ANNALES
POLONICI MATHEMATICI
XLIX (1989)

# Capacities associated to the Siciak extremal function

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**Abstract.** Certain property of the Siciak extremal function is proved which implies that two logarithmic capacities associated to this function are Choquet capacities.

**1. Introduction.** The purpose of this paper is to give a proof of a theorem concerning the nature of the Siciak extremal function which allows us to show that two capacities associated to this function are Choquet capacities.

Let $L$ be the family of plurisubharmonic functions on $\boldsymbol{C}^n$ which satisfy

$$u(z) \leqslant \log^+ |z| + O(1), \quad z \in \boldsymbol{C}^n.$$

Denote by $H$ the subclass of $\exp L$ which consists of non-zero homogeneous plurisubharmonic functions, i.e.,

$$u(\lambda z) = |\lambda| u(z), \quad z \in \boldsymbol{C}^n, \quad \lambda \in \boldsymbol{C},$$

for every $u \in H$. Let $E$ be a bounded subset of $\boldsymbol{C}^n$. Following Siciak [9], we may define two extremal functions in $\boldsymbol{C}^n$ setting

$$\Phi_E(z) = \sup\{u(z)\colon\ u \in \exp L,\ u \leqslant 1 \text{ on } E\},$$

$$\Psi_E(z) = \sup\{u(z)\colon\ u \in H,\ u \leqslant 1 \text{ on } E\}.$$

(In a different way the extremal functions were defined earlier in [6], [7].) They are related by the formula

$$\Psi_{1 \times E}(1, z) = \Phi_E(z), \quad z \in \boldsymbol{C}^n,$$

where $\Psi_{1 \times E}(t, z)$, $(t, z) \in \boldsymbol{C} \times \boldsymbol{C}^n$, is the homogeneous extremal function of the set $\{1\} \times E \subset \boldsymbol{C} \times \boldsymbol{C}^n$. If $E$ is $L$-regular, i.e. $\Phi_E$ is continuous, then also $\Psi_{1 \times E}$ is continuous in $\boldsymbol{C} \times \boldsymbol{C}^n$. Given a function $f\colon \Omega \to [-\infty, +\infty]$,

$$f^*(z) = \limsup_{\zeta \to z} f(\zeta)$$

denotes its upper semicontinuous regularization.

The main result of the paper is the following theorem:

THEOREM 1.1. *Given $E$ a compact subset of $C^n$, we have*

$$\limsup_{(t,\zeta)\to(0,z)} \Psi_{1\times E}(t,\zeta) = \limsup_{\zeta\to z} \Psi_{1\times E}(0,\zeta), \qquad z\in C^n.$$

Several capacities in $C^n$ have been defined by means of the extremal functions (see e.g. [4], [8], [9], [11]). It is an easy consequence of Theorem 1.1 that two of them

$$c(E) := \big(\max_{|z|=1} \Psi^*_{1\times E}(0,z)\big)^{-1}, \qquad \bar{c}(E) := \exp\Big(-\int_{|z|=1} \log \Psi^*_{1\times E}(0,z)\, d\sigma\Big)$$

(where $\sigma$ denotes the normalized Lebesgue measure on the unit sphere) are Choquet capacities, i.e., they satisfy

(1) $\qquad c(E) \leqslant c(F) \quad$ if $E\subset F$,

(2) $\qquad c(K_j)\downarrow c(K) \quad$ if $K_j\downarrow K$ as $j\to\infty$,

(3) $\qquad c(E_j)\uparrow c(E) \quad$ if $E_j\uparrow E$ as $j\to\infty$

for $K$, $K_j$ compact and $E$, $F$, $E_j$ arbitrary subsets of $C^n$. In the case of complex plane the definitions above yield the classical logarithmic capacity.

Theorem 1.1 also implies that, given $E$, a compact subset of $C^n$, the extremal function $\log \Phi^*_E$ has the directional limit

$$\lim_{|\lambda|\to\infty} \log(\Phi^*_E(\lambda z)/|\lambda|), \qquad \text{where } z\in C^n,\ \lambda\in C,$$

in every complex line through the origin which lies outside some pluripolar cone.

**2. Preliminaries.** To prove our theorem we need some results of the Monge–Ampère operator theory developed recently by Bedford and Taylor [1], [2]. Let $\Omega$ be an open subset of $C^n$ and $u$ a locally bounded plurisubharmonic function on $\Omega$. Then

(2.1) $\qquad (dd^c u)^k = dd^c u \wedge \ldots \wedge dd^c u \quad$ ($k$ factors), $k\leqslant n$,

is a positive current on $\Omega$ with measure coefficients, where $d^c = i(\bar{\partial}-\partial)$.

CONVERGENCE THEOREM ([2]). *Let $\{u^i_j\} \subset \mathrm{PSH}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ be a sequence converging almost everywhere on $\Omega$ to a function $u^i\in \mathrm{PSH}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ for $0\leqslant i\leqslant k\leqslant n$. If all but one of the sequences $\{u^0_j\},\ldots,\{u^k_j\}$ are monotone, either decreasing or increasing, then*

$$\lim_{j\to\infty} dd^c u^1_j \wedge \ldots \wedge dd^c u^k_j = dd^c u^1 \wedge \ldots \wedge dd^c u^k,$$

$$\lim_{j\to\infty} u^0_j\, dd^c u^1_j \wedge \ldots \wedge dd^c u^k_j = u^0\, dd^c u^1 \wedge \ldots \wedge dd^c u^k,$$

$$\lim_{j\to\infty} du^0_j \wedge d^c u^1_j \wedge dd^c u^2_j \wedge \ldots \wedge dd^c u^k_j = du^0 \wedge d^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^k$$

*(in the sense of currents).*

COROLLARY 2.1. *Let* $\{u_j\} \subset \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ *be a sequence converging almost everywhere to* $u \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ *and let* $\{v_j\} \subset \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ *converge locally uniformly to* $v \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$. *If* $\mathrm{supp}(dd^c v_j)^n \subset K \subset\subset \Omega$ *for every j, then*

$$\int_{\Omega} u_j (dd^c v_j)^n \to \int_{\Omega} u (dd^c v)^n.$$

(From every subsequence of $\{v_j\}$ one may pick out its subsequence $\{v_{j_k}\}$ so that $v_{j_k} + 1/k$ decreases to $v$ in a neighbourhood of $K$ and apply the Convergence Theorem.)

The extremal function $u_E = \log \Phi_E^*$ of a compact non-pluripolar set $E$ has the following properties:

(2.2) $$(dd^c u_E)^n = 0 \quad \text{on} \quad C^n \setminus E \quad \text{(see [2])},$$

(2.3) $$\int_{C^n} (dd^c u_E)^n = c_n \quad (c_n \text{ independent of } E) \quad \text{(see [10])},$$

(2.4) $$\int_{C^n} u_E (dd^c u_E)^n = 0 \quad \text{(see [2])},$$

(2.5) $$u_F = \max\{0, u_E - \log R\}$$

if $F$ is the closure of $\{z \in C^n \colon u_E(z) < \log R\}$ (see [3]).

For later reference we give also

PROPOSITION 2.2 ([3]). *Let* $u \in L \cap C^{\infty}$ *and* $\varphi \in C_0^{\infty}$. *Put*

$$u^r = \max\{0, u - r\}.$$

*If the set* $\Omega = \{u < r\}$ *is bounded and has a smooth boundary, then*

$$\int_{\partial\Omega} \varphi d^c u \wedge (dd^c u)^{n-1} = \int_{\Omega} \varphi (dd^c u^r)^n.$$

Finally, we recall a result of Levenberg [4] (see also [12]).

THEOREM 2.3. *Let E be a Borel subset of a compact* $K \subset C^n$. *Write* $\mu = (dd^c u_K^*)^n$. *If*

$$u_E^* = 0 \quad \textit{a.e. with respect to } \mu,$$

*then*

$$u_E^* = u_K^*.$$

**3. Convergence of subharmonic functions.** We begin with an auxiliary the proposition.

PROPOSITION 3.1. *Let D be an open disc centered at zero in the complex plane and G a circled, compact subset of* $C^n$. *If* $u \colon D \times G \to \boldsymbol{R}$ *is a bounded function satisfying the following conditions:*

(i) $u(e^{it}\tau, e^{it}z) = u(\tau, z)$,

(ii) $u(\cdot, z)$ *is subharmonic in* $D$ *for each* $z \in G$,

(iii) $u(\tau, \cdot)$ *is upper semicontinuous in* $G$ *for each* $\tau \in D \setminus \{0\}$,

*then* $u(0, \cdot)$ *is upper semicontinuous and there exists a sequence of real numbers* $\varepsilon_j$ *decreasing to zero such that*

$$\lim_{j\to\infty} u(\varepsilon_j, z) = \lim_{\tau\to 0} u(\tau, z) = u(0, z)$$

*for almost every* $z \in G$ *with respect to the Lebesgue measure in* $\mathbf{C}^n$.

Proof. Write

$$h(z) = u(0, z).$$

Since $u(\cdot, z)$ is subharmonic, we have

$$\overline{\lim_{\tau\to 0}}\, u(\tau, z) = \inf_{\varepsilon>0} (\max_{|\tau|=\varepsilon} u(\tau, z)) = h(z).$$

By (i), $h(z) = h(e^{it}z)$ and then the maximum principle for subharmonic functions gives

$$\sup_t \{u(r, e^{it}z) - h(z)\} \downarrow 0 \quad (\text{as } r \downarrow 0) \text{ for all } z \in G. \tag{3.1}$$

This along with the third property of $u$ proves that $h$ is upper semicontinuous. Indeed, if $\{z_j\}$ is any sequence of points of $G$ converging to $z$, then

$$\max_{|\tau|=\varepsilon} u(\tau, z) \geqslant \overline{\lim_{j\to\infty}}\, (\max_{|\tau|=\varepsilon} u(\tau, z_j)) \geqslant \overline{\lim_{j\to\infty}}\, h(z_j). \tag{3.2}$$

Fix a sequence $\{q_j\}$ of positive real numbers with

$$\sum_{j=1}^{\infty} q_j = 1.$$

By Luzin theorem, for every $j$ one can find an open subset $E_j$ of $G$ such that

$$\lambda(E_j) \leqslant q_j \tag{3.3}$$

(where $\lambda$ denotes the normalized by 1 Lebesgue measure on $G$) and $h$ restricted to $K_j = G \setminus E_j$ is continuous. Since $G$ is a circled and $h(e^{it}z) = h(z)$, we may assume $K_j$ to be circled. Let $r$ be the radius of $D$. Put

$$h_\varepsilon(z) = \max_{|\tau|=\varepsilon} u(\tau, z) \quad \text{for } 0 < \varepsilon < r.$$

Every $h_\varepsilon$ is then upper semicontinuous (see (3.2)) and $h_\varepsilon \downarrow h$ as $\varepsilon \downarrow 0$. It follows from Dini's theorem that for every $j$ there exists $\varepsilon_j > 0$ such that

$$h_{\varepsilon_j}(z) - h(z) \leqslant q_j^2 \quad \text{on } K_j. \tag{3.4}$$

Put

$$F_j = \{z \in K_j : u(\varepsilon_j, z) - h(z) \leqslant -q_j\}. \tag{3.5}$$

We claim that $\lambda(F_j) \leqslant q_j$. Suppose $\lambda(F_j) > q_j$. Then by (3.4)

$$u(\varepsilon_j, z) - h(z) \leqslant q_j^2 \quad \text{for } z \in K_j,$$

whence

$$\int_{K_j} (u(\varepsilon_j, z) - h(z)) d\lambda = \int_{F_j} + \int_{K_j \setminus F_j} \tag{3.6}$$

$$\leqslant -q_j \lambda(F_j) + q_j^2 \lambda(K_j \setminus F_j) < -q_j^2 (1 - \lambda(K_j \setminus F_j)).$$

Since $K_j$ is circled, we have

$$\int_{K_j} (u(\varepsilon_j, z) - h(z)) d\lambda = \int_{K_j} (u(\varepsilon_j, e^{-it} z) - h(e^{-it} z)) d\lambda, \quad t \in [0, 2\pi]. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$0 \leqslant \int_{K_j} \left((1/2\pi) \int_0^{2\pi} (u(\varepsilon_j e^{it}, z) - h(z)) dt\right) d\lambda(z) < -q_j^2 (1 - \lambda(K_j \setminus F_j)) < 0,$$

which is an absurd. Thus we have proved that

$$\lambda(F_j) \leqslant q_j \quad (j \geqslant 1). \tag{3.8}$$

Now, take

$$z \in G \setminus \bigcup_{j \geqslant k} (E_j \cup F_j).$$

Then by (3.5) we have $u(\varepsilon_j, z) \geqslant h(z) - q_j$ for $j \geqslant k$. Hence

$$\lim_{j \to \infty} u(\varepsilon_j, z) \geqslant h(z).$$

Since the inequality

$$\overline{\lim_{j \to \infty}} u(\varepsilon_j, z) \leqslant h(z) \quad \text{for } z \in G$$

is assured by assumption (ii), we get

$$\lim_{j \to \infty} u(\varepsilon_j, z) = h(z) \quad \text{for all } z \in G \setminus S,$$

where

$$S = \bigcap_{k \geqslant 1} \bigcup_{j \geqslant k} (E_j \cup F_j).$$

It remains to observe that by (3.3) and (3.8) we have

$$\lambda\Big(\bigcup_{j=k}^{\infty}(E_j \cup F_j)\Big) \leqslant 2\sum_{j=k}^{\infty} q_j \to 0 \quad \text{as } k \to \infty .$$

Thus the proposition follows.

**4. Main lemma.** The crucial step in the proof of Theorem 1.1 is the following lemma.

LEMMA 4.1. *Let $A$ and $B$ be non-pluripolar compact subsets of $\boldsymbol{C}^n$ such that $A \subset B$ and $B$ is L-regular. We claim that the operator*

$$J(\lambda, A, B) = \int_{\boldsymbol{C}^n} \log \Psi^*_{1\times A}(\lambda, z)\,[dd^c \log^+ \Psi_{1\times B}(\lambda, z)]^n,$$

*where $0 \leqslant \lambda \leqslant 1$ and $z \in \boldsymbol{C}^n$, has the following properties;*

(a) $\lim_{j\to\infty} J(\lambda, A, B_j) = 0$ *when* $B_j \downarrow A$,

(b) $J(\lambda, A, B) \leqslant J(\tau, A, B)$ *for* $0 < \lambda \leqslant \tau$,

(c) $\lim_{\lambda\to 0} J(\lambda, A, B) = J(0, A, B)$.

Proof. We begin with a proof of (b). For fixed $\tau \leqslant 1$ put

$$u(z) = \log^+ \Psi^*_{1\times A}(\tau, \tau z), \qquad v(z) = \log^+ \Psi_{1\times B}(\tau, \tau z).$$

$u$ and $v$ are the extremal functions of $\overline{\{\Phi_A < \tau^{-1}\}}$ and $\overline{\{\Phi_B < \tau^{-1}\}}$, respectively.

Let $v_\varepsilon \downarrow v$ $(\varepsilon \downarrow 0)$ be the standard smooth plurisubharmonic approximation of $v$. By Sard's theorem the boundary of the set

$$\Omega_{\varepsilon_j}(R) := \{v_{\varepsilon_j} < R\}$$

is smooth for almost every $R \in \boldsymbol{R}$, where $\varepsilon_j \downarrow 0$ is some (fixed) sequence (we drop indices $j$ in what follows). Choose $R > 0$ with the property above and put

$$\lambda := \tau \exp(-R), \qquad \tilde{v}_\varepsilon := \max(R, v_\varepsilon), \qquad \tilde{v} := \max(R, v).$$

Since $\tilde{v}$ is the extremal function of $\overline{\Omega(R)} = \overline{\{v < R\}}$, by (2.2) we have

$$\operatorname{supp}(dd^c \tilde{v})^n \subset \partial\Omega(R).$$

Take $\varphi \in C_0^\infty(\boldsymbol{C}^n)$ with $\varphi = 1$ on $\overline{\Omega(R)}$. The Convergence Theorem gives

$$\int_{\overline{\Omega(R)}} (u-R)(dd^c \tilde{v})^n = \int_{\boldsymbol{C}^n} \varphi(u-R)(dd^c \tilde{v})^n = \lim_{\varepsilon\to 0} \int_{\boldsymbol{C}^n} \varphi(u-R)(dd^c \tilde{v}_\varepsilon)^n \tag{4.1}$$

$$= \lim_{\varepsilon\to 0} \int_{\overline{\Omega(R)}} \varphi(u-R)(dd^c \tilde{v}_\varepsilon)^n + \lim_{\varepsilon\to 0} \int_{\boldsymbol{C}^n\setminus\overline{\Omega(R)}} \varphi(u-R)(dd^c \tilde{v}_\varepsilon)^n.$$

Observe that $v_\varepsilon \geq v \geq R$ on $\mathbf{C}^n \setminus \overline{\Omega(R)}$, so $\tilde{v}_\varepsilon = v_\varepsilon$ on this set. Take $\chi \in C_0^\infty(\mathbf{C}^n)$ such that $\chi = 1$ on $(\operatorname{supp}\varphi)\setminus\overline{\Omega(R)}$ and $\chi = 0$ on $\{v = 0\} \supset \operatorname{supp}(dd^c v)^n$. Now from the Convergence Theorem we conclude that the second term of (4.1) vanishes since

$$(4.2)\qquad \int_{\mathbf{C}^n\setminus\overline{\Omega(R)}} \varphi(u-R)(dd^c\tilde{v}_\varepsilon)^n = \int_{\mathbf{C}^n} \chi\varphi(u-R)(dd^c v_\varepsilon)^n$$

$$\to \int_{\mathbf{C}^n} \chi\varphi(u-R)(dd^c v)^n = 0, \qquad \varepsilon \downarrow 0.$$

The similar argument leads to

$$(4.3)\quad \lim_{\varepsilon\to 0} \int_{\overline{\Omega(R)}\setminus\overline{\Omega_\varepsilon(R)}} \varphi(u-R)(dd^c\tilde{v}_\varepsilon)^n = \lim_{\varepsilon\to 0} \int_{\overline{\Omega(R)\setminus\Omega_\varepsilon(R)}} \varphi(u-R)(dd^c v_\varepsilon)^n = 0.$$

Combining (4.1) with (4.2) and (4.3), we get

$$(4.4)\qquad \int_{\overline{\Omega(R)}} (u-R)(dd^c\tilde{v})^n = \lim_{\varepsilon\to 0} \int_{\overline{\Omega_\varepsilon(R)}} (u-R)(dd^c\tilde{v}_\varepsilon)^n.$$

Apply Proposition 2.2 to the right-hand side of (4.4) to obtain

$$(4.5)\qquad \int_{\overline{\Omega_\varepsilon(R)}} (u-R)(dd^c\tilde{v}_\varepsilon)^n = \int_{\overline{\Omega_\varepsilon(R)}} (u-v_\varepsilon)(dd^c\tilde{v}_\varepsilon)^n$$

$$= \int_{\partial\Omega_\varepsilon(R)} (u-v_\varepsilon)\, d^c v_\varepsilon \wedge (dd^c v_\varepsilon)^{n-1}.$$

By repeated use of Stokes theorem we derive from (4.4) and (4.5) that

$$(4.6)\quad \int_{\overline{\Omega(R)}} (u-R)(dd^c\tilde{v})^n = \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon(R)} (u-v_\varepsilon)\, d^c v_\varepsilon \wedge (dd^c v_\varepsilon)^{n-1}$$

$$= \lim_{\varepsilon\to 0}\Big( \int_{\Omega_\varepsilon(R)} (u-v_\varepsilon)(dd^c v_\varepsilon)^n + \int_{\Omega_\varepsilon(R)} d(u-v_\varepsilon) \wedge d^c v_\varepsilon \wedge (dd^c v_\varepsilon)^{n-1}\Big)$$

$$= \lim_{\varepsilon\to 0}\Big( \int_{\Omega_\varepsilon(R)} (u-v_\varepsilon)(dd^c v_\varepsilon)^n + \int_{\Omega_\varepsilon(R)} dv_\varepsilon \wedge d^c(u-v_\varepsilon) \wedge (dd^c v_\varepsilon)^{n-1}\Big)$$

$$= \lim_{\varepsilon\to 0}\Big( \int_{\Omega_\varepsilon(R)} (u-v_\varepsilon)(dd^c v_\varepsilon)^n + \int_{\partial\Omega_\varepsilon(R)} v_\varepsilon\, d^c(u-v_\varepsilon) \wedge (dd^c v_\varepsilon)^{n-1} -$$

$$- \int_{\Omega_\varepsilon(R)} v_\varepsilon\, dd^c(u-v_\varepsilon) \wedge (dd^c v_\varepsilon)^{n-1}\Big)$$

$$= \lim_{\varepsilon \to 0} \Big[ \int_{\Omega_\varepsilon(R)} u (dd^c v_\varepsilon)^n + R \Big( \int_{\Omega_\varepsilon(R)} dd^c u \wedge (dd^c v_\varepsilon)^{n-1} - \int_{\Omega_\varepsilon(R)} (dd^c v_\varepsilon)^n \Big) - \\ - \int_{\Omega_\varepsilon(R)} v_\varepsilon \, dd^c u \wedge (dd^c v_\varepsilon)^{n-1} \Big]$$

$$\leqslant \int_{C^n} u (dd^c v)^n .$$

To justify the last inequality note that the second term in square brackets tends to non-positive constant, because by [10] we have

$$\int_{\Omega_\varepsilon(R)} (dd^c v_\varepsilon)^n \to \int_{\Omega(R)} (dd^c v)^n = c_n \geqslant \lim_{\varepsilon \to 0} \int_{C^n} dd^c u \wedge (dd^c v_\varepsilon)^{n-1} .$$

Besides

$$\int_{\Omega_\varepsilon(R)} u (dd^c v_\varepsilon)^n \to \int_{C^n} u (dd^c v)^n$$

as it was shown in the proof of Lemma 3.4 [3]. Finally, by (2.1),

$$\int_{C^n} v_\varepsilon \, dd^c u \wedge (dd^c v_\varepsilon)^{n-1} \geqslant 0 .$$

Observe that (4.6) holds true for every $R > 0$. Indeed, if $\partial\Omega(R)$ is not smooth, then one can apply the Convergence Theorem to the left-hand side of (4.6) with $R$ replaced by some $R'$ arbitrary close to $R$ and such that $\partial\Omega(R')$ is smooth.

Now, by (2.2), (2.5) and the homogeneity of $\Psi$, we have

$$\int_{\overline{\Omega(R)}} (u - R)(dd^c \tilde{v})^n = \int_{C^n} \big(\log \Psi^*_{1 \times A}(\tau, \tau z) - \log \Psi_{1 \times B}(\tau, \tau z)\big) \times \\ \times \big(dd^c \max(\log \Psi_{1 \times B}(\tau, \tau z), \log(\tau/\lambda))\big)^n \\ = \int_{C^n} \big(\log \Psi^*_{1 \times A}(1, z) - \log \Psi_{1 \times B}(1, z)\big)\big(dd^c \log^+ \Psi_{1 \times B}(\lambda, \lambda z)\big)^n .$$

Hence after the change of variables $z \to z/\lambda$ one obtains

$$\int_{\overline{\Omega(R)}} (u - R)(dd^c \tilde{v})^n = J(\lambda, A, B). \tag{4.7}$$

Similarly, the transformation $z \to z/\tau$ leads to

$$\int_{C^n} u (dd^c v)^n = \int_{C^n} \log^+ \Psi^*_{1 \times A}(\tau, \tau z)\big(dd^c \log^+ \Psi_{1 \times B}(\tau, \tau z)\big)^n \\ = J(\tau, A, B). \tag{4.8}$$

The second part of our lemma follows from (4.6), (4.7) and (4.8).

Now we turn to the proof of (c). Setting $\log \Psi^*_{1 \times A}(\lambda, z)$ for $u(\lambda, z)$ in Proposition 3.1, one can find a sequence $\varepsilon_j \downarrow 0$ such that

$$\log \Psi^*_{1 \times A}(0, z) = \lim_{j \to \infty} \log \Psi^*_{1 \times A}(\varepsilon_j, z)$$

almost everywhere in $\{|z| < M\} \setminus \{|z| < \delta\}$, where $\delta$ and $M$ are so chosen that

$$\{|z| < \delta\} \subset \{z \in C^n: \Psi^*_{1\times A}(0, z) \leqslant 1\} \subset \{|z| < M\}.$$

Since $\Psi_{1\times B}$ is continuous and homogeneous, $\log \Psi_{1\times B}(\lambda, z)$ tends to $\log \Psi_{1\times B}(0, z)$ uniformly in $C^n \setminus \{|z| < \delta\}$ as $\lambda \to 0$. Thus we may apply Corollary 2.1 to obtain

$$\lim_{j\to\infty} J(\varepsilon_j, A, B) = J(0, A, B).$$

This equality combined with (2) gives (3).

It remains to show the first part of the lemma. By (2.2), (2.4) and the Convergence Theorem we have

$$\lim_{j\to\infty} J(1, A, B_j) = 0,$$

but from (2) and (3) it follows that

$$J(\lambda, A, B_j) \leqslant J(1, A, B_j) \quad \text{for } 0 \leqslant \lambda \leqslant 1$$

and every $j$. So

$$\lim_{j\to\infty} J(\lambda, A, B_j) = 0 \quad \text{where } 0 \leqslant \lambda \leqslant 1.$$

This completes the proof of Lemma 3.1.

**5. Proof of Theorem 1.1.** For $E$ a compact subset of $C^n$, let $E_k$ be a decreasing to $E$ sequence of compact $L$-regular sets. It is well known (see [9]) that

$$(5.1) \qquad \begin{aligned} v(z) := \lim_{k\to\infty} \log \Psi_{1\times E_k}(0, z) &= \sup_{k\geqslant 1} \log \Psi_{1\times E_k}(0, z) \\ &= \log \Psi_{1\times E}(0, z). \end{aligned}$$

Setting $u(z) := \log \Psi^*_{1\times E}(0, z)$, we may rewrite the conclusion of Theorem 1.1 as

$$(5.2) \qquad u = v^*.$$

It is clear that $u \geqslant v^*$.

When $E$ is pluripolar,

$$\sup_{|z|=1} \Phi_{E_k}(z) \to +\infty \quad \text{as } k \to \infty$$

(see [8]) and, by the result of Taylor [10],

$$\limsup_{|z|\to\infty} (\Phi_{E_k}(z)/|z|) \geqslant A(\sup_{|z|=1} \Phi_{E_k}(z))^{1/n}$$

for some $A > 0$ and every $k$. It means that

$$\sup_{|z|=1} \Psi_{1\times E_k}(0, z) = \limsup_{|z|\to\infty} \Psi_{1\times E_k}(1/|z|, z/|z|)$$
$$\geqslant A\big(\sup_{|z|=1} \Psi_{1\times E_k}(1, z)\big)^{1/n}, \quad k \geqslant 1,$$

so

$$\sup_{|z|=1} \Psi_{1\times E_k}(0, z) \to +\infty \quad \text{as } k \to \infty,$$

and by [8], $+\infty \equiv v^* \leqslant u = +\infty$. This proves the theorem for pluripolar sets.

Now assume $E$ not to be pluripolar. By the first part of Lemma 4.1 and the Convergence Theorem we have

$$\int_{C^n} \tilde{u}(dd^c \tilde{v}^*)^n = 0, \tag{5.3}$$

where $\tilde{u} = \max(0, u)$ and $\tilde{v}^* = \max(0, v^*)$. Because of identities

$$\begin{matrix} v(\lambda z) = v(z) + \log|\lambda| \\ u(\lambda z) = u(z) + \log|\lambda| \end{matrix} \quad \text{for } z \in C^n \text{ and } \lambda \in C, \tag{5.4}$$

$\tilde{u}$ and $\tilde{v}^*$ are the extremal functions of $F_1 = \{u \leqslant 0\}$ and $F_2 = \{v^* \leqslant 0\}$, respectively. Moreover, the set of points where $v^*$ discontinuous is pluripolar by (5.1) and the theorem of Bedford and Taylor [2] which states that negligible sets are pluripolar. Hence $\tilde{v}^*$ is also the upper semicontinuous regularization of the extremal function of the set $K = \overline{\{v^* < 0\}}$ (see [9]). Now $F_1 \subset K$ by (5.4) and we may apply Theorem 2.2 to derive (5.2) from (5.3). Thus Theorem 1.1 follows.

**6. Capacities.** One may define two capacities, $c$ and $\bar{c}$, corresponding to the extremal function $u_E = \log \Phi_E$, then former introduced by Siciak [8] and Zaharjuta [11], latter studied by Levenberg [5]. Both are extensions of the notion of the logarithmic capacity to the multi-dimensional case. For $E$ a compact subset of $C^n$ put

$$\gamma(E) = \overline{\lim_{|z|\to\infty}} \big(u_E(z) - \log|z|\big) = \max_{|z|=1}\big(\log \Psi^*_{1\times E}(0, z)\big),$$
$$\bar{\gamma}(E) = \lim_{R\to\infty} \int_{|z|=1} \big(u_E(Rz) - \log R\big)\, d\sigma = \int_{|z|=1} \log \Psi^*_{1\times E}(0, z)\, d\sigma$$

(where $\sigma$ is the unitary invariant normalized surface measure on the sphere $\{|z| = 1\}$) and define

$$c(E) = \exp(-\gamma(E)), \quad \bar{c}(E) = \exp(-\bar{\gamma}(E)).$$

Note that by Theorem 1.1 $\Psi_{1\times E}(0, z) = \Psi^*_{1\times E}(0, z)$ for every $z \in C^n \setminus A$, where $A$ is a pluripolar cone. The definitions above may thus be stated as follows:

$$(6.1) \qquad c(E) = \big(\sup_{|z|=1} \Psi_{1\times E}(0, z)\big)^{-1}, \qquad \bar{c}(E) = \exp\Big(-\int_{|z|=1} \log \Psi_{1\times E}(0, z)\, d\sigma\Big).$$

Now it is easy to see that

COROLLARY 6.1. *$c$ and $\bar{c}$ are Choquet capacities.*

Proof. The statement is a direct consequence of (6.1) and properties of the homogeneous extremal function proved in [9].

From the Theorem 1.1 follows also

COROLLARY 6.2. *For $E$ a compact subset of $C^n$ and for every $z$ from the unit sphere which lie outside some pluripolar cone, there exists the directional limit of $u_E^*$*

$$\lim_{|\lambda|\to\infty} \big(u_E^*(\lambda z) - \log|\lambda|\big) = \log \Psi_{1\times E}(0, z), \qquad \lambda \in C,$$

*in the complex line through $z$ and the origin.*

Proof. If we denote

$$A = \{z\colon\ \Psi_{1\times E}(0, z) \neq \Psi^*_{1\times E}(0, z)\},$$

then $A$ is a pluripolar cone by Theorem 1.1. For $z \notin A$ we have

$$u_E(\lambda z) - \log|\lambda| = \log \Psi_{1\times E}(1/|\lambda|, z),$$

and, since $\Psi_{1\times E}$ is lower semicontinuous,

$$\begin{aligned}\Psi_{1\times E}(0, z) &\leqslant \varliminf_{t\to 0} \Psi_{1\times E}(t, z) \leqslant \varlimsup_{t\to 0} \Psi_{1\times E}(t, z) \\ &\leqslant \varlimsup_{t\to 0} \Psi^*_{1\times E}(t, z) \leqslant \Psi^*_{1\times E}(0, z) = \Psi_{1\times E}(0, z),\end{aligned}$$

which was to be proved.

**Acknowledgments.** The author wishes to express his appreciation to Professor Józef Siciak for helpful discussions and comments.

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*Reçu par la Rédaction le 24.04.1987*