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Some theorems on differentiable solutions of a system of functional equations of n-th order

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Abstract. This paper contains two theorems concerning the existence and the uniqueness of C^r-solutions of a system of functional equations

$$\varphi_i(x) = h_i(x, \{\varphi_j[f_k(x)]\}), \quad x \in \langle 0, a), \ i, j = 1, \ldots, m, \quad k = 1, \ldots, n,$$

where f_k, h_i denote real functions of one and $nm+1$ real variables, respectively, and φ_i denote unknown functions (Section 2, Theorems 1, 2). In Section 3 there are given some conditions for the continuous dependence of C^r-solutions on given functions (Theorems 3, 4).

1. We start from some notational conventions:

(a) m, n, r denote fixed positive integers;

(b) the indices i and j run over the set $\{1, \ldots, m\}$, while the indices k, σ, p and ν run over the sets $\{1, \ldots, n\}$, $\{1, \ldots, r\}$, $\{1, 2, \ldots\}$ and $\{0, 1, 2, \ldots\}$, respectively;

(c) the symbol $\{y_{jk}\}$ denotes a vector belonging to the space R^{mn}, namely

$$\{y_{jk}\} := (y_{11}, \ldots, y_{m1}, \ldots, y_{1n}, \ldots, y_{mn})$$

and $0 := (0, \ldots, 0) \in R^{mn}$;

(d) the Greek letters $\gamma, \varphi, \psi, \Phi, \Psi$ stand for vector-functions defined on subsets of R and with values in R^m, for example

$$\varphi(x) := (\varphi_1(x), \ldots, \varphi_m(x));$$

(e) by $C^r_m[A]$, $A \subset R$, we mean the class of functions $\gamma\colon A \mapsto R^m$ such that the functions γ_i have continuous derivatives up to order r in A. In the case $m = 1$ we write $C^r[A] := C^r_1[A]$;

(f) the letter I denotes the present interval $\langle 0, a\rangle$, $a > 0$, and $\Omega := I \times R^{mn}$.

The purpose of the present paper is to prove some theorems concerning the existence of C^r-solutions of a system of functional equations

$$\varphi_i(x) = h_i(x, \{\varphi_j[f_k(x)]\}), \quad x \in I, \tag{1}$$

and the continuous dependence of the solutions on the given functions. In this system h_i and f_k are given and φ_i are unknown functions. In the proof of the main result on the existence of C^r-solution of the system (1) we shall use Schauder's theorem and apply the following

LEMMA 1 ([13]). *Let $a_{ij} \geqslant 0$. The system of inequalities*

$$\sum_j a_{ij} x_j < x_i$$

has a solution $R_i > 0$ iff all the characteristic roots of the matrix $[a_{ij}]$ are less than one in absolute value.

Remark 1 ([12], [13]). Each of the equivalent statements in Lemma 1 is also equivalent to the following one:

$$c^l_{\lambda\lambda} > 0 \quad \text{for } l = 0, 1, \ldots, m-1, \quad \lambda = 1, \ldots, m-l,$$

where

$$c^0_{\lambda\mu} := \begin{cases} 1 - a_{\lambda\mu}, & \lambda = \mu, \\ a_{\lambda\mu} & \lambda \neq \mu, \quad \lambda, \mu = 1, \ldots, m, \end{cases}$$

$$c^{l+1}_{\lambda\mu} := \begin{cases} c^l_{11} c^l_{\lambda+1\,\mu+1} + c^l_{\lambda+1\,1} c^l_{1\,\mu+1}, & \lambda \neq \mu, \\ c^l_{11} c^l_{\lambda+1\,\mu+1} - c^l_{\lambda+1\,1} c^l_{1\,\mu+1}, & \lambda = \mu, \end{cases}$$

$l = 0, 1, \ldots, m-2, \quad \lambda, \mu = 1, \ldots, m-l-1.$

2. The fundamental theorems regarding the uniqueness and the existence of C^r-solutions in the case $m = n = 1$ are due to Choczewski [3], [4] (see also [8], Chapter IV). This theory has been further extended by J. Matkowski [10], [11], who obtained some result on the uniqueness of C^r-solutions of a system of functional equations. Other results regarding these problems can be found in [7] and [5]. The theorems presented in this paper generalize all the preceding ones except of that contained in [7], which is of slightly different nature.

We assume that:

(2.i) $f_k \in C^r[I]$, $0 < f_k(x) < x$ for $x \in (0, a)$;

(2.ii) $h_i \in C^r[\Omega]$, $h_i(0, \{0\}) = 0$;

(2.iii) all the characteristic roots of the matrix $[\sum_k a_{ijk}]$, where

$$a_{ijk} := \left| \frac{\partial h_i}{\partial y_{jk}} (0, \{0\}) [f'_k(0)]^r \right| \tag{2.1}$$

are less than one in absolute value;

(2.iv) there exist a number L and a set $U_1 := \langle 0, c_1 \rangle \times \langle -d_1, d_1 \rangle^{mn}$ such that for all $(x, \{y_{jk}\}) \in U_1$, $l = 0, 1, \ldots, r$, $s_{jk} = 0, 1, \ldots, r-l$, $s_{11} + \ldots + s_{mn} = r-l$, the inequalities

$$\left|\frac{\partial^r h_i}{\partial x^l \partial y_{11}^{s_{11}} \dots \partial y_{mn}^{s_{mn}}}(x, \{y_{jk}\}) - \frac{\partial^r h_i}{\partial x^l \partial y_{11}^{s_{11}} \dots \partial y_{mn}^{s_{mn}}}(x, \{\bar{y}_{jk}\})\right| \leqslant L \sum_j \sum_k |y_{jk} - \bar{y}_{jk}|$$

are satisfied.

Let us define functions h_i^σ by the recurent relations

$$h_i^0(x, \{y_{jk}^0\}) := h_i(x, \{y_{jk}^0\}), \quad (x, \{y_{jk}^0\}) \in \Omega;$$

$$h_i^\sigma(x, \{y_{jk}^0\}, \dots, \{y_{jk}^\sigma\}) := \frac{\partial h_i^{\sigma-1}}{\partial x}(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{\sigma-1}\}) +$$

$$+ \sum_{q=1}^m \sum_{t=1}^n \sum_{s=0}^{\sigma-1} \frac{\partial h_i^{\sigma-1}}{\partial y_{qt}^s}(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{\sigma-1}\}) y_{qt}^{s+1} f_t'(x),$$

$$(x, \{y_{jk}^0\}, \dots, \{y_{jk}^\sigma\}) \in \Omega \times R^{mn\sigma}.$$

By induction we prove

LEMMA 2. *If we have $f_k \in C^r[I]$, $f_k(I) \subset I$, $h_i \in C^r[\Omega]$ and if $\varphi \in C_m^r[I]$ is a solution of system* (1) *in the interval I, then*

$$\varphi_i^{(\sigma)}(x) = h_i^\sigma(x, \{\varphi_j[f_k(x)]\}, \dots, \{\varphi_j^{(\sigma)}[f_k(x)]\}), \quad x \in I. \tag{2.2}$$

In particular, (2.2) *implies that if $\varphi \in C_m^r[I]$ is a solution of system* (1) *such that $\varphi_i(0) = 0$, $\varphi_i^{(\sigma)}(0) = \eta_i^\sigma$ and if conditions* (2.i) *and* (2.ii) *are fulfilled, then the system of numbers η_i^σ must be a solution of the system of equations*

$$\eta_i^\sigma = h_i(0, \{0\}, \{\eta_{jk}^1\}, \dots, \{\eta_{jk}^\sigma\}), \tag{2.3}$$

where $\eta_{jk}^s = \eta_j^s$, $s = 1, \dots, \sigma$.

The following lemma shows the structure of the functions h_i^σ.

LEMMA 3. *If the functions f_k and h_i belong to $C^r[I]$ and $C^r[\Omega]$, respectively, then $h_i \in C^r[\Omega \times R^{mn\sigma}]$ and*

$$h_i(x, \{y_{jk}^0\}, \dots, \{y_{jk}^\sigma\}) = Z_i^\sigma(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{\sigma-1}\}) + Q_i(x, \{y_{jk}^0\}, \{y_{jk}^\sigma\}), \tag{2.4}$$

where

$$Q_i(x, \{y_{jk}^0\}, \{y_{jk}^\sigma\}) := \sum_{q=1}^m \sum_{t=1}^n \frac{\partial h_i}{\partial y_{qt}}(x, \{y_{jk}\}) y_{qt}^\sigma [f_t'(x)]^\sigma \tag{2.5}$$

and $Z_i \in C^{r-\sigma}[\Omega \times R^{mn(\sigma-1)}]$.

Proof. Induction.

Now suppose that $\varphi \in C_m^r[I]$ is a solution of system (1). Write φ_i in the form

$$\varphi_i(x) = P_i(x) + \gamma_i(x), \quad x \in I, \tag{2.6}$$

where

$$P_i(x) := \sum \frac{\eta_i^\sigma}{\sigma!} x^\sigma, \quad x \in I. \tag{2.7}$$

Defining the functions

$$H_i(x, \{y_{jk}\}) := h_i\big(x, \{P_j[f_k(x)] + y_{jk}\}\big) - P_i(x), \quad x \in I,$$

we observe that if $f_k \in C^r[I]$, $h_i \in C^r[\Omega]$, then $H_i \in C^r[\Omega]$. Moreover, $\gamma \in C_m^r[I]$ is a solution of the system of functional equations

$$\gamma_i(x) = H_i\big(x, \{\gamma_j[f_k(x)]\}\big), \quad x \in I, \tag{2.8}$$

fulfilling the conditions

$$\gamma_i(0) = \gamma_i^{(\sigma)}(0) = 0. \tag{2.9}$$

Conversely, if $\gamma \in C_m^r[I]$ fulfilling (2.9) is a solution of system (2.8) in the interval I, then for every system of numbers η_i^σ fulfilling (2.3) the function $\varphi \in C_m^r[I]$ defined by (2.6) and (2.7) is a solution of system (1) in the interval I and the conditions

$$\varphi_i(0) = 0, \quad \varphi_i^{(\sigma)}(0) = \eta_i^\sigma$$

are fulfilled. Thus we have the following

LEMMA 4. *Let a system of numbers η_i^σ be a fixed solution of* (2.3). *If* (2.i) *and* (2.ii) *are fulfilled, then system* (1) *has exactly one solution $\varphi \in C_m^r[I]$ in the interval I fulfilling the conditions $\varphi_i(0) = \varphi_i^{(\sigma)}(0) = 0$ iff system* (2.8) *has exactly one solution $\gamma \in C_m^r[I]$ in the interval I fulfilling* (2.9). *These solutions are interrelated by formulas* (2.6) *and* (2.7).

It is easily seen that all the characteristic roots of the matrix $\left[\sum_k \left|\frac{\partial h_i}{\partial y_{jk}}(0, \{0\})[f_k'(0)]^r\right|\right]$ are less than one in absolute value and $h_i(0, \{0\}) = 0$ iff all the characteristic roots of the matrix $\left[\sum_k \left|\frac{\partial H_i}{\partial y_{jk}}(0, \{0\})[f_k'(0)]^r\right|\right]$ are less than one in absolute value and $H_i(0, \{0\}) = 0$. Therefore, in the sequel we shall study solutions $\varphi \in C_m^r[I]$ of system (1) fulfilling

$$\varphi_i(0) = \varphi_i^{(\sigma)}(0) = 0. \tag{2.10}$$

If $\varphi \in C_m^r[I]$ is a solution of system (1) in the interval I and if φ_i satisfy (2.10), then, in virtue of (2.2), (2.i) and (2.ii), we obtain

$$h_i^\sigma(0, \{0\}, \ldots, \{0\}) = 0. \tag{2.11}$$

Hence, on account of (2.4) and (2.5), we have

$$Z_i^\sigma(0, \{0\}, \ldots, \{0\}) = 0. \tag{2.12}$$

THEOREM 1. *If assumptions* (2.i), (2.ii), (2.iii) *and conditions* (2.11) *are fulfilled and if, moreover, there exists a* $\xi > 0$ *such that*

$$|f_k'(x)| \leqslant 1 \quad \text{for } x \in \langle 0, \xi \rangle, \tag{2.13}$$

then the system of equations (1) *has exactly one solution* $\varphi \in C_m^r[I]$ *in the interval* I, *fulfilling conditions* (2.10).

Proof. (2.iii) and Lemma 1 yield the existence of $R_i > 0$ and $\eta > 0$ such that

$$\sum_j \sum_k (a_{ijk} + \eta) R_j < R_i, \qquad 0 < R_i < 1. \tag{2.14}$$

Since the functions Z_i^r (see Lemma 3) and f_k' are continuous in $\Omega \times \times R^{mn(\sigma-1)}$ and in I, respectively, then on account of (2.12), (2.1) and (2.14), there exist numbers $c_1 > 0$ and $d > 0$ such that

$$\begin{gathered} |Z_i^r(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{r-1}\})| \leqslant R_i - \sum_j \sum_k (a_{ijk} + \eta) R_j, \\ \left| \frac{\partial h_i}{\partial y_{jk}^0} (x, \{y_{jk}^0\}) [f_k'(x)]^r \right| \leqslant a_{ijk} + \eta \end{gathered} \tag{2.15}$$

$$\text{for all } 0 \leqslant x \leqslant c_1, \ |y_{jk}^s| \leqslant d, \ s = 0, 1, \dots, r-1.$$

Put

$$c := \min(c_1, 1, \xi, d) \tag{2.16}$$

and define the sets $D \subset \Omega \times R^{mn(r-1)}$, $D' \subset \Omega$ as follow:

$$D := \{(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{r-1}\});\ 0 \leqslant x \leqslant c,\ |y_{jk}^s| \leqslant R_j,\ s = 0, 1, \dots, r-1\},$$
$$D' := \{(x, \{y_{jk}^0\});\ 0 \leqslant x \leqslant c,\ |y_{jk}^0| \leqslant R_j\}.$$

The functions Z_i^r, f_k' and $\partial h_i / \partial y_{jk}^0$ are uniformly continuous in the sets D, $\langle 0, c \rangle$ and D', respectively. Hence, for arbitrary $\varepsilon_i > 0$ fulfilling the system of inequalities

$$\sum_j \sum_k (a_{ijk} + \eta) \varepsilon_j < \varepsilon_i,$$

there exists a $\delta > 0$ such that

for every $(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{r-1}\})$, $(\bar{x}, \{\bar{y}_{jk}^0\}, \dots, \{\bar{y}_{jk}^{r-1}\}) \in D$ and $|x - \bar{x}| < \delta$, $|y_{jk}^s - \bar{y}_{jk}^s| < \delta$, $s = 0, 1, \dots, r-1$ we have the inequalities

$$\begin{gathered} |Z_i^r(x, \{y_{jk}^0\}, \dots, \{y_{jk}^{r-1}\}) - Z_i^r(\bar{x}, \{\bar{y}_{jk}^0\}, \dots, \{\bar{y}_{jk}^{r-1}\})| \\ \leqslant \tfrac{1}{2}\Big[\varepsilon_i - \sum_j \sum_k (a_{ijk} + \eta) \varepsilon_j\Big], \\ \sum_{q=1}^m \sum_{t=1}^n \left| \frac{\partial h_i}{\partial y_{qt}^0} (x, \{y_{jk}^0\}) [f_t'(x)]^r - \frac{\partial h_i}{\partial y_{qt}^0} (\bar{x}, \{\bar{y}_{jk}^0\}) [f_t'(\bar{x})]^r \right| \\ \leqslant \tfrac{1}{2}\Big[\varepsilon_i - \sum_j \sum_k (a_{ijk} + \eta) \varepsilon_j\Big]. \end{gathered} \tag{2.17}$$

In the space $C^r[\langle 0, c\rangle]$ we introduce the norm

$$\|u\| := \max\{\sup_{\langle 0,c\rangle} |u(x)|, \ldots, \sup_{\langle 0,c\rangle} |u^{(r)}(x)|\}. \tag{2.18}$$

$(C^r[\langle 0, c\rangle], \|\cdot\|)$ is a normed vector space over the field R and the convergence of a sequence of its elements (u_p) is equivalent to the uniform convergence of (u_p) jointly with all the derivatives up to the order r in $\langle 0, c\rangle$. As is well known it is a Banach space.

Now, let $X_j \subset C^r[\langle 0, c\rangle]$ denote the class of all functions fulfilling the following conditions:

$$\varphi_j(0) = \varphi_j^{(\sigma)}(0) = 0; \tag{2.19}$$

$$|\varphi_j^{(r)}(x)| \leqslant R_j \quad \text{for all } x \in \langle 0, c\rangle; \tag{2.20}$$

(2.21) for arbitrary numbers $\varepsilon_j > 0$ such that $\sum_j \sum_k (a_{ijk} + \eta)\varepsilon_j < \varepsilon_i$, the conditions x, $\bar{x} \in \langle 0, c\rangle$ and $|x - \bar{x}| < \delta$, where $\delta = \delta(\varepsilon_1, \ldots, \varepsilon_m)$ is just as in (2.17), imply the inequality $|\varphi_j^{(r)}(x) - \varphi_j^{(r)}(\bar{x})| \leqslant \varepsilon_j$.

Now we define a transformation T by the formula

$$T(\varphi)(x) := \big(T_1(\varphi)(x), \ldots, T_m(\varphi)(x)\big), \quad \varphi \in X_1 \times \ldots \times X_m, \quad x \in \langle 0, c\rangle, \tag{2.22}$$

where

$$T_i(\varphi)(x) := h_i\big(x, \{\varphi_j[f_k(x)]\}\big), \quad x \in \langle 0, c\rangle. \tag{2.23}$$

We shall show that the set $X_1 \times \ldots \times X_m$ and the transformation T satisfy all the assumptions of Schauder's fixed point theorem. First we prove that for $\varphi_j, \bar{\varphi}_j \in X_j$ we have

$$\|\varphi_j - \bar{\varphi}_j\| = \sup_{\langle 0,c\rangle} |\varphi_j^{(r)}(x) - \bar{\varphi}_j^{(r)}(x)|. \tag{2.24}$$

In fact, according to the mean-value theorem and (2.19), we obtain the existence of an $\bar{x} \in (0, x)$ such that

$$|\varphi_j^{(\sigma-1)}(x) - \bar{\varphi}_j^{(\sigma-1)}(x)| = x|\varphi_j^{(\sigma)}(\bar{x}) - \bar{\varphi}_j^{(\sigma)}(\bar{x})|.$$

Hence and by (2.16) (because of $c \leqslant 1$) we get

$$\sup_{\langle 0,c\rangle} |\varphi_j^{(\sigma-1)}(x) - \bar{\varphi}_j^{(\sigma-1)}(x)| \leqslant \sup_{\langle 0,c\rangle} |\varphi_j^{(\sigma)}(x) - \bar{\varphi}_j^{(\sigma)}(x)|,$$

which means, in virtue of (2.18), that (2.24) holds. Furthermore, it follows from (2.19), (2.20), (2.21), (2.24) and from the Arzelà theorem that $X_j \subset C^r[\langle 0, c\rangle]$ are compact. The convexity of the sets X_j is obvious. Thus $X_1 \times \ldots \times X_m$ is a convex and compact subset of the Banach space $(C^r[\langle 0, c\rangle], \|\cdot\|)^m$.

Now we prove that T transforms the set $X_1 \times \ldots \times X_m$ into itself. Similarly as above we assert that

$$(2.25) \qquad |\varphi_j^{(s)}(x)| \leqslant R_j x, \quad x \in \langle 0, c\rangle, \ s = 0, 1, \ldots, r-1.$$

Differentiating (2.23) σ times we obtain (see Lemma 2) that

$$(2.26) \qquad T_i^{(\sigma)}(\varphi)(x) = h_i^\sigma\big(x, \{\varphi_j[f_k(x)]\}, \ldots, \{\varphi_j^{(\sigma)}[f_k(x)]\}\big).$$

According to Lemma 3, the definition of X_j and (2.i), the functions h_i are continuous in the interval $\langle 0, c\rangle$. Therefore $T_i(\varphi) \in C^r[\langle 0, c\rangle]$. Putting $x = 0$ in (2.23) and (2.26) we get, in view of (2.ii), (2.19) and (2.11), $T_i(\varphi)(0) = 0$ as well as $T_i^{(\sigma)}(\varphi)(0) = 0$. Thus T_i fulfil (2.19).

To obtain an estimate for $T_i^{(r)}(\varphi)(x)$, we make use of (2.26) and Lemma 3:

$$|T_i^{(r)}(\varphi)(x)| \leqslant \big|Z_i^r\big(x, \{\varphi_j[f_k(x)]\}, \ldots, \{\varphi_j^{(r-1)}[f_k(x)]\}\big)\big| + \\ + \sum_{q=1}^{m}\sum_{t=1}^{n}\left|\frac{\partial h_i}{\partial y_{jk}^0}\big(x, \{\varphi_j[f_k(x)]\}\big)[f_t'(x)]^r \varphi_q^{(r)}[f_t(x)]\right|.$$

It follows from (2.i), (2.25), (2.16) and (2.14) that

$$|\varphi_j^{(s)}[f_k(x)]| \leqslant R_j f_k(x) \leqslant R_j x < x \leqslant d, \qquad s = 0, 1, \ldots, r-1.$$

So, by (2.15) and (2.20),

$$|T_i^{(r)}(\varphi)(x)| \leqslant R_i$$

which means that $T_i(\varphi)$ fulfil (2.20).

To see that $T_i(\varphi)$ satisfy also (2.21), take an arbitrary system of numbers $\varepsilon_i > 0$ satisfying the system of inequalities

$$\sum_j \sum_k (a_{ijk} + \eta)\varepsilon_j < \varepsilon_i.$$

Let $|x - \bar{x}| < \delta$, x, $\bar{x} \in \langle 0, c\rangle$, where $\delta = \delta(\varepsilon_1, \ldots, \varepsilon_m)$ is just as in (2.17). Then, on account of (2.26) and Lemma 3, we get

$$(2.27) \quad |T_i^{(r)}(\varphi)(x) - T_i^{(r)}(\varphi)(\bar{x})| \leqslant \big|Z_i^r\big(x, \{\varphi_j[f_k(x)]\}, \ldots, \{\varphi_j^{(r-1)}[f_k(x)]\}\big) - \\ - Z_i^r\big(\bar{x}, \{\varphi_j[f_k(\bar{x})]\}, \ldots, \{\varphi_j^{(r-1)}[f_k(\bar{x})]\}\big)\big| + \\ + \sum_{q=1}^{m}\sum_{t=1}^{n}\left|\frac{\partial h_i}{\partial y_{qt}^0}\big(x, \{\varphi_j[f_k(x)]\}\big)[f_t'(x)]^r\big(\varphi_q^{(r)}[f_t(x)] - \varphi_q^{(r)}[f_t(\bar{x})]\big)\right| + \\ + \sum_{q=1}^{m}\sum_{t=1}^{n}\left|\varphi_q^{(r)}[f_t(\bar{x})]\left(\frac{\partial h_i}{\partial y_{qt}^0}\big(\bar{x}, \{\varphi_j[f_k(\bar{x})]\}\big)[f_t'(\bar{x})]^r - \right.\right. \\ \left.\left. - \frac{\partial h_i}{\partial y_{qt}^0}\big(x, \{\varphi_j[f_k(x)]\}\big)[f_t'(x)]^r\right)\right|.$$

By the mean value theorem, (2.13) and (2.16)

$$|f_k(\bar{x})-f_k(x)| \leqslant |\bar{x}-x|, \quad x, \bar{x} \in \langle 0, c\rangle. \tag{2.28}$$

In virtue of (2.25), (2.18), (2.20) and (2.14)

$$|\varphi_j^{(s)}[f_k(x)]-\varphi_j^{(s)}[f_k(\bar{x})]| < \delta, \quad s = 0, 1, \ldots, r-1, \quad x \in \langle 0, c\rangle.$$

Moreover, (2.21) and (2.28) imply

$$|\varphi_j^{(r)}[f_k(x)]-\varphi_j^{(r)}[f_k(\bar{x})]| \leqslant \varepsilon_j$$

and (2.i), (2.25), (2.14) and (2.16) show that

$$|\varphi_j[f_k(x)]| \leqslant d,$$
$$|\varphi_j^{(s)}[f_k(x)]| \leqslant R_j, \quad s = 0, 1, \ldots, r-1.$$

Thus we may use relations (2.17), (2.15) and (2.17), (2.20), (2.14) and (2.17), in order to estimate the first, second and the last summand of (2.27), respectively. Thus we have

$$|T_i^{(r)}(\varphi)(x)-T_i^{(r)}(\varphi)(\bar{x})| \leqslant \varepsilon_i$$

and hence $T_i(\varphi)$ fulfil (2.21).

It remains to show that the transformation T is continuous. Let $\underset{p}{\varphi} \in X_1 \times \ldots \times X_m$, $\underset{0}{\varphi} \in X_1 \times \ldots \times X_m$ and let $\lim\limits_{p\to\infty} \underset{p}{\varphi_i} = \underset{0}{\varphi_i}$ in the sense of the convergence in the set X_i. Since T_i maps $X_1 \times \ldots \times X_m$ into X_i, we get $T_i(\underset{p}{\varphi})$, $T_i(\underset{0}{\varphi}) \in X_i$. On account of (2.26) and Lemma 3 we infer

$$\begin{aligned}
&|T_i^{(r)}(\underset{p}{\varphi})(x)-T_i^{(r)}(\underset{0}{\varphi})(x)| \\
&\qquad \leqslant \big|Z_i^r\big(x, \{\underset{p}{\varphi_j}[f_k(x)]\}, \ldots, \{\underset{p}{\varphi_j^{(r-1)}}[f_k(x)]\}\big) - \\
&\qquad\qquad -Z_i^r\big(x, \{\underset{0}{\varphi_j}[f_k(x)]\}, \ldots, \{\underset{0}{\varphi_j^{(r-1)}}[f_k(x)]\}\big)\big| + \\
&+\sum_{q=1}^{m}\sum_{t=1}^{n}\Big[\Big|\frac{\partial h_i}{\partial y_{qt}^0}\big(x, \{\underset{p}{\varphi_j}[f_k(x)]\}\big)[f_t'(x)]^r\Big|\,|\underset{p}{\varphi_q^{(r)}}[f_t(x)]-\underset{0}{\varphi_q^{(r)}}[f_t(x)]| + \\
&+|\underset{0}{\varphi_q^{(r)}}[f_t(x)][f_t'(x)]^r|\,\Big|\frac{\partial h_i}{\partial y_{qt}^0}\big(x, \{\underset{p}{\varphi_j}[f_k(x)]\}\big)-\frac{\partial h_i}{\partial y_{qt}^0}\big(x, \{\underset{0}{\varphi_j}[f_k(x)]\}\big)\Big|\Big].
\end{aligned}$$

From the uniform continuity of the functions Z_i^r in D and $\partial h_i/\partial y_{jk}^0$ in D' and from the fact that the sequences $(\underset{p}{\varphi_j^{(s)}})$, $s = 0, 1, \ldots, r$, are uniformly convergent in the interval $\langle 0, c\rangle$ to the functions $\underset{0}{\varphi_j^{(s)}}$ we infer that the sequences $\big(T_i^{(r)}(\underset{p}{\varphi})\big)$ are uniformly convergent to $\big(T_i^{(r)}(\underset{0}{\varphi})\big)$ in $\langle 0, c\rangle$, which, in virtue of (2.24), means that $\|T_i(\underset{p}{\varphi})-T_i(\underset{0}{\varphi})\| \to 0$ whenever $\|\underset{p}{\varphi_j}-\underset{0}{\varphi_j}\| \to 0$. Therefore the transformations T_i are continuous in $X_1 \times \ldots \times X_m$.

It follows from Schauder's theorem that there exists at least one solution $\varphi \in C^r_m[\langle 0, c\rangle]$ of the system of equations (1) satisfying conditions (2.10). This solution can be extended to a solution defined on the whole interval I in the same way as in [2], and this extension preserves the class of regularity. The uniqueness of solutions follows from the theorem contained in [6]. Thus the proof is complete.

For the functions $f_k \in C^r[I]$ and $h_i \in C^r[\Omega]$ it follows from (2.1) that for arbitrarily fixed number $\eta > 0$ there exists a set $U_2 := \langle 0, c_2\rangle \times \times \langle -d_2, d_2\rangle^{mn}$ in which the inequalities

$$\left|\frac{\partial h_i}{\partial y^0_{jk}}(x, \{y^0_{jk}\})[f'_k(x)]^r\right| \leqslant a_{ijk} + \frac{\eta}{2} \tag{2.29}$$

hold.

The following lemma is true:

LEMMA 5. *If* (2.i), (2.ii) *and* (2.iv) *are fulfilled, then to an arbitrary positive number d, there exist constants $L^s_{ijk} > 0$ such that for every $(x, \{y^0_{jk}\}, \ldots$ $\ldots, \{y^r_{jk}\})$, $(x, \{\bar{y}^0_{jk}\}, \ldots, \{\bar{y}^r_{jk}\}) \in Z := (U_1 \cap U_2) \times \langle -d, d\rangle^{mnr}$ we have*

$$|h^r_i(x, \{y^0_{jk}\}, \ldots, \{y^r_{jk}\}) - h^r_i(x, \{\bar{y}^0_{jk}\}, \ldots, \{\bar{y}^r_{jk}\})| \leqslant \sum_{q=1}^{m}\sum_{t=1}^{n}\sum_{s=0}^{r} L^s_{iqt}|y^s_{qt} - \bar{y}^s_{qt}|, \tag{2.30}$$

where $L^r_{ijk} = a_{ijk} + \eta/2$.

The proof of this lemma is quite similar to that of Lemma 4.5 in [8], p. 94. It is based on Lemma 3 and on the fact that the functions Z^σ_i occurring in (2.4) can be written in the form

$$Z^1_i(x, \{y^0_{jk}\}) = \frac{\partial h_i}{\partial x}(x, \{y^0_{jk}\}),$$

$$Z^s_i(x, \{y^0_{jk}\}, \ldots, \{y^s_{jk}\}) = P^s_i(x, \{y^0_{jk}\}, \{y^1_{jk}\}) + R^s_i(x, \{y^0_{jk}\}, \ldots, \{y^{s-1}_{jk}\}), \quad s = 2, \ldots, r,$$

where

$$P^s_i(x, \{y^0_{jk}\}, \{y^1_{jk}\}) = \frac{\partial^s h_i}{\partial x^s}(x, \{y^0_{jk}\}) + \sum_{\alpha=1}^{s}\binom{s}{\alpha}\sum_{j_1,\ldots,j_\alpha=1}^{m}\sum_{k_1,\ldots,k_\alpha=1}^{n}\frac{\partial^s h_i}{\partial x^{s-\alpha}\partial y^0_{j_1k_1}\ldots\partial y^0_{j_\alpha k_\alpha}}(x, \{y^0_{jk}\}) \times y^1_{j_1k_1}\ldots y^1_{j_\alpha k_\alpha}f'_{k_1}(x)\ldots f'_{k_\alpha}(x),$$

are polynomials of the variables $(x, \{y^0_{jk}\})$ and $R^s_i(x, \{y^0_{jk}\}, \ldots, \{y^{s-1}_{jk}\})$ are of the class C^{r-s} with respect to x and of the class C^{r-s+1} with respect to y^0_{jk}. The above formulas can be obtained by induction.

THEOREM 2. *Let assumptions* (2.i)–(2.iv) *and condition* (2.11) *be fulfilled. Then there exists exactly one solution* $\varphi \in C_m^r[I]$ *of system* (1) *in the interval* I *satisfying* (2.10).

Proof. By Lemma 1 we have the existence of constants $\eta > 0$ and $R_i > 0$ such that

$$\sum_j \sum_k (a_{ijk} + \eta) R_j < R_i. \tag{2.31}$$

Having chosen such an $\eta > 0$, we shall regard it as fixed in inequalities (2.29). The continuity of h_i^r in $\Omega \times R^{mnr}$, (2.11) and (2.31) guarantee the existence of a number $c_3 > 0$ such that

$$|h_i^r(x, \{0\}, \ldots, \{0\})| < R_i - \sum_j \sum_k (a_{ijk} + \eta) R_j \quad \text{for } x \in \langle 0, c_3 \rangle. \tag{2.32}$$

Let us choose a number $c > 0$ such that

$$0 < c < \min\left\{c_1, c_2, c_3, 1, \min_i \frac{d_3}{R_i}, \min_{i,j,k} \frac{\eta}{2\sum_{s=0}^{r-1} L_{ijk}^s}\right\}. \tag{2.33}$$

Let X_i denote the space of all functions of class $C^r[\langle 0, c \rangle]$ satisfying the conditions

$$\varphi_i(0) = \varphi_i^{(\sigma)}(0) = 0; \tag{2.34}$$

$$|\varphi_i^{(r)}(x)| \leqslant R_i \quad \text{for all } x \in \langle 0, c \rangle, \tag{2.35}$$

and let T_i be the transformation defined on $X_1 \times \ldots \times X_m$ by the formula

$$T_i(\varphi)(x) := h_i(x, \{\varphi_j[f_k(x)]\}), \quad x \in \langle 0, c \rangle. \tag{2.36}$$

We shall show that the spaces X_i and the transformations T_i satisfy all the assumptions of the theorem contained in [12]. To this aim, we introduce the metric

$$\varrho(\varphi_i, \overline{\varphi}_i) := \sup_{\langle 0, c \rangle} |\varphi_i^{(r)}(x) - \overline{\varphi}_i^{(r)}(x)| \quad \text{in } X_i. \tag{2.37}$$

Then, X_i are complete metric spaces. In virtue of the mean-value theorem and (2.33), for $\varphi_i \in X_i$ we have

$$|\varphi_i^{(s)}(x)| = |\varphi_i^{(s)}(x) - \varphi_i^{(s)}(0)| \leqslant R_i c \leqslant d_3, \quad s = 0, 1, \ldots, r-1, \quad x \in \langle 0, c \rangle. \tag{2.38}$$

Take an arbitrary $\varphi_i \in X_i$. On account of (2.36), (2.i), (2.34) and (2.ii) we get

$$T_i(\varphi)(0) = 0.$$

Differentiating (2.36) σ times, we obtain as in Lemma 2

$$T_i^{(\sigma)}(\varphi)(x) = h_i^\sigma(x, \{\varphi_j[f_k(x)]\}, \ldots, \{\varphi_j^{(\sigma)}[f_k(x)]\}), \quad x \in \langle 0, c \rangle. \tag{2.39}$$

Putting $x = 0$ in this equality and making use of (2.i), (2.34) and (2.11) we have

$$T_i^{(c)}(\varphi)(0) = 0.$$

Thus $T_i(\varphi)$ fulfil (2.34).

It follows from (2.39) that

$$|T_i^{(r)}(\varphi)(x)| \leqslant \big|h_i^r(x, \{\varphi_j[f_k(x)]\}, \ldots, \{\varphi_j^{(r)}[f_k(x)]\}) - \\ -h_i^r(x, \{0\}, \ldots, \{0\})\big| + |h_i^r(x, \{0\}, \ldots, \{0\})|.$$

Since

$$\sup_{\langle 0,c\rangle} |\varphi_j^{(s)}[f_k(x)]| \leqslant \sup_{\langle 0,c\rangle} |\varphi_j^{(s)}(x)|, \qquad s = 0, 1, \ldots, r-1,$$

then by (2.38), (2.33), we may use inequalities (2.30) and (2.32). Therefore

$$|T_i^{(r)}(\varphi)(x)| \leqslant \sum_j \sum_k \Big[\sum_{s=0}^{r-1} L_{ijk}^s |\varphi_j^{(s)}[f_k(x)]| + \\ + (a_{ijk} + \tfrac{1}{2}\eta)\, |\varphi_j^{(r)}[f_k(x)]|\Big] + R_i - \sum_j \sum_k (a_{ijk} + \eta) R_j.$$

According to (2.i) and (2.38), we have

(2.40) $$\sup_{\langle 0,c\rangle} |\varphi_j^{(s)}[f_k(x)]| \leqslant \sup_{\langle 0,c\rangle} |\varphi_j^{(s)}(x)| \leqslant R_j c, \qquad s = 0, 1, \ldots, r-1,$$

which, in virtue (2.33), (2.i) and (2.35), implies the estimation

$$|T_i^{(r)}(\varphi)(x)| \leqslant R_i, \qquad x \in \langle 0, c\rangle.$$

Thus $T_i(\varphi)$ satisfies (2.35).

For $\varphi_i, \bar{\varphi}_i \in X_i$ we have

$$\varrho\big(T_i(\varphi),\ T_i(\bar{\varphi})\big) \leqslant \sum_j \sum_k (a_{ijk} + \eta)\, \varrho(\varphi_j, \bar{\varphi}_j).$$

In fact, since the transformations T_i map $X_1 \times \ldots \times X_m$ onto X_i, (2.37) and (2.39) imply that

$$\varrho\big(T_i(\varphi), T_i(\bar{\varphi})\big) \\ = \sup_{\langle 0,c\rangle} \big|h_i^r(x, \{\varphi_j[f_k(x)]\}, \ldots, \{\varphi_j^{(r)}[f_k(x)]\}) - \\ - h_i^r(x, \{\bar{\varphi}_j[f_k(x)]\}, \ldots, \{\bar{\varphi}_j^{(r)}[f_k(x)]\})\big|.$$

Using (2.40) and (2.33) for $\varphi_j \in X_j$ we obtain

$$|\varphi_j^{(s)}[f_k(x)]| \leqslant d_s, \qquad x \in \langle 0, c\rangle,\ s = 0, 1, \ldots, r-1,$$

and by (2.i), (2.35) and (2.33)

$$|\varphi_j^{(r)}[f_k(x)]| \leqslant d, \qquad x \in \langle 0, c\rangle.$$

Therefore, we may use inequalities (2.30). Thus

$$(2.41)\qquad \varrho\big(T_i(\varphi), T_i(\bar\varphi)\big) \leqslant \sum_j \sum_k \Big[\sum_{s=0}^{r-1} L^s_{ijk} \sup_{\langle 0,c\rangle} |\varphi_j^{(s)}[f_k(x)] - \bar\varphi_j^{(s)}[f_k(x)]| + \left(a_{ijk} + \frac{\eta}{2}\right) \sup_{\langle 0,c\rangle} |\varphi_j^{(r)}[f_k(x)] - \bar\varphi_j^{(r)}[f_k(x)]|\Big].$$

Because of $f_k(\langle 0, c\rangle) \subset \langle 0, c\rangle$ (see (2.i)) and in virtue of (2.37) we have

$$(2.42)\qquad \sup_{\langle 0,c\rangle} |\varphi_j^{(r)}[f_k(x)] - \bar\varphi_j^{(r)}[f_k(x)]| \leqslant \varrho(\varphi_j, \bar\varphi_j).$$

Similarly, using additionally (2.34) and the mean-value theorem, we obtain

$$(2.43)\qquad \sup_{\langle 0,c\rangle} |\varphi_j^{(s)}[f_k(x)] - \bar\varphi_j^{(s)}[f_k(x)]| \leqslant \sup_{\langle 0,c\rangle} |\varphi_j^{(s)}(x) - \bar\varphi_j^{(s)}(x)| \leqslant \sup_{\langle 0,c\rangle} |\varphi_j^{(r-1)}(x) - \bar\varphi_j^{(r-1)}(x)| \leqslant c\varrho(\varphi_j, \bar\varphi_j), \qquad s = 0, 1, \ldots, r-1.$$

According to (2.42), (2.43) and (2.33) we may go on with the estimation (2.41)

$$\varrho\big(T_i(\varphi), T_j(\bar\varphi)\big) \leqslant \sum_j \sum_k \Big[\varrho(\varphi_j, \bar\varphi_j)\frac{\eta}{2} + \left(a_{ijk} + \frac{\eta}{2}\right)\varrho(\varphi_j, \bar\varphi_j)\Big] = \sum_j \sum_k (a_{ijk} + \eta)\varrho(\varphi_j, \bar\varphi_j).$$

It follows from (2.31), Lemma 1 and Remark 1 that the remaining assumptions of the theorem contained in [12] are fulfilled. Thus there exists a unique solution $\varphi \in C^r_m[\langle 0, c\rangle]$ of the system of equations (1) in the interval $\langle 0, c\rangle$ satisfying conditions (2.10). The existence and the uniqueness of solutions in the whole interval I can be obtained just as in Theorem 1.

3. In this section we consider the problem of the continuous dependence of C^r_m-solutions of system (1), satisfying conditions (2.10), on the given functions. For this purpose we shall consider the sequence of the systems of equations;

$$(3.1)\qquad \varphi_i(x) = \underset{\nu}{h_i}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}\big), \qquad x \in I.$$

As in Section 2, we assume that

$$(3.\mathrm{i})\qquad \underset{\nu}{f_k} \in C^r[I], \qquad 0 < \underset{\nu}{f_k}(x) < x \qquad \text{for } x \in (0, a);$$

$$(3.\mathrm{ii})\qquad \underset{\nu}{h_i} \in C^r[\Omega], \qquad \underset{\nu}{h_i}(0, \{0\}) = 0;$$

(3.iii) all the characteristic roots of the matrix $[\sum_k \underset{\nu}{a}_{ijk}]$, where

$$\underset{\nu}{a}_{ijk} := \left| \frac{\partial \underset{\nu}{h}_i}{\partial y_{jk}} (0, \{0\}) [\underset{\nu}{f}'_k(0)]^r \right|, \tag{3.2}$$

are less than one in absolute value;

(3.iv) the sequences of functions $(\underset{\nu}{f}_k)$ and $(\underset{\nu}{h}_i)$ are almost uniformly convergent to the functions $\underset{0}{f}_k$ and $\underset{0}{h}_i$, together with all their derivatives up to the order r in I and Ω, respectively.

We define the functions $\underset{\nu}{h}_i^\sigma$ similarly to the former definition of h_i^σ, namely:

$$\underset{\nu}{h}_i^0(x, \{y_{jk}^0\}) := \underset{\nu}{h}_i(x, \{y_{jk}^0\}), \quad (x, \{y_{jk}^0\}) \in \Omega,$$

$$\underset{\nu}{h}_i^\sigma(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^\sigma\}) := \frac{\partial \underset{\nu}{h}_i^{\sigma-1}}{\partial x} (x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{\sigma-1}\}) + \tag{3.3}$$

$$+ \sum_{q=1}^{m} \sum_{t=1}^{n} \sum_{s=0}^{\sigma-1} \frac{\partial \underset{\nu}{h}_i^{\sigma-1}}{\partial y_{qt}^s} (x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{\sigma-1}\}) y_{qt}^{s+1} \underset{\nu}{f}'_t(x),$$

$$(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^\sigma\}) \in \Omega \times R^{mn\sigma}.$$

The functions $\underset{\nu}{h}_i^\sigma$ are assumed to satisfy (cf. (2.11)) the conditions

$$\underset{\nu}{h}_i^\sigma(0, \{0\}, \ldots, \{0\}) = 0. \tag{3.v}$$

If $\underset{\nu}{f}_k \in C^r[I]$ and $\underset{\nu}{h}_i \in C^r[\Omega]$, then Lemma 3 implies that

$$\underset{\nu}{h}_i^\sigma(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^\sigma\}) = \underset{\nu}{Z}_i^\sigma(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{\sigma-1}\}) + \tag{3.4}$$

$$+ \sum_{q=1}^{m} \sum_{t=1}^{n} \frac{\partial \underset{\nu}{h}_i}{\partial y_{qt}^0} (x, \{y_{jk}^0\}) y_{qt}^\sigma [\underset{\nu}{f}'_t(x)]^\sigma,$$

and

$$\underset{\nu}{h}_i \in C^{r-1}[\Omega \times R^{mn\sigma}], \qquad \underset{\nu}{Z}_i^\sigma \in C^{r-\sigma}[\Omega \times R^{mn(\sigma-1)}].$$

Hence and by (3.3) we get

$$\underset{\nu}{Z}_i^\sigma(0, \{0\}, \ldots, \{0\}) = 0. \tag{3.5}$$

LEMMA 6. *If the functions* $\underset{\nu}{f}_k \in C^r[I]$, $\underset{\nu}{h}_i \in C^r[\Omega]$ *satisfy* (3.iv), *then the sequences of the functions* $(\underset{\nu}{h}_i^\sigma)$ *and* $(\underset{\nu}{Z}_i^\sigma)$ *are almost uniformly convergent to the functions* $\underset{0}{h}_i^\sigma$ *and* $\underset{0}{Z}_i^\sigma$ *in the sets* $\Omega \times R^{mn\sigma}$ *and* $\Omega \times R^{mn(\sigma-1)}$, *respectively.*

Proof. Induction.

THEOREM 3. *If assumptions* (3.1), (3.ii), (3.iii), (3.v) *are fulfilled and if there exists a number* $\xi > 0$ *such that*

$$|f_k'(x)| \leqslant 1 \quad \text{for } x \in \langle 0, \xi\rangle, \tag{3.6}$$

then for an arbitrary ν *system* (3.1) *has exactly one solution* $\underset{\nu}{\varphi} \in C_m^r[I]$ *in the interval* I *satisfying conditions* (2.10). *If, moreover,* (3.iv) *is fulfilled, then the sequences* $(\underset{\nu}{\varphi})$ *and* $(\underset{\nu}{\varphi}^{(\sigma)})$ *are almost uniformly convergent to* $\underset{0}{\varphi}$ *and* $\underset{0}{\varphi}^{(\sigma)}$, *respectively, in the interval* I.

Proof. For any fixed ν, the existence and the uniqueness of a solution $\underset{\nu}{\varphi} \in C_m^r[I]$ satisfying (2.10) are ensured by Theorem 1. To prove the second part of our theorem we use J. Matkowski's lemma contained in [9] (Lemma 2). Let

$$A := X_1 \times \ldots \times X_m,$$

where X_j are defined just as in Theorem 1. From (3.iii) and Lemma 1 it follows that there exist an $\eta > 0$ and R_i such that

$$\sum_j \sum_k (\underset{0}{a}_{ijk} + \eta) R_j < R_i, \qquad 0 < R_i < 1.$$

According to (3.iv) and Lemma 6, we infer that for arbitrarily fixed numbers c_2, $0 < c_2 < a$ and $d_1 > 0$ there exists a positive integer q_1 such that for $\nu \geqslant q_1$, $x \in \langle 0, c_2\rangle$, $|y_{jk}^s| \leqslant d_1$, $s = 0, 1, \ldots, r-1$, the following inequalities hold:

$$|\underset{\nu}{Z_i^r}(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\})| \leqslant |\underset{0}{Z_i^r}(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\})| + \tfrac{1}{2}\Big[R_i - \sum_j \sum_k (\underset{0}{a}_{ijk} + \eta) R_j\Big]$$

and

$$\left|\frac{\partial \underset{\nu}{h_i}}{\partial y_{jk}^0}(x, \{y_{jk}^0\})[\underset{\nu}{f_k'}(x)]^r\right| \leqslant \left|\frac{\partial \underset{0}{h_i}}{\partial y_{jk}^0}(x, \{y_{jk}^0\})[\underset{0}{f_k'}(x)]^r\right| + \frac{\eta}{2}.$$

On account of (3.5) and (3.2) there exist numbers c_1 and d, $0 < d \leqslant d_1$, $0 < c_1 < c_2$, such that for all $\nu \geqslant q_1$, $x \in \langle 0, c_1\rangle$, $|y_{jk}^s| \leqslant d$, $s = 0, 1, \ldots, r-1$ we have

$$|\underset{\nu}{Z_i^r}(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\})| \leqslant R_i - \sum_j \sum_k (\underset{0}{a}_{ijk} + \eta) R_j,$$

and

$$\left|\frac{\partial \underset{\nu}{h_i}}{\partial y_{jk}^0}(x, \{y_{jk}^0\})[\underset{\nu}{f_k}(x)]^r\right| \leqslant \underset{0}{a}_{ijk} + \eta.$$

Put

$$c := \min\{1, c_1, \xi, d\},$$
$$D := \{(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\});\ 0 \leqslant x \leqslant c,\ |y_{jk}^l| \leqslant R_j,\ l = 0, 1, \ldots, r-1\},$$
$$D' := \{(x, \{y_{jk}^0\});\ |y_{jk}^0| \leqslant R_j\}.$$

Take arbitrary numbers $\varepsilon_i > 0$ fulfilling the system of inequalities

$$\sum_j \sum_k (\underset{0}{a}_{ijk} + \eta)\varepsilon_j < \varepsilon_i.$$

By Lemma 6 and in virtue of the uniform continuity of the functions $\underset{0}{Z}{}_i^r$ in the set D, $\partial \underset{0}{h}_i/\partial y_{jk}^0$ in the set D' and $\underset{0}{f}_k$ in the interval $\langle 0, c\rangle$, we get the existence of a $\delta > 0$ (depending on ε_i) and a positive integer q_2 such that for all $(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\})$, $(\bar{x}, \{\bar{y}_{jk}^0\}, \ldots, \{\bar{y}_{jk}^{r-1}\}) \in D$, $|x - \bar{x}| < \delta$, $|y_{jk}^l - \bar{y}_{jk}^l| < \delta$, $l = 0, 1, \ldots, r-1$ and $\nu \geqslant q_2$ or $\nu = 0$ we have

$$\begin{aligned}
&|\underset{\nu}{Z}{}_i^r(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\}) - \underset{\nu}{Z}{}_i^r(\bar{x}, \{\bar{y}_{jk}^0\}, \ldots, \{\bar{y}_{jk}^{r-1}\})| \\
&\qquad \leqslant |\underset{\nu}{Z}{}_i^r(x, \{y_{jk}^0\}, \ldots, \{y_{jk}^{r-1}\}) - \underset{0}{Z}{}_i^r(\bar{x}, \{\bar{y}_{jk}^0\}, \ldots, \{\bar{y}_{jk}^{r-1}\})| + \\
&\qquad\quad + |\underset{\nu}{Z}{}_i^r(\bar{x}, \{\bar{y}_{jk}^0\}, \ldots, \{\bar{y}_{jk}^{r-1}\}) - \underset{0}{Z}{}_i^r(\bar{x}, \{\bar{y}_{jk}^0\}, \ldots, \{\bar{y}_{jk}^{r-1}\})| \\
&\qquad \leqslant \tfrac{1}{2}\Big[\varepsilon_i - \sum_j \sum_k (\underset{0}{a}_{ijk} + \eta)\varepsilon_j\Big]
\end{aligned}$$

and, similarly,

$$\sum_{q=1}^{m} \sum_{t=1}^{n} \left| \frac{\partial \underset{\nu}{h}_i}{\partial y_{qt}^0}(x, \{y_{jk}^0\})[\underset{\nu}{f}{}_i'(x)]^r - \frac{\partial \underset{\nu}{h}_i}{\partial y_{qt}^0}(\bar{x}, \{\bar{y}_{jk}^0\})[\underset{\nu}{f}{}_i'(\bar{x})]^r \right|$$
$$\leqslant \tfrac{1}{2}\Big[\varepsilon_i - \sum_j \sum_k (\underset{0}{a}_{ijk} + \eta)\varepsilon_j\Big].$$

Having chosen such R_j, c, ε_j, δ, we define the sets X_j by conditions (2.19), (2.20) and (2.21) and the transformations $\underset{\nu}{T}$ on the set $X_1 \times \ldots \times X_m$ by the formulas

$$\underset{\nu}{T}(\varphi)(x) := \big(\underset{\nu}{T}_1(\varphi)(x), \ldots, \underset{\nu}{T}_m(\varphi)(x)\big), \tag{3.7}$$

where

$$\underset{\nu}{T}_i(\varphi)(x) := \underset{\nu}{h}_i\big(x, \{\varphi_j[\underset{\nu}{f}_k(x)]\}\big), \qquad x \in \langle 0, c\rangle. \tag{3.8}$$

We shall show that the set A and the transformations $\underset{\nu}{T}$ for all $\nu \geqslant q := \max\{q_1, q_2\}$ or $\nu = 0$ fulfil all the assumptions of J. Matkowski's lemma [9]. To this purpose we define the following metric in the set A:

$$\varrho(\varphi, \bar{\varphi}) := \sum_i \sup_{\langle 0, c\rangle} |\varphi_i^{(r)}(x) - \bar{\varphi}_i^{(r)}(x)|.$$

It is easily seen that A is compact. For $\nu \geqslant q$ or $\nu = 0$ we infer as in the proof of Theorem 1 that $\underset{\nu}{T}$ are continuous transformations mapping A into A, which have exactly one fixed point $\underset{\nu}{\varphi}$ in the set A. Now we shall show that the sequence $(\underset{\nu}{T})$ is uniformly convergent to $\underset{0}{T}$ in the set A.

From the definition of the metric ϱ and on account of (3.7), (3.8) and (2.26) (replacing $T_i^{(r)}$ by $\underset{\nu}{T_i^{(r)}}$, h_i^r by $\underset{\nu}{h_i^r}$ and f_k by $\underset{\nu}{f_k}$) we get

$$\begin{aligned}(3.9)\qquad \varrho\big(\underset{\nu}{T}(\varphi), \underset{0}{T}(\varphi)\big) \leqslant \sum_i \sup_{\langle 0,c\rangle} \Big|&\underset{\nu}{h_i^r}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{\nu}{f_k}(x)]\}\big) - \\ &- \underset{0}{h_i^r}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{\nu}{f_k}(x)]\}\big)\Big| + \\ + \sum_i \sup_{\langle 0,c\rangle} \Big|&\underset{0}{h_i^r}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{\nu}{f_k}(x)]\}\big) - \\ &- \underset{0}{h_i^r}\big(x, \{\varphi_j[\underset{0}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{0}{f_k}(x)]\}\big)\Big|.\end{aligned}$$

Since $x \in \langle 0, c\rangle$, then by (2.25), (3.i) and (2.20) the sequences $(\underset{\nu}{h_i^r})$ tend uniformly to $\underset{0}{h_i^r}$ in the set $\langle 0, c\rangle \times \langle -1, 1\rangle^{mn(r+1)}$. Therefore, for every $\varepsilon > 0$ there exists a positive integer N_1 such that for all $\nu \geqslant N_1$ and $x \in \langle 0, c\rangle$ the following inequalities hold:

$$\begin{aligned}(3.10)\qquad \Big|&\underset{\nu}{h_i^r}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{\nu}{f_k}(x)]\}\big) - \\ &- \underset{0}{h_i^r}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{\nu}{f_k}(x)]\}\big)\Big| \leqslant \frac{\varepsilon}{2m}.\end{aligned}$$

Moreover, from the facts that X_j are compact and $\underset{0}{h_i^r}$ are uniformly continuous in $\langle 0, c\rangle \times \langle -1, 1\rangle^{mn(r+1)}$ it follows that there exists a positive integer N_2 such that for every $\varphi \in A$, $\nu \geqslant N_2$ and $x \in \langle 0, c\rangle$ we have

$$\begin{aligned}(3.11)\qquad \Big|&\underset{0}{h_i^r}\big(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{\nu}{f_k}(x)]\}\big) - \\ &- \underset{0}{h_i^r}\big(x, \{\varphi_j[\underset{0}{f_k}(x)]\}, \ldots, \{\varphi_j^{(r)}[\underset{0}{f_k}(x)]\}\big)\Big| < \frac{\varepsilon}{2m}.\end{aligned}$$

Inserting inequalities (3.10) and (3.11) into (3.9), we get

$$\varrho\big(\underset{\nu}{T}(\varphi), \underset{0}{T}(\varphi)\big) < \varepsilon$$

for every $\nu \geqslant \max\{N_1, N_2\}$ and $\varphi \in A$.

Thus the sequence $(\underset{\nu}{T})$ tends uniformly to $\underset{0}{T}$ in the set A. By J. Matkowski's lemma [9] the sequences $(\underset{\nu}{\varphi}^{(s)})$, $s = 0, 1, \ldots, r$, tend uniformly to $\underset{0}{\varphi}^{(s)}$ in the interval $\langle 0, c\rangle$.

It remains to show that $(\underset{\nu}{\varphi}^{(s)})$, $s = 0, 1, \dots, r$, tend almost uniformly to $\underset{0}{\varphi}^{(s)}$ in the whole interval I.

Let d be the supremum of all $b < a$ such that the sequences $(\underset{\nu}{\varphi}^{(s)})$, $s = 0, 1, \dots, r$, tend uniformly to $\underset{0}{\varphi}^{(s)}$ in the interval $\langle 0, b\rangle$. For the indirect proof suppose that $d < a$. Assumptions (3.i) and (3.iv) imply that there exists a $\zeta > 0$ such that

$$\underset{\nu}{f}_k(\langle 0, d+\zeta\rangle) \subset \langle 0, d-\zeta\rangle.$$

The functions $\underset{\nu}{\varphi}_i^{(s)}$, $s = 0, 1, \dots, r$, are uniformly continuous in $\langle 0, d-\zeta\rangle$; the sequences $(\underset{\nu}{\varphi}_i^{(s)})$ and $(\underset{\nu}{f}_k)$, $s = 0, 1, \dots, r$, tend uniformly to $\underset{0}{\varphi}_i^{(s)}$ and $\underset{0}{f}_k$ in the intervals $\langle 0, d-\zeta\rangle$ and $\langle 0, d+\zeta\rangle$, respectively. Therefore for an arbitrary $\varepsilon > 0$ there exists a positive integer N_3 such that for every $\nu \geqslant N_3$ and $x \in \langle 0, d+\zeta\rangle$ we have

$$|\underset{\nu}{\varphi}_i^{(s)}[\underset{\nu}{f}_k(x)] - \underset{0}{\varphi}_i^{(s)}[\underset{0}{f}_k(x)]| \leqslant |\underset{\nu}{\varphi}_i^{(s)}[\underset{\nu}{f}_k(x) - \underset{\nu}{\varphi}_i^{(s)}[\underset{0}{f}_k(x)]| + \\ + |\underset{\nu}{\varphi}_i^{(s)}[\underset{0}{f}_k(x)] - \underset{0}{\varphi}_i^{(s)}[\underset{0}{f}_k(x)]| \leqslant \varepsilon,$$

$s = 0, 1, \dots r$, which means that $(\underset{\nu}{\varphi}_i^{(s)}[\underset{\nu}{f}_k])$ tend uniformly to $\underset{0}{\varphi}_i^{(s)}[\underset{0}{f}_k]$, $s = 0, 1, \dots, r$, in $\langle 0, d+\zeta\rangle$. Hence and from the fact that the functions $\underset{\nu}{h}_i^{(s)}$, $s = 0, 1, \dots, r$, are uniformly continuous in $\langle 0, d+\zeta\rangle \times \times \langle -1, 1\rangle^{mn(s+1)}$ we infer that for any $\varepsilon > 0$ there exists a positive integer N_4 such that for every $\nu \geqslant N_4$, $x \in \langle 0, d+\zeta\rangle$, $s = 0, 1, \dots, r$, we have

$$\text{(3.12)} \quad \Big|\underset{\nu}{h}_i^s(x, \{\underset{\nu}{\varphi}_j[\underset{\nu}{f}_k(x)]\}, \dots, \{\underset{\nu}{\varphi}_j^{(s)}[\underset{\nu}{f}_k(x)]\}) - \\ - h_i^s(x, \{\underset{0}{\varphi}_j[\underset{0}{f}_k(x)]\}, \dots, \{\underset{0}{\varphi}_j^{(s)}[\underset{0}{f}_k(x)]\})\Big| \leqslant \frac{\varepsilon}{2}.$$

By Lemma 6 there is a positive integer N_5 such that for every $\nu \geqslant N_5$, $x \in \langle 0, d+\zeta\rangle$, $s = 0, 1, \dots, r$, we have

$$\Big|\underset{\nu}{h}_i^s(x, \{\underset{0}{\varphi}_j[\underset{0}{f}_k(x)]\}, \dots, \{\underset{0}{\varphi}_j^{(s)}[\underset{0}{f}_k(x)]\}) - \underset{0}{h}_i^s(x, \{\underset{0}{\varphi}_j[\underset{0}{f}_k(x)]\}, \dots, \{\underset{0}{\varphi}_j^{(s)}[\underset{0}{f}_k(x)]\})\Big| < \frac{\varepsilon}{2}.$$

These inequalities together with (3.12) and (2.2) (with φ_i replaced by $\underset{\nu}{\varphi}_i$, h_i by $\underset{\nu}{h}_i$ and f_k by $\underset{\nu}{f}_k$) imply that for every $\nu \geqslant \max\{N_3, N_4, N_5\}$, $s = 0, 1, \dots, r$,

$$\sup_{\langle 0,d+\zeta\rangle} |\underset{\nu}{\varphi}{}_i^{(s)}(x)-\underset{0}{\varphi}{}_i^{(s)}(x)| \leqslant \sup_{\langle 0,d+\zeta\rangle} \Big(\big|\underset{\nu}{h}{}_i^s(x, \{\underset{\nu}{\varphi}_j[f_k(x)]\}, \ldots, \{\underset{\nu}{\varphi}{}_j^{(s)}[f_k(x)]\}) -$$
$$- \underset{\nu}{h}{}_i^s(x, \{\underset{0}{\varphi}_j[\underset{0}{f}_k(x)]\}, \ldots, \{\underset{0}{\varphi}{}_j^{(s)}[\underset{0}{f}_k(x)]\})\big| +$$
$$+ \big|\underset{\nu}{h}{}_i^s(x, \{\underset{0}{\varphi}_j[\underset{0}{f}_k(x)]\}, \ldots, \{\underset{0}{\varphi}{}_j^{(s)}[\underset{0}{f}_k(x)]\}) -$$
$$- \underset{0}{h}{}_i^s(x, \{\underset{0}{\varphi}_j[\underset{0}{f}_k(x)]\}, \ldots, \{\underset{0}{\varphi}{}_j^{(s)}[\underset{0}{f}_k(x)]\})\big|\Big) < \varepsilon,$$

which proves that $(\underset{\nu}{\varphi}{}^{(s)})$ tend uniformly to $\underset{0}{\varphi}{}^{(s)}$ in $\langle 0, d+\zeta\rangle$, contrary to the definition of d. This contradiction finishes the proof of the uniform convergence of $(\underset{\nu}{\varphi}{}^{(s)})$ to $\underset{0}{\varphi}{}^{(s)}$, $s = 0, 1, \ldots, r$, in the whole interval I, and thus also the proof of Theorem 3.

Remark 2. If we assumed condition (3.iii) for $\nu = 0$ only, we would get the first part of our assertion for ν sufficiently large. This can be seen by Lemma 1 and from the fact that for an arbitrarily small positive constant η the inequalities

$$\underset{\nu}{a}_{ijk} \leqslant \underset{0}{a}_{ijk} + \eta$$

hold for ν sufficiently large.

THEOREM 4. *Let assumptions* (3.i), (3.ii), (3.iii), (3.iv) (3.v) *be fulfilled. If, moreover,*

(3.13) *there exist a number $L > 0$ and a set $U_1 := \langle 0, c_1\rangle \times \langle -d_1, d_1\rangle^{mn}$ in which the inequalities*

$$\left|\frac{\partial^r \underset{\nu}{h}_i}{\partial x^l\, \partial y_{mn}^{s_{11}} \ldots \partial y_{mn}^{s_{mn}}}(x, \{y_{jk}\}) - \frac{\partial^r \underset{\nu}{h}_i}{\partial x^l\, \partial y_{11}^{s_{11}} \ldots \partial y_{mn}^{s_{mn}}}(x, \{\bar{y}_{jk}\})\right| \leqslant L \sum_j \sum_k |y_{jk} - \bar{y}_{jk}|$$

hold for $l = 0, 1, \ldots, r$, $s_{jk} = 0, 1, \ldots, r-l$, $s_{11} + \ldots + s_{mn} = r - l$,

then for an arbitrary ν system (3.1) *has exactly one solution $\underset{\nu}{\varphi} \in C_m^r[I]$ in I satisfying conditions* (2.10) *and the sequences of the solutions $(\underset{\nu}{\varphi})$ and their derivatives $(\underset{\nu}{\varphi}{}^{(\sigma)})$ tend almost uniformly to $\underset{0}{\varphi}$ and $\underset{0}{\varphi}{}^{(\sigma)}$ in the interval I.*

Proof. For a fixed ν, the existence and the uniqueness follows from Theorem 2. It remains to show that $(\underset{\nu}{\varphi})$ and $(\underset{\nu}{\varphi}{}^{(\sigma)})$ are almost uniformly convergent to $\underset{0}{\varphi}$ and $\underset{0}{\varphi}{}^{(\sigma)}$ in I.

By (3.iii), (3.2) and Lemma 1 there exist numbers $\eta > 0$ and $R_i > 0$ such that

$$\sum_j \sum_k (\underset{0}{a}_{ijk} + \eta)\, R_j < R_i. \tag{3.14}$$

On account of (3.iv) and (3.2) there exists a positive integer N_1 and a set $U_2 := \langle 0, c_2\rangle \times \langle -d_2, d_2\rangle^{mn}$, $0 < c_2 < a$, $d_2 > 0$, such that for all $(x, \{y^0_{jk}\}) \in U_2$ and $\nu \geqslant N_1$ or $\nu = 0$ we have

$$(3.15) \qquad \left| \frac{\partial \underset{\nu}{h_i}}{\partial y^0_{jk}} (x, \{y^0_{jk}\}) [\underset{\nu}{f'_k}(x)]^r \right| \leqslant \underset{0}{a_{ijk}} + \frac{\eta}{2}.$$

According to (3.13) and Lemma 5, the functions $\underset{\nu}{h^r_i}$ fulfil the Lipschitz condition with respect to y^s_{jk} with the constants $\underset{\nu}{L^s_{ijk}}$, $s = 0, 1, \ldots, r$, in the set $Z := (U_1 \cap U_2) \times \langle -d, d\rangle^{mn}$, $d \geqslant \max\limits_i R_i$, and, by (3.15) and (3.iv), we may assume without loss of generality that the numbers $\underset{\nu}{L^s_{ijk}}$, $s = 0, 1, \ldots, r$, where $\underset{\nu}{L^r_{ijk}} = \underset{0}{a_{ijk}} + \eta/2$, are independent of the choice of the number $\nu \geqslant N_1$. Thus

$$(3.16) \quad |\underset{\nu}{h^r_i}(x, \{y^0_{jk}\}, \ldots, \{y^r_{jk}\}) - \underset{\nu}{h^r_i}(x, \{\bar{y}^0_{jk}\}, \ldots, \{\bar{y}^r_{jk}\})| \leqslant \sum_j \sum_k \sum_{s=0}^{r} L^s_{ijk} |y^s_{jk} - \bar{y}^s_{jk}|,$$

where $L^r_{ijk} = \underset{0}{a_{ijk}} + \eta/2$, $\nu \geqslant N_1$ or $\nu = 0$.

In virtue of (3.14), (3.iv) and (3.v) there exist a $c_3 > 0$ and a positive integer N_2 such that for all $x \in \langle 0, c_3\rangle$ and $\nu \geqslant N_2$ or $\nu = 0$ the inequalities

$$(3.17) \qquad |\underset{\nu}{h^r_i}(x, \{0\}, \ldots, \{0\})| \leqslant R_i - \sum_j \sum_k (\underset{0}{a_{ijk}} + \eta) R_j$$

are fulfilled.

We define (as in the proof of Theorem 2) the number

$$c := \min\left\{c_1, c_2, c_3, 1, \min_i \frac{d_3}{R_i}, \min_{i,j,k} \frac{\eta}{2\sum\limits_{s=0}^{r-1} L^s_{ijk}}\right\},$$

where $d_3 := \min\{d_1, d_2, d\}$.

Let (X_i, ϱ) be the metric spaces of all functions of class C^r in $\langle 0, c\rangle$ fulfilling conditions (2.34) and (2.35), the metric ϱ being defined by (2.37). Let $\underset{\nu}{T_i}$ be the transformations defined for $\varphi \in X_1 \times \ldots \times X_m$ by the formula

$$\underset{\nu}{T_i}(\varphi)(x) := \underset{\nu}{h_i}(x, \{\varphi_j[\underset{\nu}{f_k}(x)]\}), \qquad x \in \langle 0, c\rangle.$$

We shall show that the spaces X_i and the transformations $\underset{\nu}{T_i}$, $\nu \geqslant N := \max\{N_1, N_2\}$ or $\nu = 0$, satisfy all the assumptions of the lemma of K. Baron [1]. Making use of (3.16) instead of (2.30) and of (3.17) instead of (2.32), just as in the proof of Theorem 2, we can show that for all $\nu \geqslant N$

or $\nu = 0$, the transformations T_i are of the type $X_1 \times \ldots \times X_m \to X_i$ and that for all $\varphi, \bar{\varphi} \in X_1 \times \ldots \times X_m$

$$\varrho\big(\underset{\nu}{T_i}(\varphi), \underset{\nu}{T_i}(\bar{\varphi})\big) \leqslant \sum_j \sum_k (\underset{0}{a_{ijk}} + \eta)\,\varrho(\varphi_j, \bar{\varphi}_j).$$

Putting

$$a_{ij} := \sum_k (\underset{0}{a_{ijk}} + \eta)$$

and using (3.14), Lemma 1 and Remark 1, we infer that

$$c^l_{\lambda\lambda} > 0, \quad l = 0, 1, \ldots, m-1, \quad \lambda = 1, \ldots, m-l.$$

On account of (2.37) and (2.39) we get

$$\varrho\big(\underset{\nu}{T_i}(\varphi), \underset{0}{T_i}(\varphi)\big) = \sup_{\langle 0, c\rangle} |\underset{\nu}{T_i^{(r)}}(\varphi)(x) - \underset{0}{T_i^{(r)}}(\varphi)(x)| = \sup_{\langle 0, c\rangle} \big|\underset{\nu}{h_i^r}\big(x, \{\underset{\nu}{\varphi_j}[f_k(x)]\}, \ldots$$
$$\ldots, \{\underset{\nu}{\varphi_j^{(r)}}[f_k(x)]\}\big) - \underset{0}{h_i^r}\big(x, \{\underset{0}{\varphi_j}[f_k(x)]\}, \ldots, \{\underset{0}{\varphi_j^{(r)}}[f_k(x)]\}\big)\big|.$$

By Lemma 5 the sequences $(\underset{\nu}{h_i^r})$ tend almost uniformly to $\underset{0}{h_i^r}$ in the set $\Omega \times R^{mnr}$. Hence and from the uniform continuity of $\underset{0}{h_i^r}$ in the set $\langle 0, c\rangle \times \times [\langle -R_1, R_1\rangle \times \ldots \times \langle -R_m, R_m\rangle]^{(r+1)n}$ it follows that $\varrho\big(\underset{\nu}{T_i}(\varphi), \underset{0}{T_i}(\varphi)\big)$ tends to zero. By a lemma of K. Baron [1] the sequences $(\underset{\nu}{\varphi_i})$ are convergent to $\underset{0}{\varphi_i}$. In our case this means that $(\underset{\nu}{\varphi_i^{(s)}})$ tend uniformly to $\underset{0}{\varphi_i^{(s)}}$, $s = 0, 1, \ldots, r$, in the interval $\langle 0, c\rangle$. In the same manner as in the proof of Theorem 3 we can show that $(\underset{\nu}{\varphi_i^{(s)}})$ tend almost uniformly to $\underset{0}{\varphi_i^{(s)}}$, $s = 0, 1, \ldots, r$, in the whole interval I. Thus the proof of Theorem 4 is finished.

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