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On L^p-estimates for solutions of the Cauchy problem for parabolic differential equations

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In the present paper we deal with a system of second order semi-linear parabolic equations in an unbounded zone and prove theorems on estimates in L^p-norm for the difference of solutions of the Cauchy problem and for the solution itself. Theorems of this kind for linear systems of even order, parabolic in the sense of Petrowskiĭ, can be found in Eĭdelman's monograph [3]. However, they are derived there from properties of the fundamental solution and, consequently, they require assumptions ensuring its existence. We do not make use of a fundamental solution; our method is patterned on that applied in [1] and [2] and require weaker assumptions. In particular we assume a weak parabolicity instead of the stronger one following the Petrowski's definition. Moreover, the coefficients of the equations are allowed to approach infinity whereas in [3] they were assumed to be bounded. The Hölder continuity of the coefficients assumed in [3] is not needed here but, on the other hand, since we make use of the adjoint operator we have to assume the coefficients to have suitable derivatives.

As a consequence of the result obtained here one can get a theorem on continuous dependence of solutions (in L^p-norm) on the initial data and on the non-linear terms appearing in the equations. This implies in tern some uniqueness theorems similar to those of paper [2].

Denote by $x = (x_1, \ldots, x_n)$ points in n-dimensional Euclidean space $\mathscr{E}^n$ $(n \geqslant 1)$ and by t points of the interval $\langle t_0, t_1\rangle$. Let $S = (t_0, t_1\rangle \times \mathscr{E}^n$, $\bar{S} = \langle t_0, t_1\rangle \times \mathscr{E}^n$. Define the differential operators

$$\mathscr{L}_0^i u^i \equiv \sum_{j,k=1}^{n} a_{jk}^i(t, x) u^i_{x_j x_k} + \sum_{j=1}^{n} b_j^i(t, x) u^i_{x_j}.$$

We shall treat the following systems:

$$(1) \qquad u_t^i = \mathscr{L}_0^i u^i + f^i(t, x, u)$$

$$(2) \qquad v_t^i = \mathscr{L}_0^i v^i + g^i(t, x, v) \qquad (i = 1, \ldots, m),\ (t, x) \in S.$$

A vector-valued function $u(t,x) = \{u^1(t,x), \dots, u^m(t,x)\}$ will be called a *solution of system* (1) (or (2)) if the components $u^i(t,x)$ are continuous in $\bar{S}$ and their derivatives appearing in (1) exist at each point of S and satisfy system (1) (or (2)). The derivatives are assumed to be integrable in any finite cylinder $S_R = (t_0, t_1) \times (|x| < R)$.

We make the following preliminary assumptions:

(A$_1$) the coefficients a^i_{jk}, b^i_j and their derivatives $(a^i_{jk})_{x_j}$, $(a^i_{jk})_{x_k}$, $(a^i_{jk})_{x_j x_k}$, $(b^i_j)_{x_j}$ are measurable and bounded in any finite cylinder $\bar{S}_R = \langle t_0, t_1 \rangle \times (|x| \leqslant R)$,

(A$_2$) $\sum_{j,k=1}^{n} a^i_{jk}(t,x)\xi_j\xi_k \geqslant 0$ $(i = 1, \dots, m)$ for any real vector $(\xi_1, \dots, \xi_n)$ and for $(t,x) \in \bar{S}$,

(A$_3$) functions $f^i(t,x,u)$, $g^i(t,x,u)$ $(i = 1, \dots, m)$ are defined for $(t,x) \in \bar{S}$, $u = \{u^1, \dots, u^m\}$ arbitrary and satisfy the inequalities

$$[f^i(t,x,u) - g^i(t,x,v)]\operatorname{sgn}(u^i - v^i) \leqslant \sum_{s=1}^{m} c^i_s(t,x)|u^s - v^s| + h^i(t,x)$$

$(i = 1, \dots, m)$ almost everywhere in $\bar{S}$, where $h^i(t,x)$ and $c^i_s(t,x)$ are continuous functions in $\bar{S}$, $h^i(t,x) \geqslant 0$, $c^i_s(t,x) \geqslant 0$ for $s \neq i$, and $\sum_{s=1}^{m} c^i_s(t,x) \leqslant c = \text{const} \geqslant -1$.

Let $u(t,x) = \{u^1(t,x), \dots, u^m(t,x)\}$, $\Phi(t,x) = \{\Phi^1(t,x), \dots, \Phi^m(t,x)\}$ be functions continuous in $\bar{S}$ and $\Phi^i(t,x) \geqslant 0$ $(i = 1, \dots, m)$. A function $u(t,x)$ will be said to belong to the space $L^p(\mathscr{E}^n)$, $1 \leqslant p < \infty$, with the weight function $\Phi(t,x)$ if the norm

$$\|u(t,\cdot)\|_{p,\Phi} = \Big(\sum_{i=1}^{m} \int_{\mathscr{E}^n} |u^i(t,x)|^p \Phi^i(t,x)\,dx\Big)^{1/p}$$

is finite for any $t \in \langle t_0, t_1 \rangle$.

Denoting by $\tilde{\mathscr{L}}^i_0$ the operator adjoint to $\mathscr{L}^i_0$;

$$\tilde{\mathscr{L}}^i_0 \Phi^i \equiv \sum_{j,k=1}^{n} \big(a^i_{kj}(t,x)\Phi^i\big)_{x_j x_k} - \sum_{j=1}^{n} \big(b^i_j(t,x)\Phi^i\big)_{x_j},$$

put

$$\tilde{\mathscr{L}}^i \Phi \equiv \tilde{\mathscr{L}}^i_0 \Phi^i + \sum_{s=1}^{m} c^s_i \Phi^s + \Phi^i_t \quad (i = 1, \dots, m).$$

THEOREM 1. *Let $u = \{u^i\}$, $v = \{v^i\}$ be solutions of systems* (1), (2) *respectively and let assumptions* (A$_1$) - (A$_3$) *be satisfied. Assume there is a vector function $\Phi = \{\Phi^i\}$ such that $\Phi^i \in C^2(\bar{S})$, $\Phi^i > 0$ in every compact subset of $\bar{S}$ and*

(3) $$\tilde{\mathscr{L}}^i \Phi \leqslant 0 \quad (i = 1, \dots, m)$$

almost everywhere in $\bar{S}$. *Set*

$$w^i = u^i - v^i, \quad w = \{w^1, \ldots, w^m\},$$
$$\psi^i = \max_j \sum_k |(a^i_{jk} + a^i_{kj})\Phi^i_{x_k}| + \Phi^i \Big[\max_{jk} |a^i_{jk}| + \max_j \Big| \sum_k (a^i_{jk} + a^i_{kj})_{x_k} - b^i_j \Big|\Big],$$
$$\Psi = (\psi^1, \ldots, \psi^m), \quad h = (h^1, \ldots, h^m)$$

and assume, moreover, that

$$(4) \qquad \int_{t_0}^{t_1} \|w(\tau, \cdot)\|^p_{p,\Psi} d\tau < \infty, \qquad \int_{t_0}^{t_1} \|h(\tau, \cdot)\|^p_{p,\Phi} d\tau < \infty.$$

Under the above assumptions if $\|w(t_0, \cdot)\|_{p,\Phi}$ *is finite, then* $\|w(t, \cdot)\|_{p,\Phi}$ *is finite for any* $t \in \langle t_0, t_1 \rangle$ *and we have the estimate*

$$(5) \qquad \max_{\langle t_0, t_1 \rangle} \|w(t, \cdot)\|_{p,\Phi} \leqslant [\|w(t_0, \cdot)\|_{p,\Phi} + (t_1 - t_0)^{1/p} \max_{\langle t_0, t_1 \rangle} \|h(t, \cdot)\|_{p,\Phi}] e^{\frac{p-1}{p}(c+1)(t_1 - t_0)}$$

or in a more precise form

$$(6) \qquad \|w(t, \cdot)\|_{p,\Phi} \leqslant \|w(t_0, \cdot)\|_{p,\Phi} \cdot e^{\frac{p-1}{p}(c+1)(t-t_0)} + \Big(\int_{t_0}^{t} \|h(\tau, \cdot)\|^p_{p,\Phi} \cdot e^{(p-1)(c+1)(t-\tau)} d\tau\Big)^{1/p}$$

for every $t \in \langle t_0, t_1 \rangle$.

Proof. Estimate (5) is an immediate consequence of (6), so we prove (6). To this end we take advantage of the Green's identity

$$(7) \qquad \sum_i (z^i \varphi^i)_t \equiv \sum_i z^i (\tilde{\mathscr{L}}^i_0 \varphi^i + \varphi^i_t) - \sum_i \varphi^i (\mathscr{L}^i_0 z^i - z^i_t) + \sum_i \sum_j \Big[\varphi^i \sum_k a^i_{jk} z^i_{x_k} - z^i \sum_k (a^i_{kj} \varphi^i)_{x_k} + b^i_j z^i \varphi^i\Big]_{x_j}$$

which is valid for any functions $z(t, x) = \{z^i(t, x)\}$, $\varphi(t, x) = \{\varphi^i(t, x)\}$ having the derivatives occurring in operators $\tilde{\mathscr{L}}^i_0$. The functions $\varphi^i(t, x)$ will be suitably chosen later on. In our case they will be smooth and non-negative in $\bar{S}$, with compact support as functions of x in $\mathscr{E}^n$. Then integrating (7) over the strip $(t_0, t) \times \mathscr{E}^n$, $t \in (t_0, t_1\rangle$, yields

$$(8) \qquad \sum_i \int_{\mathscr{E}^n} z^i \varphi^i dx = \sum_i \int_{\mathscr{E}^n} z^i \varphi^i \Big|_{t=t_0} dx + \int_{t_0}^{t} d\tau \int_{\mathscr{E}^n} \sum_i [z^i (\tilde{\mathscr{L}}^i_0 \varphi^i + \varphi^i_t) - \varphi^i (\mathscr{L}^i_0 z^i - z^i_t)]\Big|_{t=\tau} dx.$$

Now we take in (8) $z^i = (\bar{w}^i)^p$, where $\bar{w}^i = [(w^i)^2 + \varepsilon]^{1/2}$, $\varepsilon > 0$. By direct computation we get

$$\mathscr{L}^i_0 z^i - z^i_t = p(\bar{w}^i)^{p-2} w^i (\mathscr{L}^i_0 w^i - w^i_t) + p(\bar{w}^i)^{p-4} [(p-1)(w^i)^2 + \varepsilon] \sum_{j,k} a^i_{jk} w^i_{x_j} w^i_{x_k}.$$

Hence, taking into account systems (1), (2) and their parabolicity we find

$$\mathscr{L}_0^i z^i - z_t^i \geqslant -p(\overline{w}^i)^{p-2}|w^i|[f^i(t,x,u) - g^i(t,x,v)]\operatorname{sgn} w^i.$$

Furthermore, assumption (A_3) implies

$$\mathscr{L}_0^i z^i - z_t^i \geqslant -p(\overline{w}^i)^{p-2}|w^i|\sum_s c_s^i|w^s| - p(\overline{w}^i)^{p-2}|w^i|h^i. \tag{9}$$

From assumption $c_s^i \geqslant 0$ for $s \neq i$ it follows

$$\sum_s c_s^i(\overline{w}^i)^{p-2}|w^i||w^s| \leqslant \sum_s c_s^i(\overline{w}^i)^{p-1}\overline{w}^s - \varepsilon c_i^i(\overline{w}^i)^{p-2}. \tag{10}$$

By Young's inequality we have the relation

$$(\overline{w}^i)^{p-1}\overline{w}^s \leqslant \frac{p-1}{p}(\overline{w}^i)^p + \frac{1}{p}(\overline{w}^s)^p \tag{11}$$

which is true for any $1 \leqslant p < \infty$. Note that for $s = i$, (11) turns into the identity. We also have

$$(\overline{w}^i)^{p-2}|w^i|h^i \leqslant (\overline{w}^i)^{p-1}h^i \leqslant \frac{p-1}{p}(\overline{w}^i)^p + \frac{1}{p}(h^i)^p. \tag{12}$$

Combining (9), (10), (11) and (12) we get

$$\mathscr{L}_0^i z^i - z_t^i \geqslant -\sum_s c_s^i z^s - (p-1)\Big(\sum_s c_s^i + 1\Big)z^i - (h^i)^p + \varepsilon p\, c_i^i(\overline{w}^i)^{p-2} \tag{13}$$

$(i = 1, \ldots, m)$. These inequalities imply the relation

$$\sum_i [z^i(\tilde{\mathscr{L}}_0^i\varphi^i + \varphi_t^i) - \varphi^i(\mathscr{L}_0^i z^i - z_t^i)] \leqslant \sum_i z^i\Big(\tilde{\mathscr{L}}_0^i\varphi^i + \sum_s c_i^s\varphi^s + \varphi_t^i\Big) + \tag{14}$$
$$+ (p-1)\sum_i\Big(\sum_s c_s^i + 1\Big)z^i\varphi^i + \sum_i (h^i)^p\varphi^i - \varepsilon p\sum_i c_i^i(\overline{w}^i)^{p-2}\varphi^i.$$

From (8) we obtain, taking advantage of (14) and passing to the limit as $\varepsilon \to 0$,

$$\|w(t,\cdot)\|_{p,\varphi}^p \leqslant \|w(t_0,\cdot)\|_{p,\varphi} + (p-1)(c+1)\int_{t_0}^t \|w(\tau,\cdot)\|_{p,\varphi}^p d\tau + \tag{15}$$
$$+ \int_{t_0}^t \|h(\tau,\cdot)\|_{p,\varphi}^p d\tau + \int_{t_0}^t \|w(\tau,\cdot)\|_{p,\tilde{\mathscr{L}}\varphi}^p d\tau,$$

where $\tilde{\mathscr{L}}\varphi = \{\tilde{\mathscr{L}}^1\varphi, \ldots, \tilde{\mathscr{L}}^m\varphi\}$. Now we solve the integral inequality (15) with respect to $\|w(t,\cdot)\|_{p,\varphi}^p$ (considering $\|w(t,\cdot)\|_{p,\tilde{\mathscr{L}}\varphi}^p$ as known). Hence (see [4] or [5])

(16) $$\|w(t,\cdot)\|_{p,\varphi}^p \leqslant \|w(t_0,\cdot)\|_{p,\varphi}^p \cdot e^{(p-1)(c+1)(t-t_0)} + \int_{t_0}^{t} \|h(\tau,\cdot)\|_{p,\varphi}^p \cdot e^{(p-1)(c+1)(t-\tau)} d\tau + \int_{t_0}^{t} \|w(\tau,\cdot)\|_{p,\tilde{\mathscr{L}}\varphi}^p \cdot e^{(p-1)(c+1)(t-\tau)} d\tau.$$

Let us substitute in (16) $\varphi^i(t,x) = \gamma^R(x)\Phi^i(t,x)$, where Φ^i are the functions occurring in assumption (3) and $\gamma^R(x)$, $R > 1$, is a function of class $C^2(\mathscr{E}^n)$ such that $\gamma^R(x) = 1$ for $|x| \leqslant R-1$, $\gamma^R(x) = 0$ for $|x| \geqslant R$, $0 \leqslant \gamma^R(x) \leqslant 1$ in $\mathscr{E}^n$ and $\sum_{j,k}|\gamma^R_{x_jx_k}| + \sum_j |\gamma^R_{x_j}|$ is bounded in $\mathscr{E}^n$ by a constant independent of R. We have

(17) $$\tilde{\mathscr{L}}^i\varphi = \gamma^R\tilde{\mathscr{L}}^i\Phi + \sum_{j,k}(a^i_{jk}+a^i_{kj})\gamma^R_{x_j}\Phi^i_{x_k} + \Phi^i\Big\{\sum_{j,k} a^i_{kj}\gamma^R_{x_jx_k} + \sum_j\Big[\sum_k (a^i_{jk}+a^i_{kj})_{x_k} - b^i_j\Big]\gamma^R_{x_j}\Big\}.$$

By assumptions (3), (4) and by (17) it follows that the upper limit as $R \to \infty$ of the last integral in (16) is less than or equal to zero. Thus if we let $R \to \infty$ in (16) and then use the elementary inequality $(a+b)^{1/p} \leqslant a^{1/p} + b^{1/p}$ ($a \geqslant 0$, $b \geqslant 0$, $p \geqslant 1$), we obtain the required estimate (6).

Now we introduce the assumption

(A_3') functions $f^i(t,x,u)$ ($i = 1,\ldots,m$) are defined for $(t,x)\in\bar{S}$, $u = (u^1,\ldots,u^m)$ arbitrary and satisfy inequalities

$$f^i(t,x,u)\,\mathrm{sgn}(u^i) \leqslant \sum_{s=1}^{m} c^i_s(t,x)|u^s| + h^i(t,x) \qquad (i = 1,\ldots,m)$$

almost everywhere in $\bar{S}$, c^i_s and h^i satisfying the conditions specified in assumption (A_3).

The following theorem is an analogue of Theorem 1 for the estimate of the solution itself.

THEOREM 2. *Let $u = \{u^i\}$ ($i = 1,\ldots,m$) be a solution of system* (1) *and let assumptions* (A_1), (A_2), (A_3') *be satisfied. Suppose functions Φ and Ψ to be defined as in Theorem* 1. *If, moreover,*

(18) $$\int_{t_0}^{t_1} \|u(\tau,\cdot)\|_{p,\Psi}^p d\tau < \infty, \qquad \int_{t_0}^{t_1} \|h(\tau,\cdot)\|_{p,\Phi}^p d\tau < \infty,$$

then estimates (6) *and* (5) *hold true with $w(t,x)$ replaced by $u(t,x)$.*

Proof of Theorem 2 is similar to that of Theorem 1 except for few evident changes.

The following corollaries are some particular cases of Theorems 1 and 2.

Corollary 1. *Let u, v be solutions of systems* (1), (2) *respectively, and let* (A_1) - (A_3) *be satisfied. Suppose that for some constants* M_1, M_2, M_3, $\lambda \geqslant 0$, *the coefficients* a^i_{jk}, b^i_j *satisfy the growth conditions*

$$(19)\qquad \begin{gathered} |a^i_{jk}| \leqslant M_1(|x|^2+1)^{(2-\lambda)/2}, \\ \Big|\sum_k (a^i_{jk}+a^i_{kj})_{x_k} - b^i_j\Big| \leqslant M_2(|x|^2+1)^{1/2}, \\ \sum_{j,k}(a^i_{kj})_{x_j x_k} - \sum_j (b^i_j)_{x_j} \leqslant M_3(|x|^2+1)^{\lambda/2} \end{gathered}$$

($j, k = 1, \ldots, n$; $i = 1, \ldots, m$) *almost everywhere in* $\bar{S} = \langle t_0, t_1\rangle \times \mathscr{E}^n$, *where*

$$t_1 - t_0 \leqslant 1/2\mu$$

and

$$\mu = nM_1K\lambda^2 + \sqrt{n}M_2\lambda + (M_3/K), \quad K > 0.$$

Put

$$\nu = 2n\, M_1K\lambda(|\lambda-2|+1)+c$$

and define

$$(20)\quad \Phi^i(t,x) = \exp\left\{-\frac{K(|x|^2+1)^{\lambda/2}}{1-\mu(t-t_0)} - \nu(t-t_0)\right\}, \quad \Phi = \{\Phi^1, \ldots, \Phi^m\}.$$

If, moreover,

$$(21)\qquad \int_{t_0}^{t_1} \|w(\tau,\cdot)\|^p_{p,|x|^2\Phi}\, d\tau < \infty \quad \text{and} \quad \int_{t_0}^{t_1} \|h(\tau,\cdot)\|^p_{p,\Phi}\, d\tau < \infty,$$

where $w = u - v$, *then estimate* (6) *holds true with* Φ *defined by* (20).

Proof. To deduce the corollary from Theorem 1 it is enough to show that under assumptions (19), the function Φ defined by (20) satisfies (3) and the convergence of the first integral in (21) implies the convergence of the first integral in (4). We omit the easy computational details.

Similarly one can derive the following

Corollary 2. *Assume u is a solution of* (1) *and* (A_1), (A_2), (A'_3) *hold true. Then* (19) *and* (21) *imply estimate* (6) $(t_1 - t_0 \leqslant 1/2\mu)$ *with* $w(t,x)$ *replaced by* $u(t,x)$ *and* Φ *defined by* (20).

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