned}
&\lambda_\mu F_2(x;\lambda_\mu)\leqslant -4\varepsilon+\tfrac{1}{4}\varepsilon \quad \text{if } \lambda_\mu\cdot 2\pi x_0\in[2\pi l,\ 2\pi l+\tfrac{1}{2}\pi]\\
&\qquad\qquad \text{and } x\in\left[\tfrac{3}{4}-\frac{\varepsilon}{8a\pi},\ \tfrac{3}{4}+\frac{\varepsilon}{8a\pi}\right] \text{ or if}\\
&\qquad \lambda_\mu\cdot 2\pi x_0\in[2\pi l+\tfrac{1}{2}\pi,\ 2\pi l+\pi] \text{ and } x\in\left[\tfrac{1}{4}-\frac{\varepsilon}{8a\pi},\ \tfrac{1}{4}+\frac{\varepsilon}{8a\pi}\right],\\
&\lambda_\mu F_2(x;\lambda_\mu)\geqslant 4\varepsilon-\tfrac{1}{4}\varepsilon \quad \text{if } \lambda_\mu\cdot 2\pi x_0\in[2\pi l+\pi,\ 2\pi l+\tfrac{3}{2}\pi]\\
&\qquad\qquad \text{and } x\in\left[\tfrac{3}{4}-\frac{\varepsilon}{8a\pi},\ \tfrac{3}{4}+\frac{\varepsilon}{8a\pi}\right] \text{ or if}\\
&\qquad \lambda_\mu\cdot 2\pi x_0\in[2\pi l+\tfrac{3}{2}\pi,\ 2\pi l+2\pi] \text{ and } x\in\left[\tfrac{1}{4}-\frac{\varepsilon}{8a\pi},\ \tfrac{1}{4}+\frac{\varepsilon}{8a\pi}\right],
\end{aligned} \tag{4.8}$$

where l is an integer.

Now, we choose a subsequence $\{\mu_\beta\}_{\beta=1,2,\ldots}$ of $\{\mu\}_{\mu=1,2,\ldots}$ in such a way that, for instance,

$$\lambda_{\mu_\beta}\cdot 2\pi x_0 \in [2\pi l, 2\pi l + \tfrac{1}{2}\pi] \quad \text{for } \beta = 1, 2, \ldots \tag{4.9}$$

(in all the remaining cases the reasoning would be the same). Thus, in view of (4.1) and from (4.7), (4.2), (4.3), in which λ_μ and λ are replaced by λ_{μ_β}, we obtain the inequality

$$\left|\lambda_{\mu_\beta}\left[f\left(x_0+\frac{x}{\lambda_{\mu_\beta}}\right)-f(x_0)\right]-k_{\mu_\beta-1}\cdot x\right| \leqslant \tfrac{3}{2}\varepsilon \tag{4.10}$$
$$\text{for } \mu_\beta \geqslant 3,\ x \in \left[1-\frac{\varepsilon}{8a\pi},\, 1\right].$$

Similarly, owing to (4.1) and from (4.8), (4.9), (4.2), (4.3), we have

$$k_{\mu_\beta-1}\cdot x - 2a - \tfrac{5}{4}\varepsilon \leqslant \lambda_{\mu_\beta}\left[f\left(x_0+\frac{x}{\lambda_{\mu_\beta}}\right)-f(x_0)\right] \leqslant k_{\mu_\beta-1}\cdot x - \tfrac{5}{2}\varepsilon \tag{4.11}$$
$$\text{for } \mu_\beta \geqslant 3,\ x \in \left[\tfrac{3}{4} - \frac{\varepsilon}{8a\pi},\, \tfrac{3}{4}+\frac{\varepsilon}{8a\pi}\right].$$

From (4.10) and (4.11) it follows that whenever $\mu_\beta \geqslant 3$, the graph of the function

$$x \to \lambda_{\mu_\beta}\left[f\left(x_0+\frac{x}{\lambda_{\mu_\beta}}\right)-f(x_0)\right]$$

passes (in the sense of Definition 3) through the following two parallelograms:

$$\mathscr{R}_{1\lambda_{\mu_\beta}} = \left\{(x, y) \in R^2;\ 1-\frac{\varepsilon}{8a\pi} \leqslant x \leqslant 1,\ k_{\mu_\beta-1}\cdot x - \tfrac{3}{2}\varepsilon \leqslant y \leqslant k_{\mu_\beta-1}\cdot x + \tfrac{3}{2}\varepsilon\right\},$$

$$\mathscr{R}_{2\lambda_{\mu_\beta}} = \left\{(x, y) \in R^2;\ \tfrac{3}{4} - \frac{\varepsilon}{8a\pi} \leqslant x \leqslant \tfrac{3}{4} + \frac{\varepsilon}{8a\pi},\ k_{\mu_\beta-1}\cdot x - 3a \leqslant y \leqslant k_{\mu_\beta-1}\cdot x - \tfrac{5}{2}\varepsilon\right\}.$$

These parallelograms satisfy the assumptions of Lemma 3 for any fixed $\mu_\beta \geqslant 3$ and for ε fixed in the construction of the function f (see Section 2).

Thus, by Lemma 3, for any linear function $y = kx$ ($k \in \boldsymbol{R}$) the inequality

$$\left|\lambda_{\mu_\beta}\left[f\left(x_0+\frac{x}{\lambda_{\mu_\beta}}\right)-f(x_0)\right]-kx\right| \geqslant \frac{\varepsilon}{2} \tag{4.12}$$

is satisfied either for $x \in \left[\tfrac{3}{4}-\frac{\varepsilon}{8a\pi},\, \tfrac{3}{4}+\frac{\varepsilon}{8a\pi}\right]$ or for $x \in \left[1-\frac{\varepsilon}{8a\pi},\, 1\right]$, provided μ_β is sufficiently large.

Hence and from Lemma 1 and Lemma 4, respectively, it follows that the function f does not possess a finite usual derivative or a finite approximative derivative at the point x_0.

Since the point x_0 was arbitrary in $\boldsymbol{R}$, the proof of the theorem in the case of the usual and the approximative derivative is completed.

In the remaining case of the distributional derivative, notice that we can choose another subsequence of $\{\mu_\beta\}_{\beta=1,2,\dots}$ (preserving notation for simplicity) in such a way that inequality (4.12) is fulfilled always in the same of the two intervals, say, the second. Thus without loss of generality we may assume that

$$\left|\lambda_{\mu_\beta}\left[f\left(x_0+\frac{x}{\lambda_{\mu_\beta}}\right)-f(x_0)\right]-kx\right| \geqslant \frac{\varepsilon}{2} \quad \text{if } x \in \left[1-\frac{\varepsilon}{8a\pi}, 1\right] \tag{4.13}$$

for μ_β sufficiently large and for any $k \in \boldsymbol{R}$.

Now, let $\varphi \in \mathcal{D}$ be such that

$$\varphi(x)\begin{cases} = 0 & \text{if } x \in \boldsymbol{R}\setminus\left(1-\dfrac{\varepsilon}{8a\pi}, 1\right), \\ > 0 & \text{if } x \in \left(1-\dfrac{\varepsilon}{8a\pi}, 1\right). \end{cases}$$

From (4.13) and from the continuity of f we obtain that for μ_β sufficiently large and for any $k \in \boldsymbol{R}$ the integral

$$\int_{-\infty}^{+\infty}\left(\lambda_{\mu_\beta}\left[f\left(x_0+\frac{x}{\lambda_{\mu_\beta}}\right)-f(x_0)\right]-kx\right)\cdot\varphi(x)\,dx$$

is either greater than the positive number M or less than the negative number $(-M)$, where

$$M = \frac{\varepsilon}{2}\int_{-\infty}^{+\infty}\varphi(x)\,dx.$$

Hence, condition (3.11) fails, so by Lemma 5 the function f does not possess a finite distributional derivative at the point x_0. Since the point x_0 was arbitrary in $\boldsymbol{R}$, the proof of the theorem is completed.

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