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Extended rotation of the covariant vector density

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Abstract. In the present note a notion of extended rotation G_{ab} of an arbitrary vector density w_i of weight $-p, p \neq 0$ is defined axiomatically (Definition 1).

Under assumption 1°, 2°, 3° it is proved that such a rotation has the form

$$G_{ab} = C^i_{ab} w_i + w_{[a,b]},$$

where C^i_{ab} is an object with transformation formula

$$C^{i'}_{a'b'} = A^{i'}_i A^a_{a'} A^b_{b'} C^i_{ab} + p(\ln|J|)_{,[a'}\delta^{i'}_{b']}.$$

The object C^i_{ab} can be prescribed in the following form:

$$C^i_{ab} = W^i_{ab} - \frac{2}{n-1} C_{[a}\delta^i_{b]},$$

where W^i_{ab} is a tensor and C_a has transformation rule:

$$C_{a'} = A^a_{a'} C_a - \frac{n-1}{2} p(\ln|J|)_{,a'}.$$

Some further properties of the extended rotation are investigated by means of objects $W^i_{ab}, C_a, D_a = \dfrac{2}{n-1} C_a$.

Introduction. Let w_i be a covariant vector and let G_{ab} be a differential concomitant of the first order of w_i.

We assume that the G_{ab} satisfies the following conditions:

(i) G_{ab} is additive with respect to $(w_i, \partial_j w_i)$,

(ii) G_{ab} satisfies a certain Leibniz rule concerning the product of w_i by a scalar σ and

(iii) G_{ab} is a covariant tensor of valence $(2, 0)$.

In [4] the notion of rotation has been derived from the above hypotheses.

In the present note we define axiomatically the extended rotation of an arbitrary covariant vector density w_i of weight $-p, p \neq 0$, but we do not assume that it is a differential concomitant of the first order of w_i.

Let X^n be an n-dimensional manifold. If transformation of the coordinate system has the form

$$x^{i'} = x^{i'}(x^i), \quad i, i' = 1, 2, \ldots, n,$$

then we put

$$A_i^{i'} = \frac{\partial x^{i'}}{\partial x^i},$$

$$A_{i'}^i = \frac{\partial x^i}{\partial x^{i'}}, \quad J = \operatorname{Det}\|A_i^{i'}\|.$$

1. Let w_i be a covariant vector density of weight $-p, p \neq 0$. Then the transformation rule of w_i has the following form:

(1.1) $$w_{i'} = \varphi(J) A_{i'}^i w_i,$$

where

$$\varphi(J) = \begin{cases} |J|^p & \text{for a } W\text{-density}, \\ (\operatorname{sgn} J)|J|^p & \text{for a } G\text{-density}. \end{cases}$$

If we denote by $U_{,j}$ the partial derivatives of a function U, then we have from (1.1)

(1.2) $$w_{i',j'} = \varphi(J)_{,j'} A_{i'}^i w_i + \varphi(J) A_{i'j'}^i w_i + \varphi(J) A_{i'}^i A_{j'}^j w_{i,j}.$$

The object $(w_i, w_{i,j})$ is called the *differential extension of the first order of* w_i. Its transformation rule is defined by (1.1) and (1.2).

Now we formulate the following definition:

Definition 1. A system of n^2 functions

$$G_{ab}, \quad a, b = 1, 2, \ldots, n,$$

is called the *extended rotation of the covariant* W- (or G-) vector density (1.1) of weight $-p, p \neq 0$, if it satisfies:

1° G_{ab} depends on $w_i, w_{i,j}$:

(1.3) $$G_{ab} = G_{ab}(w_i, w_{i,j});$$

2° G_{ab} is additive with respect to $(w_i, w_{i,j})$:

(1.4) $$G_{ab}(\underset{1}{w_i} + \underset{2}{w_i}, \underset{1}{w_{i,j}} + \underset{2}{w_{i,j}}) = G_{ab}(\underset{1}{w_i}, \underset{1}{w_{i,j}}) + G_{ab}(\underset{2}{w_i}, \underset{2}{w_{i,j}})$$

($\underset{1}{w_i}$ and $\underset{2}{w_i}$ are covariant vector densities of weight $-p$);

3° G_{ab} satisfies the following Leibniz rule for the product of w_i by a scalar σ:

(1.5) $$G_{ab}[\sigma w_i, (\sigma w_i)_{,j}] = \sigma G_{ab}(w_i, w_{i,j}) + w_{[a}\sigma_{,b]};$$

4° G_{ab} is a W- (or G-) tensor density of weight $-p$ and valence $(0, 2)$:

(1.6) $$G_{a'b'} = \varphi(J) A_{a'}^a A_{b'}^b G_{ab}.$$

From (1.4) and (1.5) it follows that G_{ab} is a linear function with respect to $(w_i, w_{i,j})$. In fact, putting $\sigma = a = \text{const}$ into (1.5) we have

$$G_{ab}(aw_i, aw_{i,j}) = aG_{ab}(w_i, w_{i,j}).$$

Hence we obtain the following form of G_{ab}:

$$G_{ab}(w_i, w_{i,j}) = C^i_{ab}w_i + C^{ij}_{ab}w_{i,j}, \tag{1.7}$$

where

$$C^i_{ab} = \text{const}, \qquad C^{ij}_{ab} = \text{const}, \qquad a, b, i, j = 1, \ldots, n.$$

Putting (1.7) into (1.5) we find

$$(C^{ij}_{ab} - \delta^i_{[a}\delta^j_{b]})w_i\sigma_{,j} = 0,$$

where δ^i_a is the general Kronecker delta. Since w_i and $\sigma_{,j}$ are arbitrary, we have

$$C^{ij}_{ab} = \delta^i_{[a}\delta^j_{b]}. \tag{1.8}$$

Inserting (1.8) into (1.7) we obtain

$$G_{ab} = C^i_{ab}w_i + w_{[a,b]}. \tag{1.9}$$

Now we determine the transformation rule of the C^i_{ab}. From (1.6) and (1.9) it follows that

$$C^{i'}_{a'b'}w_{i'} + w_{[a',b']} = \varphi(J)A^a_{a'}A^b_{b'}(C^i_{ab}w_i + w_{[a,b]}). \tag{1.10}$$

Putting (1.1) and (1.2) into (1.10) it is easily seen that

$$C^{i'}_{a'b'}A^{i'}_{i'}w_i = [A^a_{a'}A^b_{b'}C^i_{ab} + p(\ln|J|)_{,[a'}A^i_{b']}]w_i.$$

Since here w_i are arbitrary, we get

$$C^{i'}_{a'b'} = A^{i'}_i\{A^a_{a'}A^b_{b'}C^i_{ab} + p(\ln|J|)_{,[a'}A^i_{b']}\}. \tag{1.11}$$

We have thus obtained the following

THEOREM 1. *Every extended rotation of a covariant W- (or G-) vector density of weight $-p$, $p \neq 0$, has the form*

$$G_{ab}(w_i, w_{i,j}) = C^i_{ab}w_i + w_{[a,b]},$$

where C^i_{ab} is an object with transformation formula (1.11).

Let C^i_{ab} be the object with transformation formula (1.11) and

$$C_a \overset{\text{df}}{=} C^i_{ia}. \tag{1.12}$$

From (1.11) it follows that C_a is an object with transformation rule

$$C_{a'} = A^a_{a'}C_a - \frac{n-1}{2}p(\ln|J|)_{,a'}. \tag{1.13}$$

Now we assume that $n \geqslant 2$ and put

$$W^i_{ab} \overset{\mathrm{df}}{=} C^i_{ab} + \frac{2}{n-1} C_{[a}\delta^i_{b]}.$$

It is easily seen that W^i_{ab} is a tensor of valence (1, 2). Thus we have the following corollaries:

COROLLARY 1. *The object C^i_{ab} has the form*

$$C^i_{ab} = W^i_{ab} - \frac{2}{n-1} C_{[a}\delta^i_{b]}, \tag{1.14}$$

where W^i_{ab} is a tensor of valence (1.2) *and C_a is an object with transformation* (1.13).

COROLLARY 2. *$C^i_{(ab)} \overset{\mathrm{df}}{=} \frac{1}{2}(C^i_{ab} + C^i_{ba})$ is a tensor, $C^i_{[ab]} \overset{\mathrm{df}}{=} \frac{1}{2}(C^i_{ab} - C^i_{ba})$ has precisely the same transformation rule as that of C^i_{ab}.*

COROLLARY 3. *The extended rotation is an antysymmetric tensor density if and only if*

$$W^i_{ab} = W^i_{[ab]}.$$

2. Let us consider a field of an object C^i_{ab} of class C^1 on X^n, $n \geqslant 2$. The object

$$D_a \overset{\mathrm{df}}{=} \frac{2}{n-1} C_a, \tag{2.1}$$

where $C_a = C^i_{ia}$ has the following transformation formula:

$$D_{a'} = A^a_{a'} D_a - p(\ln |J|)_{,a'}. \tag{2.2}$$

M. Kucharzewski has proved in [2] that every covariant derivative of a density $\mathfrak{g}$ of weight $-p$, $p \neq 0$, has the form

$$F_i(\mathfrak{g}, \mathfrak{g}_{,j}) = \mathfrak{g}_{,i} + K_i \mathfrak{g}, \tag{2.3}$$

where K_i is an object with transformation rule (2.2).

Now we introduce the following notation:

$$\begin{aligned} w_{a|b} &\overset{\mathrm{df}}{=} C^i_{ab} w_i + w_{[a,b]}, \\ \mathfrak{g}_{;a} &\overset{\mathrm{df}}{=} \mathfrak{g}_{,a} + D_a \mathfrak{g}, \\ V_{ab} &\overset{\mathrm{df}}{=} D_{[a,b]}. \end{aligned} \tag{2.4}$$

It is known (cf. [3], p. 83) that the following equality is true:

$$\mathrm{Rot\,Grad}\,\sigma = 0,$$

where σ is a scalar field of class C^1. We shall require that the extended rotation in the sense of Definition 1 satisfies

$$(2.5)\qquad \mathfrak{g}_{;a|b} = 0,$$

for any density $\mathfrak{g}$ of weight $-p, p \neq 0$.

We prove the following

THEOREM 2. *The extended rotation fulfils* (2.5) *if and only if*

$$(2.6)\qquad W^i_{ab} = 0, \quad V_{ab} = 0.$$

Proof. From (1.9), (2.1), (2.3) and (2.4) it follows that

$$\mathfrak{g}_{;a|b} = (C^i_{ab} + D_{[a}\delta^i_{b]})\mathfrak{g}_{,i} + (C^i_{ab}D_i + V_{ab})\mathfrak{g}.$$

If $\mathfrak{g}_{;a|b} = 0$ for every density $\mathfrak{g}$ of weight $-p, p \neq 0$, then we obtain

$$C^i_{ab} + D_{[a}\delta^i_{b]} = 0, \quad C^i_{ab}D_i + V_{ab} = 0.$$

Thus we have (2.8).

If (2.6) holds, then it is easily seen that $\mathfrak{g}_{;a|b} = 0$.

This completes the proof.

Remarks.

I. Let X^n be an L^n(1) and let Λ^i_{ab} be an object of the linear displacement in L^n. Then we can define the extended rotation as follows:

$$W^i_{ab} \stackrel{\mathrm{df}}{=} S^i_{ab}, \quad C_a \stackrel{\mathrm{df}}{=} \frac{n-1}{2}p\Lambda_a,$$

where $S^i_{ab} = \Lambda^i_{[ab]}$, $\Lambda_a = \Lambda^i_{ai}$.

If w_i is a covariant vector density of weight $-p$, then we put

$$w_{a|b} \stackrel{\mathrm{df}}{=} (S^i_{ab} - p\Lambda_{[a}\delta^i_{b]})w_i - w_{[a,b]}.$$

Such an extended rotation coincides with the alternation of the covariant derivative of w_i with respect to Λ^i_{ab}.

II. Now we assume that in X^n there is given a field of an object Λ_a with transformation formula

$$\Lambda_{a'} = A^a_{a'}\Lambda_a - (\ln|J|)_{,a'}.$$

Then we can define the following extended rotation of a covariant vector density of weight, $-p\ p \neq 0$:

$$w_{a|b} \stackrel{\mathrm{df}}{=} -p\Lambda_{[a}\delta^i_{b]}w_i + w_{[a,b]}.$$

In this case we have $W^i_{ab} = 0$.

(1) In X^n a field of an object Λ^i_{ab} with transformation rule

$$\Lambda^{i'}_{a'b'} = A^{i'}_i A^a_{a'} A^b_{b'} \Lambda^i_{ab} + A^{i'}_s A^s_{a'b'}$$

is given.

We notice that $W^i_{ab} = 0$ if and only if $\mathfrak{g}_{,a|b} = V_{ab}\mathfrak{g}$ for every density of weight $-p, p \neq 0$.

III. Let w_i be a covariant vector density of weight $p = 0$ (i. e. w_i is a covariant vector or a J-vector). It is easily seen (1°, 2°, 3°, (1.9) and (1.11)) that

$$G_{ab}(w_i, w_{i,j}) = C^i_{ab} w_i + w_{[a,b]},$$

where C^i_{ab} is an arbitrary tensor of valence (1.2).

If we assume (cf. [4]) that G_{ab} is a differential concomitant of the first order of w_i, then we have

$$C^i_{ab} = 0,$$

i.e.

$$G_{ab}(w_i, w_{i,j}) = w_{[a,b}\,.$$

References

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[2] M. Kucharzewski, *Kovariante Ableitung der Skalare Dichten*, Prace Naukowe U. Śl., Prace Mat. 1 (1969), p. 61–70.

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[4] J. A. Schouten, *Ricci-Calculus*, Berlin–Gröningen–Heidelberg 1954.

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