ANNALES
POLONICI MATHEMATICI
XXVII (1973)

Global existence for non-linear partial differential equations of the first order

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In this paper we shall deal with the Cauchy problem for the differential equation

$$z_x = f(x, Y, z, z_Y), \quad \text{where } Y = (y_1, \ldots, y_n), \qquad z_Y = (z_{y_1}, \ldots, z_{y_n}) \tag{1}$$

with the initial condition

$$z(a, Y) = w(Y), \tag{2}$$

where f and w are sufficiently regular.

We consider the existence domain for the solution of (1), (2). We give an estimation for the existence domain depending on the domains of functions f, w and the existence domain for the solutions of certain ordinary differential equations introduced by Pliś [1]. Similar results of Pliś [1] were obtained in a neighbourhood of a characteristic, but our theorem is global. In the proof we divide the domain into thin slices. We prove that in each of such slices a suitable transformation has an inverse.

Let $a < b$ be fixed and let $g\colon [a, b) \times R^n \to R$ be a C^1-function such that $g_x^2 + \sum_{i=1}^{n} g_{y_i}^2 > 0$.

Put

$$G = \{(x, Y) \in [a, b) \times R^n : g(x, Y) < 0\}, \qquad S = \{(x, Y) \in [a, b) \times R^n : g(x, Y) = 0\}. \tag{3}$$

We suppose that G is a connected (necessarily open) region and, moreover, that $G \cap \{(x, Y) : x = a\}$ is also an open region.

For an arbitrary fixed number $\varepsilon \in (0, b-a)$ we denote by g_ε the function

$$[a, b-\varepsilon) \times R^n \ni (x, Y) \to g(x, Y) + \varepsilon \in R,$$

and by G_ε the set

$$\{(x, Y)\in[a, b-\varepsilon)\times R^n : g_\varepsilon(x, Y) < 0\}.$$

We suppose that ε is such that $G_\varepsilon \cap \{(x, Y) : x = a\}$ is a non-empty, open region; it is obvious that this condition is fulfilled for every ε smaller than some positive number ε_0.

Denote by $R(\xi), R_\varepsilon(\xi)$ the intersections of sets G and G_ε with the plane $x = \xi$, and by $r(\xi)$, $r_\varepsilon(\xi)$ the projections of $R(\xi), R_\varepsilon(\xi)$ on the space $(y_1, \ldots, y_n)$. We shall denote by ∂G the boundary of the set G and by $\varrho(x, Z)$ the distance of point x from the set Z:

$$P = \{(x, Y, z, Q) : (x, Y)\in G, z, Q \text{ arbitrary}\},$$

where $Y = (y_1, \ldots, y_n)$; $Q = (q_1, \ldots, q_n)$.

We shall denote by

$$Y(x, V) = \big(y_1(x, V), \ldots, y_n(x, V)\big); \quad z(x, V);$$
$$Q(x, V) = \big(q_1(x, V), \ldots, q_n(x, V)\big) \quad (V = v_1, \ldots, v_n);$$
$$t_{ij}(x, V) \quad (i, j = 1, \ldots, n)$$

the solution of the following system of characteristic equations consisting of the system

$$\begin{aligned} y_i' &= -f_{q_i}(x, Y, z, Q), \\ z' &= f(x, Y, z, Q) - \sum_{k=1}^{n} q_k f_{q_k}(x, Y, z, Q), \\ q_i' &= f_{y_i}(x, Y, z, Q) + q_i f_z(x, Y, z, Q), \end{aligned} \tag{4}$$

and the system

$$\begin{aligned} t_{ij}' = \sum_{k=1}^{n}\sum_{m=1}^{n} f_{q_k q_m}(x, Y, z, Q) t_{ki} t_{mj} + \sum_{k=1}^{n} \big(f_{q_k y_i}(\ldots) + \\ + f_{q_k z}(\ldots) q_i\big) t_{kj} + \sum_{k=1}^{n} \big(f_{q_k y_j}(\ldots) + f_{q_k z}(\ldots) q_j\big) t_{ki} + \\ + f_z(\ldots) t_{ij} + f_{y_i y_j}(\ldots) + f_{y_j z}(\ldots) q_j + f_{y_j z}(\ldots) q_i + \\ + f_{zz}(\ldots) q_i q_j \quad (i, j = 1, \ldots, n) \end{aligned} \tag{5}$$

with the initial conditions

$$Y(a, V) = V, \quad z(a, V) = w(V), \quad q_i(a, V) = w_{y_i}(V) \quad (i = 1, \ldots, n), \tag{6}$$
$$t_{ij}(a, V) = w_{y_i y_j}(V) \quad (i, j = 1, \ldots, n).$$

(7) Let $I(a, V)$ denote the interval on which exists $Y(x, V)$ for $V \in r(a)$.

Assumption. Suppose that the function $f(x, Y, z, Q)$ defined in P is of class C^2 in P and satisfies the conditions

$$|f|,\ |Df|,\ |D^2 f| < M \text{ in } P, \tag{8}$$

where Df denotes any derivative of first order and $D^2 f$ denotes any derivative of second order.

The function $w(Y)$ defined in $r(a)$ is of class C^2 in $r(a)$ and satisfies the conditions

$$|w|,\ |Dw|,\ |D^2 w| < M \text{ in } r(a). \tag{9}$$

Suppose that there exists a solution $t_{ij}(x, V)$ of system (5) satisfying the initial conditions (6) for $V \in r(a)$, defined in $I(a, V)$.

Assume finally that at the points $(x, Y(x))$ at which $g(x, Y(x)) = 0$ the following inequality is satisfied:

$$g_x(x, Y(x)) - \sum_{i=1}^{n} g_{v_i}(x, Y(x)) \cdot f_{q_i}(x, Y(x), z(x), Q(x)) > 0. \tag{10}$$

THEOREM. *Under the above assumptions there exists a solution of the problem* (1), (2) *of class* C^2 *and defined on the set* G.

Proof. To begin with remark that in virtue of the uniqueness property it is sufficent to demonstrate that the solution of problem (1), (2) exists in the set $\bar{G}_\varepsilon$ for arbitrary $\varepsilon > 0$. Let ε be a fixed positive number.

Consider the integrals $Y(x, V), z(x, V), Q(x, V)$ for $V \in r(a)$. By assumption we have $t_{ij}(x, V)$ defined in $I(a, V)$. To prove the theorem, it is enough, in view of [1], to show that:

I. The mapping: $T(x, V) = (x, Y(x, V))$ (or explicitly

$$T(x, v_1, \ldots, v_n) = (x, y_1(x, v_1, \ldots, v_n), \ldots, y_n(x, v_1, \ldots, v_n)))$$

has an inverse mapping defined on $\bar{G}_\varepsilon$ and $\operatorname{Det}\left(\dfrac{\partial y_i}{\partial v_j}\right) \neq 0$ on $\bar{G}_\varepsilon$.

II. The integrals $y_i(x, V)$, where $V \in r(a)$ $(i = 1, \ldots, n)$, cover the whole domain $\bar{G}_\varepsilon$. In view of inequality (10) and the Borel theorem there are two constans A and B such that, if $\varrho(\eta; \partial r_\varepsilon(\xi)) < B$ for every $a \leqslant \xi \leqslant b - \varepsilon$, then

$$g_x(\xi, \eta) - \sum_{i=1}^{n} g_{v_i}(\xi, \eta) f_{q_i}(\xi, \eta, z(\xi), Q(\xi)) > A, \tag{11}$$

where $\eta \in r_\varepsilon(\xi)$ $\quad (\eta = \eta_1, \ldots, \eta_n)$.

We divide the interval $[a, b]$ into subintervals: $a = x^0 < x^1 < \ldots \ldots < x^m = b$, where $\delta_i = x^i - x^{i-1}$ $(i = 1, \ldots, m)$ will be chosen suitably. The planes $x = x^i$ $(i = 1, \ldots, m-1)$ divide the domain $G(G_\varepsilon)$ into the layers $\Delta^i(\Delta^i_\varepsilon)$.

Now we shall show that I and II are satisfied in Δ_ε^1 if δ_1 fulfils the following properties:

(i) $r_\varepsilon(x) \subset r(x^0)$ for every $x \in [x^0, x^0+\delta_1]$,

(ii) if $\varrho\big(V, \partial r_\varepsilon(x^0)\big) < B/2\big(V \in r_\varepsilon(x^0)\big)$, then the inequality

$$g_x(\xi, \eta) - \sum_{i=1}^{n} g_{v_i}(\xi, \eta) f_{q_i}\big(\bar{\xi}, Y(\bar{\xi}, V), z(\bar{\xi}, V), Q(\bar{\xi}, V)\big) > A$$

is true for $x^0 \leqslant \xi \leqslant x^0+\delta_1$, $x^0 \leqslant \bar{\xi} \leqslant x^0+\delta_1$; $v_k \leqslant \eta_k \leqslant y_k$ if $v_k < y_k$, and $y_k \leqslant \eta_k \leqslant v_k$ if $v_k > y_k$, where $y_k = y_k(x^0+\delta_1, V)$;

(iii) there exists a constant L such that, if $\varrho\big(Y, \partial r_\varepsilon(x^1)\big) < L$, then $\varrho\big(V, \partial r_\varepsilon(x^0)\big) < B/2$, where $Y = (y_1, \ldots, y_n) \in r_\varepsilon(x^1)$, $V = (v_1, \ldots, v_n) \in r_\varepsilon(x^0)$ and $v_1 = y_i \pm M(x^0-x^1)$,

$$\text{(iv)} \qquad \delta_1 < \min\left(\frac{\ln(1+1/2n)}{na}, \frac{A}{3C\beta n}, \sqrt{\frac{B}{4\beta n}}\right),$$

where a, β, and C will be chosen later.

Ad I. By [2] this will be proved if we show that the inequality

$$\left|\frac{\partial y_i(x, V)}{\partial v_j} - \frac{\partial y_i(x^0, V)}{\partial v_j}\right| < \frac{1}{n} \qquad (i, j = 1, \ldots, n) \tag{12}$$

holds true in Δ_ε^1.

From [1] the functions $u_i = u_{ij}(x) = \dfrac{\partial y_i(x, V)}{\partial v_j}$ $(i = 1, \ldots, n)$ fulfil the system of equations

$$u_i' = \sum_{k=1}^{n}\Big[-f_{q_i q_k}(Z) - f_{q_i z}(Z) q_k(x, V) - \\ - \sum_{m=1}^{n} f_{q_i q_m}(Z) t_{mk}(x, V)\Big] u_k \qquad (i = 1, \ldots, n), \tag{13}$$

where $Z = \big(x, Y(x, V), z(x, V), Q(x, V)\big)$.

It is evident that $\dfrac{\partial y_i(x^0, V)}{\partial v_j} = \delta_{ij}$. We shall write

$$\varepsilon_{ij} = \frac{\partial y_i(x, V)}{\partial v_j} - \delta_{ij}. \tag{14}$$

Let us suppose that there exists such a positive constant K that

$$|q_i| < K, \qquad |t_{ik}| < K \quad \text{in } \bar{G}_\varepsilon\ (i, k = 1, \ldots, n). \tag{15}$$

Then by (13), (14) and the theorem on differential inequalities we have

$$|\varepsilon_{ij}(x)| \leqslant e^{na|x-x^0|} - 1,$$

where $a = a(M, K, n) = M + MK + nMK$.

For $|x-x^0| = \dfrac{\ln(1+1/2n)}{na}$ inequality (12) is satisfied. From the definition of δ_1 it follows that in the layer Δ_ε^1 the mapping T has an inverse and $\mathrm{Det}\left(\dfrac{\partial y_i}{\partial v_j}\right) \neq 0$.

Ad II. Now we shall show that the integrals $y_i(x, V)$, where $V \epsilon r_\varepsilon(x^0)$, cover the set Δ_ε^1. Let $Y^*(x^*, Y^*)$ by any given point of the set Δ_ε^1, and let $V^*(x^0, V^*)$ be the projection of Y^* on the plane $x = x^0$, and $Y^* = (y_1^*, \ldots, y_n^*)$; $V^* = (v_1^*, \ldots, v_n^*)$. In view of property (i) $V^* \epsilon r(x^0)$. We shall prove that there exists a point $V \epsilon r_\varepsilon(x^0)$ such that $y_i(x^*, V) = y_i^*$. This will be a fixed point of a mapping $\overline{T}$. Let $\overline{T} = T_2 \cdot T_1$, where

$$T_1:\ r_\varepsilon(x^0) \ni V^i = (v_1^i, \ldots, v_n^i) \to Y^{i+1} = \big(y_1(x^*, V^i), \ldots, y_n(x^*, V^i)\big) \epsilon r_\varepsilon(x^*),$$

$$T_2:\ r_\varepsilon(x^*) \ni Y^{i+1} \to V^{i+1} = \big(v_1^i + v_1^* - y_1(x^*, V^i), \ldots, v_n^i + v_n^* - y_n(x^*, V^i)\big) \epsilon r(x^0)$$

and

$$v_i^0 = v_i^* + \left.\frac{dy_i(x, V)}{dx}\right|_{x=x^0} (x^0 - x). \tag{16}$$

We shall show that the mapping $\overline{T}$ satisfies two properties:

1° $\varrho\big(\overline{T}(V), \overline{T}(W)\big) \leqslant \frac{1}{2}\varrho(V, W)$,

2° $\overline{T}$ is defined on set $r_\varepsilon(x^0)$.

Ad 1°: Let $\overline{V} = \overline{T}(V)$, $\overline{W} = \overline{T}(W)$. We define $\varrho(V, W) = \sum_{i=1}^{n} |v_i - w_i|$.

Now we estimate $\varrho(\overline{V}, \overline{W})$. By the definition of the mapping $\overline{T}$ we have:

$$(\bar{v}_1 - v_1, \ldots, \bar{v}_n - v_n)$$
$$= \big(v_1^* - y_1(x^*, v_1, \ldots, v_n), \ldots, v_n^* - y_n(x^*, v_1, \ldots, v_n)\big),$$
$$(\bar{w}_1 - w_1, \ldots, \bar{w}_n - w_n)$$
$$= \big(v_1 - y_1(x^*, w_1, \ldots, w_n), \ldots, v_n^* - y_n(x^*, w_1, \ldots, w_n)\big).$$

Taking the coordinates of points $\overline{V}$, $\overline{W}$ and using the mean values theorem, we obtain from (12):

$$\varrho(\overline{V}, \overline{W}) \leqslant \tfrac{1}{2}\varrho(V, W), \tag{17}$$

which shows that property 1° is fulfilled.

Ad 2°: We consider the following difference:

$$g_\varepsilon(x^0, V^0) - g_\varepsilon(x^*, Y^1) \tag{18}$$
$$= (x^0 - x^*)\Big[g_x(\xi, \eta) - \sum_{i=1}^{n} g_{v_i}(\xi, \eta) f_{q_i}\big(\bar{\xi}, Y(\bar{\xi}, V^0), z(\bar{\xi}, V^0), Q(\bar{\xi}, V^0)\big)\Big],$$

where $x^0 \leqslant \xi \leqslant x^*$, $x^0 \leqslant \bar{\xi} \leqslant x^*$, $v_k^0 \leqslant \eta_k \leqslant y_k^1 = y_k(x^*, V^0)$. If $\varrho\big(Y^*, \partial r_\varepsilon(x^*)\big) < L$, then from (iii) $\varrho\big(V^0, \partial r_\varepsilon(x^0)\big) < B/2$ and from (ii) we have:

$$(19) \qquad g_x(\xi, \eta) - \sum_{i=1}^{n} g_{v_i}(\xi, \eta) f_{q_i}\big(\bar{\xi}, Y(\bar{\xi}, V^0), z(\bar{\xi}, V^0), Q(\bar{\xi}, V^0)\big) > A.$$

From (18), (19) we obtain:

$$(20) \qquad g_\varepsilon(x^0, V^0) < g_\varepsilon(x^*, Y^1) - A(x^* - x^0).$$

Next we estimate $\varrho(Y^*, Y^1)$.

Write $y_i(x) = v_i^0 - f_{q_i}\big(x^0, V^*, z(x^0, V^*), Q(x^0, V^*)\big)(x - x^0)$ $(i = 1, \dots, n)$, where v_i^0 are given by formula (16), and notice that

$$(21) \qquad y_i(x^0) = y_i(x^0, V^0) = v_i^0.$$

Taking the coordinates of the points Y^*, Y^1 and using (21), we have:

$$(22) \quad \begin{aligned} |y_i^* - y_i^1| &= \big|v_i^0 - f_{q_i}\big(x^0, V^*, z(x^0, V^*), Q(x^0, V^*)\big)(x^* - x^0) - y_i(x^*, V^0)\big| \\ &= \Big|\int_{x^0}^{x^*} \big\{\big[v_i^0 - f_{q_i}\big(x^0, Y(x^0, V^*), z(x^0, V^*), Q(x^0, V^*)\big)(x - x^0)\big]' - \\ &\quad - [y_i(x^*, V^0)]'\big\} dx \\ &\leqslant \int_{x^0}^{x^*} \big|f_{q_i}\big(x, Y(x, V^0), z(x, V^0), Q(x, V^0)\big) - \\ &\quad - f_{q_i}\big(x^0, Y(x^0, V^*), z(x^0, V^*), Q(x^0, V^*)\big)\big| dx. \end{aligned}$$

Write

$$N = \max_{i,j} \left(\max_{\bar{G}_\varepsilon} \left|\frac{\partial y_i(x, V)}{\partial v_j}\right|, \left|\frac{\partial z(x, V)}{\partial v_j}\right|, \left|\frac{\partial q_i(x, V)}{\partial v_j}\right|\right),$$

$$J = \max_j \big(\max_{\bar{G}_\varepsilon} |q_j(x, V)|\big);$$

then

$$(23) \quad \begin{aligned} &f_{q_i}\big(x, Y(x, V^0), z(x, V^0), Q(x, V^0)\big) - \\ &\qquad - f_{q_i}\big(x^0, Y(x^0, V), z(x^0, V), Q(x^0, V)\big) \\ &\leqslant (2MNn + MN) \sum_{j=1}^{n} |v_j^0 - v_j^*| + (M + M^2 + 2M^2 n + 2M^2 Nn)|x - x^0|. \end{aligned}$$

But from (16)

$$(24) \qquad \sum_{j=1}^{n} |v_j^0 - v_j^*| \leqslant Mn|x^* - x^0|.$$

By (22), (23), (24) we have:

$$\varrho(Y^*, Y^1) \leqslant \beta n |x^* - x^0|^2, \tag{25}$$

where $\beta = Mn(2MNn + MN) + \dfrac{M + M^2 + 2M^2 n + 2M^2 Nn}{2}$.

Let

$$C = \max_i \big(\max_{\overline{G}_\varepsilon} |q_{v_i}(x, Y)|\big). \tag{26}$$

From (26) and (25) it follows that

$$|g_\varepsilon(x^*, Y^*) - g_\varepsilon(x^*, Y^1)| \leqslant C\beta n |x^* - x^0|^2. \tag{27}$$

In view of (20), (27), (3) and (iv) we have:

$$g_\varepsilon(x^0, V^0) \leqslant g_\varepsilon(x^*, Y^1) - A(x^* - x^0) \leqslant C\beta n |x^* - x^0|^2 - A|x^* - x^0| < 0. \tag{28}$$

This shows that the point V^0 belongs to $r_\varepsilon(x^0)$. Now we will estimate the distance of point V^0 from $\partial r_\varepsilon(x^0)$. Let $\varrho(V^0, \overline{V}) = \min\limits_{V \in \partial r_\varepsilon x^0} \varrho(V^0, V)$; then

$$|g_\varepsilon(x^0, V^0)| \leqslant C \sum_{i=1}^{n} |v_i^0 - \bar{v}_i|. \tag{29}$$

By (29), (20), (26) and (28) we have:

$$\sum_{i=1}^{n} |v_i^0 - \bar{v}_i| \geqslant \frac{A|x^* - x^0| - g_\varepsilon(x^*, Y^1)}{C}. \tag{30}$$

From (17) we obtain:

$$\sum_{i=1}^{\infty} \varrho(V^{i+1}, V^i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho(V^1, V^0) = \varrho(V^1, V^0).$$

We notice that

$$\varrho(V^0, V^1) = \varrho(Y^*, Y^1). \tag{31}$$

For $\overline{T}$ to be defined on $r_\varepsilon(x^0)$ it is enough to show

$$2\varrho(V^1, V^0) < \varrho(V^0, \overline{V}). \tag{32}$$

In virtue of (iii) the inequality

$$2C\beta n |x^* - x^0|^2 < A|x^* - x^0| - C\beta n |x^* - x^0|^2 - g_\varepsilon(x^*, Y^*)$$

holds true.

Hence and from (31), (25), (27) and (30) we obtain inequality (32). Now we must consider the case where $\varrho\big(Y^*, \partial r_\varepsilon(x^*)\big) > L$. Then we have either

$$\varrho(V^0, \overline{V}) < B/2 \tag{a}$$

or

(b) $$\varrho(V^0, \bar{V}) \geqslant B/2 \text{ and } V^0 \text{ is inside } r_\varepsilon(x^0).$$

It is enough to consider case (b).

By (25) and (iv) we obtain inequality (32), i.e. we infer that the points $V^1, V^2, \ldots$ are inside $r_\varepsilon(x^0)$. In this way we have proved that the mapping $\bar{T}$ is defined on $r_\varepsilon(x^0)$ if δ_1 satisfies properties (i), (ii), (iii), (iv). From the fixed-point theorem there is a point $V \in r_\varepsilon(x^0)$ such that the integral $y_i(x, V)$ passes through the point (x^*, Y^*). Hence from [1] the solution $z(x, Y)$ of problem (1), (2) of class C^2 exists in the layer Δ_ε. This solution exists on a neighbourhood of the layer Δ_ε. It is defined for $x^1 = x^0 + \delta_1$.

Write

(33) $$z(x^1, Y) = w_1(Y).$$

Now we can consider the solution of equation (1) with condition (33) on the layer Δ_ε^2. After a finite number of steps we obtain the solution of class C^2 on $\bar{G}_\varepsilon$, which was to be proved.

The author wishes to express his sincere thanks to Professor A. Pliś for suggesting problems considered in this paper and for his valuable advice.

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Reçu par la Rédaction le 9. 9. 1971