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An area inequality for quasi-subordinate analytic functions

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Abstract. Let $f(z)$, $F(z)$ be non-zero analytic functions in $|z| < 1$, $f(0) = F(0) = 0$. We say that $f(z)$ is *quasi-subordinate to* $F(z)$ if there exist two analytic functions $\varphi(z)$ and $\omega(z)$ such that $|\varphi(z)| \leqslant 1$, $|\omega(z)| < 1$, $\omega(0) = 0$ and $f(z) = \varphi(z) F(\omega(z))$.

Let $A(r, f)$ denote the area of the region on the Riemann surface onto which the disk $|z| < r$ is mapped by $f(z)$.

In this paper we prove that if f is quasi-subordinate to F, then

(i) $$A(r, f) \leqslant T(r) A(r, F).$$

The function $T(r)$ was determined by Reich [3].

The all pairs (f, F) for which equality holds in (i) are given.

I. Introduction. Let $f(z)$ and $F(z)$ be non-zero analytic functions in $|z| < 1$, $f(0) = F(0) = 0$. Then $f(z)$ is called *quasi-subordinate to* $F(z)$ if there exist two analytic functions $\varphi(z)$ and $\omega(z)$ such that $|\varphi(z)| \leqslant 1$, $|\omega(z)| < 1$, $\omega(0) = 0$ and $f(z) = \varphi(z) F(\omega(z))$. This definition was introduced by Robertson [4]. In particular, if $\varphi(z) \equiv 1$ ($\omega(z) \equiv z$) we have the concept of subordination (majorization).

Let $A(r, f)$ denote the area of the region on the Riemann surface onto which the disk $|z| < r$ is mapped by $f(z)$. Golusin [1] has shown that, under subordination, $A(r, f) \leqslant A(r, F)$ for $r \leqslant 1/\sqrt{2}$. Reich [3] has extended this result by showing that for $0 < r < 1$,

(1) $$A(r, f) \leqslant T(r) A(r, F),$$

where

(2) $$T(r) = mr^{2m-2}$$

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in the range

$$\frac{m-1}{m} \leqslant r^2 \leqslant \frac{m}{m+1} \qquad (m = 1, 2, \ldots).$$

He also finds, for each r, all pairs (f, F) for which equality holds in (1).

In this paper we show that (1) remains true for quasi-subordinate pairs. Furthermore, the pairs (f, F) for which equality holds (for some r) in (1) remain the same, with the exception of the case $r \leqslant 1/\sqrt{2}$.

II. THEOREM. *Let $f(z)$ be quasi-subordinate to $F(z)$ in $|z| < 1$, $f(0) = F(0) = 0$. Then $A(r,f) \leqslant T(r)A(r,F)$, with equality for a given r possible only under the following circumstances*:

$$\begin{array}{lll} & \text{(a)} & r^2 < \tfrac{1}{2},\ \varphi(z) \equiv \varepsilon \ \textit{and}\ \omega(z) = \eta z. \\ & \text{(b)} & r^2 = \tfrac{1}{2},\ \text{(i) } \textit{either}\ \varphi(z) \equiv \varepsilon \ \textit{and}\ \omega(z) = \eta z \ \textit{or} \\ (3) & & \qquad \text{(ii) } f(z) = C\eta z^2,\ F(z) = Cz \ \textit{or} \\ & & \qquad \text{(iii) } f(z) = C\varepsilon(az+z^2),\ F(z) = C(z+\bar{a}\bar{\eta}z^2). \\ & \text{(c)} & \dfrac{m-1}{m} < r^2 < \dfrac{m}{m+1},\ f(z) = C\eta z^m,\ F(z) = Cz \qquad (m \geqslant 2), \\ & \text{(d)} & r^2 = \dfrac{m}{m+1} \qquad (m \geqslant 2), \\ & & \qquad \textit{either}\ f(z) = C\eta z^m,\ F(z) = Cz \\ & & \qquad \textit{or} \quad f(z) = C\eta z^{m+1},\ F(z) = Cz. \end{array}$$

In the above and henceforth C denotes an arbitrary non-zero complex constant, $|a| < 1$, $|\varepsilon| = 1$, $|\eta| = 1$. Also, it is easy to check that each of the above cases yields equality in (1). The proof of the theorem will require several lemmas. We first adopt the following notation:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad F(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \omega(z) = \sum_{n=1}^{\infty} c_n z^n, \quad \varphi(z) = \sum_{n=0}^{\infty} d_n z^n.$$

It is well known that

$$A(r,f) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \qquad A(r,F) = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}.$$

LEMMA 1. *If $\omega(z) \not\equiv \eta z$, then $|a_1| < |b_1|$, $|c_1| < 1$.*

LEMMA 2. *$|d_1| \leqslant 1-|d_0|^2$, with equality only if $\varphi(z) = \varepsilon \dfrac{z-a}{1-\bar{a}z}$.*

LEMMA 3. $\sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leqslant \sum_{k=1}^{\infty} |b_k|^2 r^{2k}$.

LEMMA 4. *If* $s_n = \sum_{k=1}^{n} |a_k|^2$, $S_n = \sum_{k=1}^{n} |b_k|^2$, *then* $s_n \leqslant S_n$, *with equality for a particular* n *implying that* $a_{n+1} = d_0 c_1^{n+1} b_{n+1}$.

LEMMA 5. *If* $\{\lambda_k\}$ *are such that* $\lambda_k \geqslant 0$, $\lambda_k \geqslant \lambda_{k+1}$ $(k = 1, 2, \ldots)$, *then*

$$\sum_{k=1}^{\infty} \lambda_k |a_k|^2 \leqslant \sum_{k=1}^{\infty} \lambda_k |b_k|^2.$$

LEMMA 5.1. *Under the hypothesis of Lemma* 5, *and with* s_n *and* S_n *defined in Lemma* 4, *equality in Lemma* 5 *implies*

$$(\lambda_k - \lambda_{k+1})(S_k - s_k) = 0 \qquad (k = 1, 2, \ldots).$$

LEMMA 6. *If*

(1) $f(z) = a_1 z + a_2 z^2 \prec F(z) = b_1 z + b_2 z^2$,

(2) $|a_1| < |b_1|$,

(3) $|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2$,

then

$$f(z) = C\eta z^2, \qquad F(z) = Cz.$$

LEMMA 7. *If* $\varphi(z) = (Az+B)(Cz+D)^{-1}$ *is a fractional linear transformation of* $|z| < 1$ *into itself, then* φ *is onto if and only if*

$$|A|^2 + |B|^2 = |C|^2 + |D|^2.$$

LEMMA 8. (See Lemma 8 in Reich [3].)

Lemmas 1 and 2 are well-known consequences of the lemma of Schwarz. Lemma 5 follows from Lemma 4, by partial summation. Lemmas 5.1, 6, and 8 are in Reich [3]. It remains to prove Lemmas 3, 4, and 7.

Proof of Lemma 3. Define $G(z) = \sum_{n=1}^{\infty} g_n z^n$ by $G(z) = F(\omega(z))$. Since $G(z)$ is subordinate to $F(z)$, we may apply the inequality of Littlewood [2] to obtain

$$\sum_{k=1}^{\infty} |g_k|^2 r^{2k} \leqslant \sum_{k=1}^{\infty} |b_k|^2 r^{2k}. \tag{4}$$

From $f(z) = \varphi(z) G(z)$, it follows that $\int_{-\pi}^{\pi} |f(z)|^2 d\theta \leqslant \int_{-\pi}^{\pi} |G(z)|^2 d\theta$. Hence,

$$\sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leqslant \sum_{k=1}^{\infty} |g_k|^2 r^{2k}. \tag{5}$$

Thus, (4) and (5) yield the desired result.

Although the next result is due to Robertson [4], we offer a proof in order to examine the case of equality.

Proof of Lemma 4. Let

$$s_n(z) = \sum_{k=1}^{n} a_k z^k, \quad S_n(z) = \sum_{k=1}^{n} b_k z^k, \quad R_n(z) = \sum_{k=n+1}^{\infty} b_k z^k.$$

Then

$$s_n(z) + \sum_{n+1}^{\infty} a_k z^k = \varphi(z) F\big(\omega(z)\big) = \varphi(z) S_n\big(\omega(z)\big) + \varphi(z) R_n\big(\omega(z)\big)$$

$$= \varphi(z) S_n\big(\omega(z)\big) + \sum_{n+1}^{\infty} b_k^{(n)} z^k, \quad \text{where } b_{n+1}^{(n)} = d_0 c_1^{n+1} b_{n+1},$$

and so on. We have

$$\varphi(z) S_n\big(\omega(z)\big) = s_n(z) + \sum_{n+1}^{\infty} p_k^{(n)} z^k, \tag{6}$$

where $p_k^{(n)} = a_k - b_k^{(n)}$. In particular, $p_{n+1}^{(n)} = a_{n+1} - d_0 c_1^{n+1} b_{n+1}$.

Since $\varphi(z) S_n\big(\omega(z)\big)$ is quasi-subordinate to $S_n(z)$, Lemma 3 applied to (6) yields

$$\sum_{k=1}^{n} |a_k|^2 r^{2k} + \sum_{n+1}^{\infty} |p_k^{(n)}|^2 r^{2k} \leqslant \sum_{k=1}^{n} |b_k|^2 r^{2k}.$$

The proof of the lemma is complete, with equality implying $a_{n+1} = d_0 c_1^{n+1} b_{n+1}$.

Proof of Lemma 7. We first assume $\varphi(z)$ is onto. It is well known that $\varphi(z)$ is of the form

$$\varphi(z) = e^{it} \frac{z-a}{1-\bar{a}z}.$$

Since $|e^{it}|^2 + |-ae^{it}|^2 = 1 + |-\bar{a}|^2$, we are done.

Conversely, suppose

$$|A|^2 + |B|^2 = |C|^2 + |D|^2. \tag{7}$$

Now, $|\varphi(z)| < 1$ implies $D \neq 0$, so that we may write

$$\varphi(z) = \frac{B}{D} + \left(\frac{A}{D} - \frac{BC}{D^2}\right) z + \ldots, \quad \text{and} \quad |B| < |D|.$$

By Lemma 2, it suffices to show that

$$\left|\frac{A}{D} - \frac{BC}{D^2}\right| = 1 - \left|\frac{B}{D}\right|^2,$$

or, what is equivalent,

(8) $$|AD-BC|^2 = (|D|^2-|B|^2)^2.$$

Now, $|\varphi(z)|<1$ implies $|Ae^{i\theta}+B|^2 \leqslant |Ce^{i\theta}+D|^2, \forall\theta$.

Hence, from (7) we have

$$\mathrm{Re}[e^{i\theta}(\bar{D}C-\bar{B}A)] \geqslant 0, \ \forall\theta,$$

and so $\bar{D}C = \bar{B}A$.

This yields

(9) $$A\bar{B}\bar{C}D = |C|^2|D|^2$$

and

(10) $$\bar{A}BC\bar{D} = |A|^2|B|^2.$$

From (7), (9) and (10), we obtain

$$\begin{aligned}|AD-BC|^2 &= |A|^2|D|^2-2\mathrm{Re}(A\bar{B}\bar{C}D)+|B|^2|C|^2\\ &= |A|^2|D^2-\mathrm{Re}(A\bar{B}\bar{C}D)-\mathrm{Re}(\bar{A}BC\bar{D})+|B|^2|C|^2\\ &= |A|^2|D|^2-|C|^2|D|^2-|A|^2|B|^2+|B|^2|C|^2\\ &= |A|^2(|D|^2-|B|^2)-|C|^2(|D|^2-|B|^2) = (|D|^2-|B|^2)^2.\end{aligned}$$

It follows from (8) that the proof is complete.

We preface the proof of the theorem with a few remarks. For subordination, Reich obtained inequality (1) using only one consequence of subordination, namely

$$\sum_{k=1}^{\infty}|a_n|^2r^{2n} \leqslant \sum_{k=1}^{\infty}|b_n|^2r^{2n}.$$

Since this also holds under quasi-subordination (see Lemma 3), his proof of (1) is valid in our case. However, examination of the circumstances under which equality holds is facilitated by a repetition of the first portion of his proof.

Proof of Theorem. For any positive integer m,

$$\frac{A(r,f)}{\pi} = \Big[mr^{2m}\sum_{k=1}^{m-1}|a_k|^2+\sum_{m}^{\infty}k|a_k|^2r^{2k}\Big]-\Big[mr^{2m}\sum_{k=1}^{m-1}|a_k|^2-\sum_{k=1}^{m-1}k|a_k|^2r^{2k}\Big],$$

hence

(11) $$\frac{A(r,f)}{\pi} = \sum_{k=1}^{\infty}\lambda_k^{(m)}|a_k|^2-\sum_{k=1}^{m-1}[(mr^{2m}-kr^{2k})|a_k|^2],$$

where

$$\lambda_k^{(m)} = \begin{cases} mr^{2m} & \text{if } 1\leqslant k\leqslant m-1,\\ kr^{2k} & \text{if } k\geqslant m.\end{cases}$$

Similarly,

$$\frac{A(r,F)}{\pi} = \sum_{k=1}^{\infty} \lambda_k^{(m)} |b_k|^2 - \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |b_k|^2]. \tag{12}$$

By Lemmas 5 and 5.1, therefore,

$$\sum_{k=1}^{\infty} \lambda_k^{(m)} |a_k|^2 \leqslant \sum_{k=1}^{\infty} \lambda_k^{(m)} |b_k|^2, \tag{13}$$

$$S_k = s_k \quad \text{for all } k \geqslant m, \quad \text{providing} \quad \frac{m-1}{m} \leqslant r^2 < \frac{m}{m+1}, \tag{14}$$

or

$$S_k = s_k \quad \text{for all } k \geqslant m+1, \quad \text{providing} \quad r^2 = \frac{m}{m+1}. \tag{15}$$

Subtracting (12) from (11) gives, by (13),

$$\begin{aligned} &\frac{1}{\pi} [A(r,f) - A(r,F)] \\ &\leqslant \sum_{k=1}^{m-1} (mr^{2m} - kr^{2k}) |b_k|^2 - \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |a_k|^2], \end{aligned} \tag{16}$$

with equality possible only if conditions (14) and (15) are met.

By Lemma 8 (ii), (iii) and (iv) the last sum of (16) is non-negative, and can vanish only if

$$a_k = 0, \quad k = 1, 2, \ldots, m-1, \quad \text{providing} \quad \frac{m-1}{m} < r^2 \leqslant \frac{m}{m+1}, \tag{17}$$

or

$$a_k = 0, \quad k = 1, 2, \ldots, m-2, \quad \text{if} \quad r^2 = \frac{m-1}{m}. \tag{18}$$

Hence

$$\begin{aligned} \frac{1}{\pi} [A(r,f) - A(r,F)] &\leqslant \sum_{k=1}^{m-1} [(mr^{2m} - kr^{2k}) |b_k|^2] \\ &= (mr^{2m-2} - 1) \sum_{k=1}^{m-1} k |b_k|^2 r^{2k} - mr^{2m-2} \sum_{k=1}^{m-1} [(kr^{2k} - r^2) |b_k|^2]. \end{aligned} \tag{19}$$

By Lemma 8 (iv), the last sum of (19) is non-negative, and can vanish only if

$$b_2 = b_3 = \ldots = b_{m-1} = 0. \tag{20}$$

Therefore

$$\begin{aligned} \frac{1}{\pi} [A(r,f) - A(r,F)] &\leqslant (mr^{2m-1} - 1) \sum_{k=1}^{m-1} k |b_k|^2 r^{2k} \\ &\leqslant (mr^{2m-2} - 1) \sum_{k=1}^{\infty} k |b_k|^2 r^{2k} = (mr^{2m-2} - 1) \frac{A(r,F)}{\pi}, \end{aligned} \tag{21}$$

where the first inequality sign may reduce to an equality sign if (14), (15), (17), (18) and (20) hold, while the second inequality may become an equality only if either $m = 1$, or $r^2 = 1/2$, or $b_m = b_{m+1} = \ldots = 0$, $r^2 > 1/2$. From (21) we immediately obtain the desired relation (1). Only the need for examining the possibility of equality in (21) remains.

Collecting the available information for this case, we are led to (3) (c) when

$$\frac{m-1}{m} < r^2 < \frac{m}{m+1}, \qquad m \geqslant 2.$$

When $r^2 = \dfrac{m}{m+1}$, $m \geqslant 2$, on the other hand, we obtain

$$f(z) = a_m z^m + a_{m+1} z^{m+1}, \qquad F(z) = b_1 z,$$

with

$$(22) \qquad |b_1|^2 = |a_m|^2 + |a_{m+1}|^2.$$

Since $f(z)$ is quasi-subordinate to $F(z)$, clearly

$$\max_{|z|=1} |f(z)| \leqslant \max_{|z|=1} |F(z)|.$$

Hence, $\max\limits_{|z|=1} |a_m + a_{m+1} z| \leqslant |b_1|$, and an appropriate choice of z, $|z| = 1$, gives

$$(23) \qquad |a_m| + |a_{m+1}| \leqslant |b_1|.$$

Squaring (23) and subtracting (22), yields the conclusion that either $|a_m| = 0$ or $|a_{m+1}| = 0$, and therefore (3) (d) follows.

For $r^2 < 1/2$ it follows from (14) that $|a_n|^2 = |b_n|^2$, $\forall n \geqslant 1$. Choose n minimally so that $a_n \neq 0$. Then $a_n = d_0 c_1^n b_n$ so that $|d_0|\,|c_1|^n = 1$. Thus, $|d_0| = 1$, $|c_1| = 1$ and we have $\varphi(z) \equiv d_0$, $\omega(z) \equiv c_1 z$. The proof of (3) (a) is complete.

The only remaining case is $r^2 = 1/2$.

Subcase (i). Suppose $\varphi(z) \equiv \varepsilon$. We leave the trivial possibility of $\omega(z) = \eta z$, and assume $\omega(z) \not\equiv \eta z$. By Lemma 1, $|a_1| < |b_1|$ and $|c_1| < 1$. Applying Lemma 4 to (15) yields

$$(24) \qquad a_k = d_0 c_1^k b_k \quad (k = 3, 4, \ldots).$$

Now, (15) implies $|a_k| = |b_k|$ $(k = 3, 4, \ldots)$, while (24) forces $|a_k| < |b_k|$. Hence $|a_k| = |b_k| = 0$ $(k = 3, 4, \ldots)$, and $f(z) = a_1 z + a_2 z^2$, $F(z) = b_1 z + b_2 z^2$. Furthermore, $|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2$, by (15); and $f(z)$ is subordinate to $\varepsilon F(z)$. Hence, the hypotheses of Lemma 6 are satisfied, with $F(z)$ replaced by $\varepsilon F(z)$; and we conclude that $f(z) = C\eta z^2$, $F(z) = Cz$.

This is the statement of (3) (b) (ii).

Subcase (ii). Suppose $\varphi(z) \not\equiv \varepsilon$. Then $|d_0| < 1$, and (15) and (24) yield $a_k = b_k = 0$ $(k = 3, 4, \ldots)$, as before. Hence, $f(z) = a_1 z + a_2 z^2$, $F(z) = b_1 z + b_2 z^2$, with $|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2$.

By quasi-subordination,

$$|a_1 z + a_2 z^2| \leqslant |b_1 \omega(z) + b_2 \omega(z)^2| = |b_1 z \Phi(z) + b_2 z^2 \Phi(z)^2|,$$

where

$$\Phi(z) = \omega(z) z^{-1}; \quad |\Phi(z)| \leqslant 1 \quad \text{by the Lemma of Schwarz.}$$

Hence,

$$|a_1 + a_2 z|^2 \leqslant |b_1 \Phi(z) + b_2 z \Phi(z)^2|^2 \leqslant |b_1 + b_2 z \Phi(z)|^2. \tag{25}$$

Replacing z by $re^{i\theta}$, (25) is equivalent to

$$Q(r) \leqslant \operatorname{Re} g(re^{i\theta}), \tag{26}$$

where

$$Q(r) = \tfrac{1}{2}\big(|a_1|^2 - |b_1|^2 + r^2(|a_2|^2 - |b_2|^2)\big)$$

and

$$g(z) = z\big(\bar{b}_1 b_2 \Phi(z) - \bar{a}_1 a_2\big). \tag{27}$$

Note that $Q(r) \leqslant 0$ with $Q(1) = 0$. Hence, $\operatorname{Re} g(z) \geqslant 0$, $|z| < 1$, for if $\operatorname{Re} g(z_0) < 0$ simply choose r near 1 so that $\operatorname{Re} g(z_0) < Q(r)$; the minimum principle applied to (26) would then yield a contradiction. Since $g(0) = 0$ it follows that $g(z) \equiv 0$ so that from (27) we have

$$\bar{b}_1 b_2 \Phi(z) - \bar{a}_1 a_2 = 0,$$

or

$$\bar{b}_1 b_2 \omega(z) = \bar{a}_1 a_2 z.$$

We leave the case $\bar{b}_1 b_2 = 0$ to the reader, only remarking that its consideration leads either to $\varphi(z) \equiv \varepsilon$ or $\omega(z) = \eta z$. Thus, we now assume $\bar{b}_1 b_2 \neq 0$ and obtain $\omega(z) = \bar{a}_1 a_2 (\bar{b}_1 b_2)^{-1} z$. To show that the coefficient has unit modulus, we obtain from (25)

$$|a_1|^2 + 2\operatorname{Re}(\bar{a}_1 a_2 e^{i\theta}) + |a_2|^2 \leqslant |b_1|^2 + 2\operatorname{Re}\left(\bar{b}_1 b_2 \frac{\bar{a}_1 a_2}{\bar{b}_1 b_2} e^{i\theta}\right) + |b_2|^2 \left|\frac{\bar{a}_1 a_2}{\bar{b}_1 b_2}\right|^2,$$

or

$$|a_1|^2 + |a_2|^2 \leqslant |b_1|^2 + |b_2|^2 \left|\frac{\bar{a}_1 a_2}{\bar{b}_1 b_2}\right|^2,$$

or

$$|b_2|^2 \leqslant |b_2|^2 \left|\frac{\bar{a}_1 a_2}{\bar{b}_1 b_2}\right|^2.$$

Since $|\Phi(z)| \leqslant 1$, we conclude $\Phi(z) \equiv \eta$, or $\omega(z) \equiv \eta z$.

We then have $f(z) = \varphi(z)F(\eta z)$, where $\varphi(z) \not\equiv \varepsilon$ and $|a_1|^2+|a_2|^2 = |b_1|^2+|b_2|^2$. Thus,

$$\varphi(z) = \frac{a_1 z + a_2 z^2}{b_1 \eta z + b_2 \eta^2 z^2} = \bar{\eta}\,\frac{a_1 + a_2 z}{b_1 + b_2 \eta z}. \tag{28}$$

$\varphi(z)$ is a fractional linear transformation of $|z| < 1$ into itself, since if $a_1 b_2 \eta = a_2 b_1$, then $\varphi(z)$ would be a constant of modulus less then unity, thus contradicting $A\left(\frac{1}{\sqrt{2}}, f\right) = A\left(\frac{1}{\sqrt{2}}, F\right)$. Furthermore, $|\bar{\eta}a_1|^2+|\bar{\eta}a_2|^2 = |b_1|^2+|b_2\eta|^2$, and so we may apply Lemma 7 to obtain

$$\varphi(z) = e^{it}\frac{z+a}{1+\bar{a}z}.$$

From (28), it follows that $f(z)$ and $F(z)$ are allowed to be of the form

$$f(z) = C\varepsilon(az+z^2), \qquad F(z) = C(z+\bar{a}\bar{\eta}z^2).$$

The proof of (3) (b) (iii), and hence the proof of the theorem is complete.

As Reich [3] remarks, $T(r)$ may written in the compact form

$$T(r) = \max_{k=1,2,\ldots} (kr^{2k-2}), \qquad 0 < r < 1.$$

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