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On the convergence of approximate iterations for an abstract equation

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1. The purpose of the present paper is to discuss the conditions ensuring the convergence of the sequence of approximate solutions $\{y_n\}$ to a solution of an abstract equation $x=f(x)$. The most simple approximate solutions of this equation are the simple iterations $x_{n+1}=f(x_n)$, $n = 0, 1, \ldots$ Because of rounding error or error in the evaluation of $f(x)$ an approximate sequence is in general produced in place of the exact sequence $\{x_n\}$. It is very important for numerical analysis to know the effect of this error.

This problem has been investigated in recent years by several authors (for more detailed discussion and bibliography see [3]).

The present paper is related to the general idea of Ważewski [4]. The results stated here are an extension of the paper [1].

2. Preliminaries and lemmas. Following Ważewski [4] we introduce

ASSUMPTION H_1.

1° *G is a partially ordered set (an ordering relation is denoted by $<$, we write $u \leqslant v$ iff $u < v$ or $u = v$), in G there exists an element 0 such that $0 \leqslant u$ for any $u \in G$;*

2° *for any $u, v \in G$ a relation $u+v$ is defined and has the following properties:*

(a) *if $u, v, w \in G$, then $u+v \in G$, $u+v = v+u$, $(u+v)+w = u+(v+w)$, $u+0 = u$,*

(b) *if $u, v, w \in G$ and $u \leqslant v$, then $u+w \leqslant v+w$,*

(c) *if $u, v, w \in G$ and $u+v \leqslant w$, then $u \leqslant w$;*

3° *for any non-increasing sequence $\{u_n\}$, $u_n \in G$, $u_{n+1} \leqslant u_n$, $n = 0, 1, \ldots$, there exists a unique element $u \in G$ called the limit of the sequence $\{u_n\}$ (we write $u = \lim_{n\to\infty} u_n$ or $u_n \searrow u$).*

The limit has the following properties:

(a) *$\lim_{n\to\infty} u_n$ is invariant with respect to the change of a finite number of elements of the sequence $\{u_n\}$,*

(b) *if* $u_n = u$, $n = 0, 1, \ldots$, *then* $\lim\limits_{n\to\infty} u_n = u$,

(c) *if* $u_n \searrow u$, $v_n \searrow v$ *and* $u_n \leqslant v_n$, *then* $u \leqslant v$,

(d) *if* $u_n \searrow u$, $v_n \searrow v$, *then* $u_n + v_n \searrow u + v$.

ASSUMPTION H_2. *The function* $a(u)$ *is defined for* $u \in \Delta \subset G$ *and has the following properties*:

1° $0 \in \Delta$ *and if* $k \in \Delta$, *then* $u \in \Delta$ *for any* $u \leqslant k$;

2° $a(\Delta) \subset G$ ($a(\Delta)$ *is the set of the values of the function* $a(u)$ *for* $u \in \Delta$);

3° *if* $u, v \in \Delta$ *and* $u \leqslant v$, *then* $a(u) \leqslant a(v)$;

4° *if* $u_n \in \Delta$, $n = 0, 1, \ldots$, *and* $u_n \searrow u$, *then* $a(u_n) \searrow a(u)$;

5° $u = 0$ *is the only solution in* Δ *of the equation* $u = a(u)$.

DEFINITION. For any $u \in \Delta$ we define the sequence $\{a_n(u)\}$ of the iterations of the element u by the recurrent formula

$$a_0(u) = u\ ,\ a_{n+1}(u) = a\big(a_n(u)\big)\ ,\ \text{if}\ a_n(u) \in \Delta\ ,\ n = 1, 2, \ldots$$

We state the following

LEMMA 1 (see [4]). *If Assumption* H_2 *is fulfilled and there exists a* $c \in \Delta$ *such that* $a(c) \leqslant c$, *then all iterations* $a_n(c)$, $n = 0, 1, \ldots$, *of the element* c *exist and*

$$a_{n+1}(c) \leqslant a_n(c) \leqslant c\ , \quad n = 0, 1, \ldots,\ \textit{and}\ a_n(c) \searrow 0\ .$$

LEMMA 2 (see [4]). *If Assumption* H_2 *is fulfilled and there exist* $q \in \Delta$ *and* $b \in \Delta$ *such that*

$$q + a(b) \leqslant b\ ,$$

then the equation

$$u = a(u) + q \tag{1}$$

has the solution $u = m(b, q) \leqslant b$, *which has the following properties*:

1° $m(b, q) = \lim\limits_{n\to\infty} b_n(b, q)$, *where* $b_0(b, q) = b$, $b_{n+1}(b, q) = q + a\big(b_n(b, q)\big)$, $n = 0, 1, \ldots$;

' *if* $p \leqslant b$ *and* $p \leqslant q + a(p)$, *then* $p \leqslant m(b, q)$.

From Lemmas 1, 2 and Lemmas 3, 4 of the paper [1] we have

LEMMA 3. *If Assumption* H_2 *is fulfilled and there exist* $q_n, b_k \in \Delta$, q_{n+1} ; $b_{k+1} \leqslant b_k$, $k, n = 0, 1, \ldots$, *such that*

$$q_n + a(b_k) \leqslant b_k\ ,$$

then the equation

$$u = a(u) + q_n$$

has a solution $u = m(b_k, q_n) \leqslant b_k$ *being the non-decreasing function of both variables* b_k, q_n, *i.e.* $m(b_k, q_{n+1}) \leqslant m(b_k, q_n)$ *and* $m(b_{k+1}, q_n) \leqslant m(b_k, q_n)$, k, $n = 0, 1, \ldots$

Moreover, if $q_n \searrow q$ *and* $b_k \searrow b$, *then* $m(b_k, q_n) \searrow m(b_k, q)$, $m(b_k, q_n) \searrow \searrow m(b, q_n)$ *and, consequently, if* $q_n \searrow 0$, *then* $m(b_k, q_n) \searrow 0$.

Definition. $m(q)$ is called the *maximal solution of equation* (1) if it satisfies this equation and for any solution $u(q)$ of (1) the inequality $u(q) \leqslant m(q)$ holds true.

Lemma 4 (see Lemma 5 of [1]). *If Assumption* H_2 *is fulfilled*, $\Delta = G$, *and for any* $q \in G$ *equation* (1) *has a maximal solution* $m(q)$, $p \in G$, *and* $p \leqslant a(p) + q$, *then* $p \leqslant m(q)$.

Moreover, if $q_n \searrow q$, $q_{n+1} \leqslant q_n$, $n = 0, 1, \ldots$, *then* $m(q_{n+1}) \leqslant m(q_n)$, $m(q_n) \searrow m(q)$, *and, consequently, if* $q_n \searrow 0$, *then* $m(q_n) \searrow 0$.

Assumption H_3. *The function* $A(u, v)$ *is defined for* $u, v \in \Delta$ *and has the following properties*:

1° $A(\Delta \times \Delta) \subset G$;

2° *if* $u, \bar{u}, v, \bar{v} \in \Delta$ *and* $u \leqslant \bar{u}$, $v \leqslant \bar{v}$, *then* $A(u, v) \leqslant A(\bar{u}, \bar{v})$;

3° *if* $u_n, v_n \in \Delta$, $n = 0, 1, \ldots$, *and* $u_n \searrow u$, $v_n \searrow v$, *then* $A(u_n, v_n) \searrow \searrow A(u, v)$;

4° $u = 0$ *is the only solution in* Δ *of the equation* $u = A(u, u)$.

Lemma 5. *If Assumption* H_3 *is fulfilled and there exist* $q \in \Delta$, $b \in \Delta$, *such that* $q + A(b, b) \leqslant b$, *then for any* $v \leqslant b$ *there exists a solution* $m_v(b, q) \leqslant b$ *of the equation*

$$u = A(u, v) + q .$$

Moreover, if $w \leqslant b$ *and* $w \leqslant A(w, v) + q$, *then*

$$w \leqslant m_v(b, q) \leqslant b .$$

Proof. Put

$$u_0 = b , \quad u_{n+1} = A(u_n, v) + q , \quad n = 0, 1, \ldots ;$$

we have

$$u_n \searrow m_v(b, q) \leqslant b , \quad w \leqslant u_n , \quad n = 0, 1, \ldots ,$$

whence we get the assertion of Lemma 5.

Lemma 6. *If the assumptions of Lemma* 5 *are fulfilled*, $q_{n+1} \leqslant q_n \leqslant q$, $n = 0, 1, \ldots$, $q_n \searrow 0$ *and* $m_{n+1}(b, q_{n+1})$ *is a solution of the equation*

$$u = A\big(u, m_n(b, q_n)\big) + q_{n+1} , \quad m_0(b, q_0) = b , \quad n = 0, 1, \ldots ,$$

then $m_n(b, q_n) \searrow 0$.

Moreover, if $w_{n+1} \leqslant A(w_{n+1}, w_n) + q_{n+1}$ *and* $w_n \leqslant b$, *then* $w_n \leqslant m_n(b, q_n)$, *and, consequently*, $w_n \searrow 0$ *(if* $w_{n+1} \leqslant w_n$*)*.

Proof. This lemma is the simple consequence of the previous one.

Lemma 7. *If Assumption* H_3 *is fulfilled*, $\Delta = G$ *and if for any* $v, q \in G$ *there exist the maximal solutions* $m(q)$, $m_v(q)$ *of the equations*

$$u = A(u, u) + q , \quad u = A(u, v) + q ,$$

and

$$d \leqslant A(d, v) + q , \quad d \in G ,$$

then $d \leqslant m_v(q)$.

Moreover, if $q_n \searrow 0$ *and* $m_{n+1}(q_{n+1})$ *is the maximal solution of the equation*

$$u = A\big(u, m_n(q_n)\big) + q_{n+1} , \quad m_0 = m(q_0) , \quad n = 0, 1, \ldots,$$

and $w_{n+1} \leqslant A(w_{n+1}, w_n) + q_{n+1}$, $w_0 \leqslant m_0$, $n = 0, 1, \ldots$, *then* $m_n(q_n) \searrow 0$ *and* $w_n \leqslant m_n(q_n)$ *and, consequently,* $w_n \searrow 0$ (*if* $w_{n+1} \leqslant w_n$).

Proof. Let u_0 be a solution of the equation

$$u = A(u, v) + q + d ;$$

then $u_0 \geqslant A(u_0, v) + q$ and $u_0 \geqslant d$.

Put $u_{n+1} = A(u_n, v) + q$, $n = 0, 1, \ldots$; we see that $u_{n+1} \leqslant u_n$, whence $u_n \searrow m_v(q)$. By induction we get $d \leqslant u_n$, $n = 0, 1, \ldots$, and consequently $d \leqslant m_v(q)$.

Further, by induction we obtain $m_{n+1}(q_{n+1}) \leqslant m_n(q_n)$, $n = 0, 1, \ldots$, hence $m_n(q_n)$ converges to an element m which satisfies the equation $u = A(u, u)$. By Assumption H_3, $m = 0$.

By virtue of induction we get $w_n \leqslant m_n(q_n)$, $n = 0, 1, \ldots$ The last assertion of lemma is obvious.

LEMMA 8. *If Assumption* H_2 *is fulfilled and the sequence* $\{\varepsilon_n\}$ *is such that* $\varepsilon_n \in G$, $\varepsilon_n \searrow 0$, $n = 0, 1, \ldots$, $\varepsilon_0 + a(b) \leqslant b$, *and the sequence* $\{z_n(\varepsilon)\}$ *is defined by the relation*

$$z_{n+1}(\varepsilon) = \varepsilon_n + a\big(z_n(\varepsilon)\big) , \quad z_0(\varepsilon) = b , \quad n = 0, 1, \ldots ,$$

then $z_{n+1}(\varepsilon) \leqslant z_n(\varepsilon)$ *and* $z_n(\varepsilon) \searrow 0$.

More general, if $\varepsilon_n \searrow \eta$, *then* $z_n(\varepsilon) \searrow z$, *where* z *is the maximal solution of the equation* $u = \eta + a(u)$.

Proof. We have $z_1(\varepsilon) = \varepsilon_0 + a(b) \leqslant b = z_0(\varepsilon)$. By induction we get $z_{n+1}(\varepsilon) \leqslant z_n(\varepsilon)$, $n = 0, 1, \ldots$ According to Assumption H_2, we obtain the assertion of the lemma.

Remark 1. Lemma 8 is a generalization of Lemma 1, if $\varepsilon_n = 0$, $n = 0, 1, \ldots$, then $z_n(0) = a_n(b)$, $n = 0, 1, \ldots$

3. The main space R. We make

ASSUMPTION H_4. *R is an abstract space such that*

1° *for some sequences* $\{x_n\}$, $x_n \in R$, $n = 0, 1, \ldots$, *there exists a uniquely determined limit* $\lim_{n\to\infty} x_n = x$, $x \in R$; $\lim_{n\to\infty} x_n$ *is invariant with respect to the change of a finite number of elements of* $\{x_n\}$ (*the relation* $\lim_{n\to\infty} x_n = x$ *will also be written* $x_n \to x$);

2° *if* $x_n = s \in R$, $n = 0, 1, \ldots$, *then* $\lim\limits_{n\to\infty} x_n = s$;

3° *the function* $r(x, y)$ *is defined on the product* $R \times R$ *and has the following properties*:

(a) $r(x, y) \in G$,

(b) $r(x, y) = 0$ *iff* $x = y$,

(c) *for any* $x, y, z \in R$,

$$r(x, y) \leqslant r(x, z) + r(y, z);$$

4° *for any* $x^* \in R$ *and* $b \in G$ *the sphere*

$$S(x^*, b) = [x\colon\ x \in R,\ r(x, x^*) \leqslant b]$$

is a closed set;

5° *the space* R *is complete in the following sense*: *if* $c_n \in G$, $n = 0, 1, \ldots$, $c_n \searrow 0$ *and for* $x_n \in R$, $n = 0, 1, \ldots$, *the Cauchy condition*

$$r(x_n, x_{n+m}) \leqslant c_n\,, \qquad n, m = 0, 1, \ldots,$$

is satisfied, then there exists a limit $y \in R$ *of sequence* $\{x_n\}$.

ASSUMPTION H_5. *The function* $f(x)$ *is defined on the sphere* $S(x^*, b) \subset R$, $x^* \in R$, $b \in \Delta$, *and has the following properties*:

1° $f(x) \in R$;

2° *for any* $x, y \in S(x^*, b)$,

$$r\big(f(x), f(y)\big) \leqslant a\big(r(x, y)\big)\,,$$

where the function $a(u)$ *satisfies Assumption* H_2 *and* $b + b \overset{\mathrm{df}}{=} 2b \in \Delta$;

3° *there exists a* $q \in \Delta$ *such that*

$$r\big(x^*, f(x^*)\big) \leqslant q \qquad \text{and} \qquad q + a(b) \leqslant b\,.$$

ASSUMPTION H_6. *Suppose that*

1° *the function* $f(x)$ *is defined for* $x \in R$, $f(x) \in R$;

2° *for any* $x, y \in R$,

$$r\big(f(x), f(y)\big) \leqslant a\big(r(x, y)\big)\,,$$

where the function $a(u)$ *satisfies Assumption* H_2 *with* $\Delta = G$;

3° *for any* $q \in G$ *the equation*

$$u = a(u) + q$$

has a maximal solution $m(q)$.

ASSUMPTION H_7. *Assume that*

1° *the functions* $f_n(x, y)$, $n = 0, 1, \ldots$, *are defined on the product* $S(x^*, b) \times S(x^*, b)$; $f_n(x, y) \in R$;

2° *for any* $x, y, s, t \in S(x^*, b)$, $n = 0, 1, \ldots$, *we have*

$$r\big(f_n(x, y), f_n(s, t)\big) \leqslant A\big(r(x, s), r(y, t)\big),$$

where the function $A(u, v)$ *satisfies Assumption* H_3, $2b \in \Delta$;

3° *there exists* $q \in \Delta$ *such that for any* $n = 0, 1, \ldots$,

$$r\big(x^*, f_n(x^*, x^*)\big) \leqslant q \quad \text{and} \quad q + A(b, b) \leqslant b .$$

Assumption H_8. *Suppose that*

1° *the functions* $f_n(x, y)$, $n = 0, 1, \ldots$, *are defined on the product* $R \times R$, $f_n(x, y) \in R$;

2° *for any* $x, y, s, t \in R$, $n = 0, 1, \ldots$,

$$r\big(f_n(x, y), f_n(s, t)\big) \leqslant A\big(r(x, s), r(y, t)\big),$$

where the function $A(u, v)$ *satisfies Assumption* H_3, $\Delta = G$;

3° *for any* $q \in G$ *there exists a maximal solution of the equation*

$$u = A(u, u) + q .$$

4. Approximate iterations, local theorems. Now we are going to formulate some basic theorems.

Theorem 1. *If Assumption* H_5 *is fulfilled and the sequence* $\{y_n\}$ *has the properties*

1° $y_n \in S(x^*, b)$, $n = 0, 1, \ldots$, $y_0 = x^*$,

2° $r\big(y_{n+1}, f(y_n)\big) \leqslant \varepsilon_n$, $\varepsilon_n \searrow 0$ *and* $\varepsilon_0 \leqslant q$,

then there exists in $S(x^*, b)$ *a unique solution* $\bar{x}$ *of the equation*

$$x = f(x) \tag{2}$$

and

$$\lim_{n\to\infty} x_n = \bar{x}, \quad \lim_{n\to\infty} y_n = \bar{x},$$

where

$$x_{n+1} = f(x_n), \quad x_0 = x^* .$$

Moreover

$$r(\bar{x}, x_n) \leqslant a_n(b), \quad r(\bar{x}, y_n) \leqslant z_n(\varepsilon), \quad n = 0, 1, \ldots,$$

$a_n(b)$ *and* $z_n(\varepsilon)$ *are defined in Lemmas* 1 *and* 8.

Proof. At first we prove the convergence of the sequence $\{x_n\}$. Observe that $x_n \in S(x^*, b)$ and $r(x_n, x_{n+m}) \leqslant a_n(b)$, $n, m = 0, 1, \ldots$ Indeed, we have $r(x_0, x^*) \leqslant b$ and further, by induction, we get

$$\begin{aligned} r(x_n, x^*) &= r\big(f(x_{n-1}), f(x^*)\big) + r\big(f(x^*), x^*\big) \leqslant a\big(r(x_{n-1}, x^*)\big) + q \\ &\leqslant a(b) + q \leqslant b . \end{aligned}$$

Similarly, we have $r(x_0, x_m) \leqslant b$ and by induction we obtain

$$r(x_n, x_{n+m}) \leqslant r\big(f(x_{n-1}), f(x_{n-1+m})\big) \leqslant a\big(r(x_{n-1}, x_{n-1+m})\big)$$
$$\leqslant a\big(a_{n-1}(b)\big) = a_n(b) .$$

Because of Lemma 1, $a_n(b) \searrow 0$. Therefore, according to Assumption H_4, there exists a limit $\bar{x}$ of the sequence $\{x_n\}$, $\bar{x} \in S(x^*, b)$. $\bar{x}$ is the solution of equation (2). In fact, we have $r\big(\bar{x}, f(\bar{x})\big) \leqslant r(\bar{x}, x_n) + r\big(f(x_{n-1}), f(\bar{x})\big) \leqslant a_n(b) + a\big(a_{n-1}(b)\big) = 2a_n(b)$. Thus, if $n \to \infty$, we obtain $r\big(\bar{x}, f(\bar{x})\big) = 0$, i.e. $\bar{x} = f(\bar{x})$. $\bar{x}$ is the unique solution of equation (2). Indeed, if there exists another solution $x' \in S(x^*, b)$ of equation (2), then we get $r(x', x_n) \leqslant a_n(b)$ and $r(x', \bar{x}) \leqslant r(x', x_n) + r(x_n, \bar{x}) \leqslant 2a_n(b)$. For $n \to \infty$ we obtain $r(x', \bar{x}) = 0$, i.e. $x' = \bar{x}$.

Now we shall prove the relation $\lim_{n\to\infty} y_n = \bar{x}$. We have

$$r(y_{n+1}, \bar{x}) \leqslant r\big(y_{n+1}, f(y_n)\big) + r\big(f(y_n), f(\bar{x})\big) \leqslant \varepsilon_n + a\big(r(y_n, \bar{x})\big) .$$

Put $r_n = r(y_n, \bar{x})$. Now we get

$$r_{n+1} \leqslant \varepsilon_n + a(r_n) , \qquad r_0 = r(y_0, \bar{x}) = r(x^*, \bar{x}) \leqslant b = z_0(\varepsilon) .$$

By induction, we easily infer that $r_n \leqslant z_n(\varepsilon)$, i.e.

$$r(y_{n+1}, \bar{x}) \leqslant z_{n+1}(\varepsilon) .$$

Moreover,

$$r(y_n, y_{n+m}) \leqslant r(y_n, \bar{x}) + r(y_{n+m}, \bar{x}) \leqslant z_n(\varepsilon) + z_{n+m}(\varepsilon) \leqslant 2z_n(\varepsilon) .$$

Because of Lemma 8, $z_n(\varepsilon) \searrow 0$, therefore, according to Assumption H_4, there exists a limit $\bar{y}$ of the sequence $\{y_n\}$, $\bar{y} \in S(x^*, b)$. We also have $r(\bar{y}, \bar{x}) \leqslant r(\bar{y}, y_n) + r(y_n, \bar{x}) \leqslant 2z_n(\varepsilon)$. Hence, if $n \to \infty$, we obtain $r(\bar{x}, \bar{y}) = 0$, i.e. $\bar{x} = \bar{y}$. Thus Theorem 1 is completely proved.

Remark 2. Instead of Assumption 2° in Theorem 1 one can assume $r\big(y_n, f(y_n)\big) \leqslant \varepsilon_n \searrow 0$, $\varepsilon_0 \leqslant q$ and $r(y_n, \bar{x}) \leqslant b$.

Indeed, we get

$$r(y_n, \bar{x}) \leqslant r\big(y_n, f(y_n)\big) + r\big(f(y_n), f(\bar{x})\big) \leqslant \varepsilon_n + a\big(r(y_n, \bar{x})\big) ,$$

whence, by Lemma 3 we obtain

$$r(y_n, \bar{x}) \leqslant m(b, \varepsilon_n) \searrow 0 .$$

However, it is obvious that condition 2° in Theorem 1 is more convenient for applications then that mentioned above; in 2° there is no requirement $r(y_n, \bar{x}) \leqslant b$.

Remark 3. If the assumptions of Theorem 1 are fulfilled and

$$\varepsilon_n + a\big(r(y_n, y_{n+1})\big) \leqslant q_n \leqslant q , \qquad q_n \searrow 0 ,$$

then

$$r(y_{n+1}, \bar{x}) \leqslant m(b, q_n) .$$

Indeed, we have

$$r(y_{n+1}, \bar{x}) \leqslant r\big(y_{n+1}, f(y_n)\big) + r\big(f(y_n), f(y_{n+1})\big) + r\big(f(y_{n+1}), f(\bar{x})\big)$$
$$\leqslant \varepsilon_n + a\big(r(y_n, y_{n+1})\big) + a\big(r(y_{n+1}, \bar{x})\big) ,$$

now by Lemma 2 we get our assertion.

The sequence $\{y_n\}$ may be generated in a variety of ways, e.g. one may define $y_n = f_{n-1}(y_{n-1})$ or y_n to be a fixed point of the function $f_n(x)$. The functions $f_n(x)$ may also be defined in a different manner, for instance, one can assume that $f_n(x) = f(x)$, $n = 1, 2, \ldots$ (then $y_n = x_n$), or $f_n(x)$ may be equal to the sum of the first n terms of a power series expansion for $f(x)$ (in some particular spaces).

THEOREM 2. *If Assumption* H_5 *is fulfilled and*

1° $f_n(x)$, $n = 0, 1, \ldots$, *are functions defined on* $S(x^*, b)$, *with values in* $S(x^*, b)$,

2° $y_{n+1} = f_n(y_n)$, $n = 0, 1, \ldots$, $y_0 = x^*$,

3° $r\big(f_n(y_n), f(y_n)\big) \leqslant \varepsilon_n$, $\varepsilon_n \searrow \varepsilon$, $\varepsilon_0 \leqslant q$ *or*

3°′ $r\big(f_n(x), f(x)\big) \leqslant \varepsilon_n(x) \searrow \varepsilon(x)$, $\varepsilon_0(x) \leqslant q$ *and Assumption* H_5 *holds for any* $f_n(x)$,

then there exists in $S(x^*, b)$ *a unique solution* $\bar{x}$ *of equation* (2) *and*

$$r(y_n, \bar{x}) \leqslant z_n(\varepsilon) \quad or \quad r(y_n, \bar{x}) \leqslant \bar{z}_n(\bar{\varepsilon}) ,$$

where $z_n(\varepsilon)$ *and* $\bar{z}_n(\bar{\varepsilon})$ *are defined in Lemma* 8.

If, furthermore, $\varepsilon_n + a\big(r(y_n, y_{n+1})\big) \leqslant q_n \leqslant q$, *then* $r(y_{n+1}, \bar{x}) \leqslant m(b, q_n)$.

Moreover, if $\varepsilon = 0$ *or* $\varepsilon(\bar{x}) = 0$, *then*

$$\lim_{n\to\infty} y_n = \bar{x} .$$

Proof. If 3° is fulfilled, then the assertion of the theorem is a simple consequence of Theorem 1. We prove the theorem in the case when 3°′ holds. We define

$$\bar{z}_{n+1}(\bar{\varepsilon}) = a\big(\bar{z}_n(\bar{\varepsilon})\big) + \bar{\varepsilon}_n , \quad \bar{z}_0(\bar{\varepsilon}) = b , \quad n = 0, 1, \ldots ,$$

where

$$\bar{\varepsilon}_n = \varepsilon_n(\bar{x}) \geqslant r\big(f_n(\bar{x}), f(\bar{x})\big) .$$

By Lemma 8 we have $\bar{z}_n(\bar{\varepsilon}) \searrow 0$. Now

$$r(y_n, \bar{x}) \leqslant r\big(f_{n-1}(y_{n-1}), f_{n-1}(\bar{x})\big) + r\big(f_{n-1}(\bar{x}), f(\bar{x})\big) \leqslant a\big(r(y_{n-1}, \bar{x})\big) + \varepsilon_{n-1} .$$

We get, by induction,

$$r(y_n, \bar{x}) \leqslant \bar{z}_n(\bar{\varepsilon}) , \quad n = 0, 1, \ldots$$

Since of $\bar{\varepsilon}_n \searrow 0$, $\bar{z}_n(\bar{\varepsilon}) \searrow 0$ and consequently $y_n \to \bar{x}$.

Remark 4. In practice it is more convenient if $\varepsilon_n(x)$ is independent of x.

THEOREM 3. *If the assumptions of Theorem 2 are fulfilled and Assumption* H_5 *holds for any* $f_n(x)$, $n = 0, 1, \ldots$, *and if the fixed point* $\tilde{y}_n$ *of the function* $f_n(x)$ ($\tilde{y}_n$*'s are uniquely determined) satisfies the condition* $r(\bar{x}, \tilde{y}_n) \leqslant b$, *then* $\tilde{y}_n \to \bar{x}$.

Proof. Since $\tilde{y}_n = f_n(\tilde{y}_n)$, we have

$$r(\tilde{y}_n, \bar{x}) \leqslant r\big(f_n(\tilde{y}_n), f_n(\bar{x})\big) + r\big(f_n(\bar{x}), f_n(y_n)\big) + r(y_{n+1}, \bar{x})$$
$$\leqslant a\big(r(\tilde{y}_n, \bar{x})\big) + a\big(r(y_n, \bar{x})\big) + r(y_{n+1}, \bar{x})\,.$$

According to Theorem 2 we get

$$r(\tilde{y}_n, \bar{x}) \leqslant a\big(r(\tilde{y}_n, \bar{x})\big) + a\big(z_n(\varepsilon)\big) + z_{n+1}(\varepsilon)\,.$$

From the relation $z_n(\varepsilon) \searrow 0$ it follows that there exists n_0 such that $a\big(z_n(\varepsilon)\big) + z_{n+1}(\varepsilon) = q_n \leqslant q$ for $n \geqslant n_0$. Now, by Lemma 2 we obtain

$$r(\tilde{y}_n, \bar{x}) \leqslant m(b, q_n) \quad \text{for } n \geqslant n_0\,.$$

If $n \to \infty$, then $m(b, q_n) \searrow 0$ and consequently $\tilde{y}_n \to \bar{x}$.

Remark 5. If it is not assumed that all the $f_n(x)$, $n = 0, 1, \ldots$, satisfy Assumption H_5 but only that there exists fixed point $\tilde{y}_n$ of $f_n(x)$ (in general not necessarily unique) and if $r(\tilde{y}_n, \bar{x}) \leqslant b$ and $r\big(\tilde{y}_n, f(\tilde{y}_n)\big) \leqslant \varepsilon_n \searrow 0$, $\varepsilon_n \leqslant q$, then $\tilde{y}_n \to \bar{x}$ (see [1]).

Remark 6. If assumptions 1°, 2°, 3°′ of Theorem 2 are fulfilled and Assumption H_5 is satisfied for any $f_n(x)$, $n = 0, 1, \ldots$, then the assertion of Theorem 2 holds true.

Indeed, from the relations

$$r\big(f(x), f(x')\big) \leqslant r\big(f(x), f_n(x)\big) + r\big(f_n(x), f_n(x')\big) + r\big(f_n(x'), f(x')\big)$$
$$\leqslant \varepsilon_n(x) + \varepsilon_n(x') + a\big(r(x, x')\big)\,,$$

together with 3°′, it follows that Assumption H_5 for $f(x)$ is fulfilled.

The following theorem is a generalization of the results of papers [2] and [5]:

THEOREM 4. *If functions* $f_n(x)$ *are defined on* $S(x^*, b)$ *with values in* $S(x^*, b)$ *and*

1° *Assumption* H_5 *for any* $f_n(x)$, $n = 0, 1, \ldots$, *is fulfilled,*

2° $\tilde{y}_n$ *is a fixed point of* $f_n(x)$ *and*

$$r\big(f_{n+1}(\tilde{y}_n), f(\tilde{y}_n)\big) \leqslant \varrho_n \searrow 0\,,$$

3° $r(\widetilde{y}_n, \bar{x}) \leqslant b$, $r(\widetilde{y}_{n+1}, \widetilde{y}_n) \leqslant p_0 \leqslant b$, $p_0 \geqslant 3a(p_0)+q$,

4° *$p = 0$ is the only solution of the equation $p = 3a(p)$,*

then

$$r(\widetilde{y}_{n+1}, f(\widetilde{y}_n)) \leqslant \varepsilon_n$$

and there exists n_0 such that $\varepsilon_n \leqslant q$ for $n \geqslant n_0$.

Consequently, $\widetilde{y}_n \to \bar{x}$, where $\bar{x}$ is a unique solution of the equation $x = f(x)$ in $S(x^*, b)$.

Proof. At first we have

$$\begin{aligned} r(\widetilde{y}_{n+1}, f(\widetilde{y}_n)) &\leqslant r(f_{n+1}(\widetilde{y}_{n+1}), f(\widetilde{y}_n)) \\ &\leqslant r(f_{n+1}(\widetilde{y}_{n+1}), f_{n+1}(\widetilde{y}_n)) + r(f_{n+1}(\widetilde{y}_n), f(\widetilde{y}_n)) \\ &\leqslant a(r(\widetilde{y}_{n+1}, \widetilde{y}_n)) + \varrho_n . \end{aligned}$$

Further, we get

$$\begin{aligned} r(\widetilde{y}_{n+1}, \widetilde{y}_n) \leqslant{}& r(f_{n+1}(\widetilde{y}_{n+1}), f_{n+1}(\widetilde{y}_n)) + r(f_{n+1}(\widetilde{y}_n), f(\widetilde{y}_n)) + \\ &+ r(f(\widetilde{y}_n), f(\widetilde{y}_{n-1})) + r(f(\widetilde{y}_{n-1}), f_n(\widetilde{y}_{n-1})) + r(f_n(\widetilde{y}_{n-1}), f_n(\widetilde{y}_n)) \\ \leqslant{}& a(r(\widetilde{y}_{n+1}, \widetilde{y}_n)) + \varrho_n + \varrho_{n-1} + 2a(r(\widetilde{y}_n, \widetilde{y}_{n-1})) . \end{aligned}$$

Put

$$p_n = m(p_0, q_n) , \quad q_n = \varrho_n + \varrho_{n-1} + 2a(p_{n-1}) , \quad n = 1, 2, \ldots$$

By assumption 3° we get $p_1 \leqslant p_0$ and further by induction and according to Lemma 3 we obtain the inequality $p_{n+1} \leqslant p_n$, $n = 0, 1, \ldots$ Now $p_n \searrow p$ and the limit p is a solution of the equation

$$p = m(p_0, 2a(p)) , \quad \text{i.e. } p = 3 \cdot a(p) .$$

By assumption 4°, $p = 0$, i.e. $p_n \searrow 0$.

Now according to Lemma 2, by induction, we obtain

$$r(\widetilde{y}_{n+1}, \widetilde{y}_n) \leqslant m(p_0, q_n) = p_n , \quad n = 0, 1, \ldots$$

Finally, we get

$$r(\widetilde{y}_{n+1}, f(\widetilde{y}_n)) \leqslant a(p_n) + \varrho_n \overset{\mathrm{df}}{=} \varepsilon_n \searrow 0 .$$

It is evident that there exists n_0 such that $\varepsilon_n \leqslant q$ for $n \geqslant n_0$. Now the relation $\widetilde{y}_n \to \bar{x}$ follows immediately from Theorem 1.

Remark 7. If Assumption H_5 is fulfilled for any $f_n(x)$, $n = 0, 1, 2, \ldots$, and

$$r(\widetilde{y}_{n+1}, f(\widetilde{y}_n)) \leqslant \varepsilon_n \searrow 0 ,$$

then

$$r(f_{n+1}(\widetilde{y}_n), f(\widetilde{y}_n)) \leqslant \delta_n \searrow 0 .$$

Indeed, we have $\widetilde{y}_n \to \bar{x}$ and, consequently, $r(\widetilde{y}_{n+1}, \widetilde{y}_n) \searrow 0$. Finally we get

$$r\big(f_{n+1}(\widetilde{y}_n), f(\widetilde{y}_n)\big) \leqslant r\big(f_{n+1}(\widetilde{y}_n), f_{n+1}(\widetilde{y}_{n+1})\big) + r\big(f_{n+1}(\widetilde{y}_{n+1}), f(\widetilde{y}_n)\big)$$
$$\leqslant a\big(r(\widetilde{y}_{n+1}, \widetilde{y}_n)\big) + \varepsilon_n \overset{\mathrm{df}}{=} \delta_n \searrow 0 .$$

THEOREM 5. *If*

1° *the function* $f(x)$ *is defined in* $S(x^*, b)$, $f(x) \in R$,

2° *Assumption* H_7 *is fulfilled,*

3° $r\big(f_n(x, x), f(x)\big) \leqslant \varepsilon_n(x) \searrow 0$, $x \in S(x^*, b)$, $\varepsilon_n(x) \leqslant q$,

then there exists in $S(x^*, b)$ *a unique solution* $\bar{x}$ *of equation* (2). *The sequence* $\{y_n\}$ *is well-defined by the relation*

$$y_n = f_n(y_n, y_{n-1}) , \quad y_0 = x^* , \quad n = 1, 2, \ldots ;$$

moreover, if $r(y_n, \bar{x}) \leqslant b$, *then* $y_n \to \bar{x}$.

Proof. First of all we prove that Assumption H_5 with $a(u) = A(u, u)$ holds true.

In fact, we have

$$r\big(f(x), f(y)\big) \leqslant r\big(f(x), f_n(x, x)\big) + r\big(f_n(x, x), f_n(y, y)\big) + r\big(f_n(y, y), f(y)\big)$$
$$\leqslant A\big(r(x, y), r(x, y)\big) + \varepsilon_n(x) + \varepsilon_n(y) ,$$

$$r\big(x^*, f(x^*)\big) \leqslant r\big(x^*, f_n(x^*, x^*)\big) + r\big(f_n(x^*, x^*), f(x^*)\big) \leqslant q + \varepsilon_n(x^*) .$$

Now if $n \to \infty$ we get our assertion. According to Theorem 1 we infer the existence and uniqueness of the solution $\bar{x}$ of equation (2).

Now we prove that y_n exists and $y_n \in S(x^*, b)$ for $n = 0, 1, \ldots$; y_n can be constructed by an ordinary iteration procedure. Put

$$y_n^{k+1} = f_n(y_n^k, v) , \quad y_n^0 = x^* , \quad v \in S(x^*, b) , \quad k = 0, 1, \ldots$$

By induction with respect to k, we get $y_n^k \in S(x^*, b)$. Indeed $y_n^0 \in S(x^*, b)$ and if $y_n^k \in S(x^*, b)$, then

$$r(x^*, y_n^{k+1}) = r\big(x^*, f_n(y_n^k, v)\big) \leqslant r\big(x^*, f_n(x^*, x^*)\big) + r\big(f_n(x^*, x^*), f_n(y_n^k v)\big)$$
$$\leqslant A\big(r(x^*, y_n^k), r(x^*, v)\big) + r\big(x^*, f_n(x^*, x^*)\big) \leqslant A(b, b) + q \leqslant b .$$

Set

$$A_k(b) = A\big(A_{k-1}(b), 0\big) , \quad A_0(b) = b .$$

It is easy to see that $A_k(b) \searrow 0$. By induction we obtain

$$r(y_n^k, y_n^{k+p}) \leqslant A_k(b) .$$

In fact, we have $r(y_n^0, y_n^p) \leqslant b$ and if $r(y_n^k, y_n^{k+p}) \leqslant A_k(b)$, then

$$r(y_n^{k+1}, y_n^{k+1+p}) \leqslant r\big(f_n(y_n^k, v), f_n(y_n^{k+p}, v)\big) \leqslant A\big(r(y_n^k, y_n^{k+p}), 0\big)$$
$$\leqslant A\big(A_k(b), 0\big) = A_{k+1}(b) .$$

Now we conclude that $y_n^k \to y_n$ for $k \to 0$. We find that the equation

$$y = f_n(y, v) , \quad v \in S(x^*, b)$$

has a unique solution y_n, $y_n \in S(x^*, b)$, $n = 0, 1, \ldots$ From this it is obvious that the sequence $\{y_n\}$ occurring in the assertion of our theorem is well-defined and $y_n \in S(x^*, b)$.

Now we show that $y_n \to \bar{x}$. We have

$$r(y_n, \bar{x}) \leqslant r\big(f_n(y_n, y_{n-1}), f_n(\bar{x}, \bar{x})\big) + r\big(f_n(\bar{x}, \bar{x}), f(\bar{x})\big)$$
$$\leqslant A\big(r(y_n, \bar{x}), r(y_{n-1}, \bar{x})\big) + \varepsilon_n(\bar{x}) .$$

Putting $r(y_n, \bar{x}) = d_n \leqslant b$ we see that

$$d_n \leqslant A(d_n, d_{n-1}) + \varepsilon_n(\bar{x}) , \quad n = 0, 1, \ldots$$

By Lemma 6 we have $d_n \searrow 0$ and, consequently, $y_n \to \bar{x}$.

Remark 8. If Assumption H_5 is fulfilled, then assumption 3° in Theorem 5 can be replaced by the following one:

$$r\big(f_n(y_n, y_{n-1}), f(y_n)\big) \leqslant \varepsilon_n \searrow 0 , \quad \varepsilon_n \leqslant q .$$

Indeed, now we get

$$r(y_n, \bar{x}) \leqslant r\big(f_n(y_n, y_{n-1}), f(y_n)\big) + r\big(f(y_n), f(\bar{x})\big) \leqslant \varepsilon_n + a\big(r(y_n, \bar{x})\big) .$$

Finally, if $r(y_n, \bar{x}) \leqslant b$, then $r(y_n, \bar{x}) \leqslant m(b, \varepsilon_n) \searrow 0$, i.e. $y_n \to \bar{x}$.

Remark 9. In our considerations we have assumed that $r(y_n, \bar{x}) \leqslant b$. It may occur that this relation is difficult to be verified. In order to avoid this difficulty it is sufficient to make the following assumption:

$$b \geqslant q + a(b) \quad \text{and} \quad 2b \geqslant q + a(2b) .$$

Under this condition our lemmas also for b replaced by $2b$ hold true and therefore the relation $r(y_n, \bar{x}) \leqslant b$ is not required.

5. Approximate iterations. Non-local theorems. Now we are going to formulate some theorems having non-local character.

THEOREM 6. *If Assumption H_6 is fulfilled and the sequence $\{y_n\}$ is such that*

$$r\big(y_{n+1}, f(y_n)\big) \leqslant \varepsilon_n \searrow 0 ,$$

then there exists a unique solution $\bar{x} \in R$ *of equation* (2) *and* $\bar{x} = \lim_{n\to\infty} x_n$, $\bar{x} = \lim_{n\to\infty} y_n$, *where* $x_{n+1} = f(x_n)$, $x_0 = x^*$, x^* *being arbitrarily fixed element in* R.

Proof. Put $b = m(q)$, $q = r(x^*, f(x^*))$. Now, if x is a solution of equation (2), then

$$r(x, x^*) \leqslant r(f(x), f(x^*)) + r(f(x^*), x^*) \leqslant a(r(x, x^*)) + q\,.$$

By Lemma 4 we get $r(x, x^*) \leqslant b$. Clearly, all the assumptions of Theorem 1 are fulfilled for the sphere $S(x^*, b)$ and, consequently, there exists a unique solution $\bar{x}$ of equation (2).

Put $k_0 = m(r(y_0, f(y_0)))$. Let w_0 be the maximal solution of the equation

$$u = a(u) + \max(\varepsilon_0, k_0)\,,$$

where

$$\max(\varepsilon_0, k_0) \overset{\mathrm{df}}{=} \begin{cases} \varepsilon_0 & \text{if} \quad \varepsilon_0 \geqslant k_0\,, \\ k_0 & \text{if} \quad k_0 > \varepsilon_0\,, \\ \varepsilon_0 + k_0 & \text{otherwise}\,. \end{cases}$$

Defining $w_{n+1} = a(w_n) + \varepsilon_n$, $n = 0, 1, \ldots$, we see that $w_n \searrow 0$. We have

$$r(y_0, \bar{x}) \leqslant r(y_0, f(y_0)) + r(f(y_0), f(\bar{x})) \leqslant r(y_0, f(y_0)) + a(r(y_0, \bar{x}))\,,$$

whence

$$r(y_0, \bar{x}) \leqslant m(r(y_0, f(y_0))) = k_0 \leqslant w_0\,.$$

Further, by induction, we get

$$\begin{aligned} r(y_n, \bar{x}) &\leqslant r(f(y_{n-1}), f(\bar{x})) + r(f(y_{n-1}), y_n) \leqslant a(r(y_{n-1}, \bar{x})) + \varepsilon_{n-1} \\ &\leqslant a(w_{n-1}) + \varepsilon_{n-1} = w_n\,, \end{aligned}$$

i.e. $r(y_n, \bar{x}) \leqslant w_n$ for $n = 0, 1, \ldots$, and the second assertion of the theorem follows.

THEOREM 7. *If Assumption* H_6 *is fulfilled and*

1° *the functions* $f_n(x)$ *are defined on* R *with values in* R,

2° $y_{n+1} \overset{\mathrm{df}}{=} f_n(y_n)$, *where* y_0 *is arbitrarily fixed element of* R,

3° $r(f_n(y_n), f(y_n)) \leqslant \varepsilon_n$,

then $r(y_n, \bar{x}) \leqslant w_n \searrow 0$ *(*w_n *is defined in the proof of Theorem* 6*), i.e.* $y_n \to \bar{x}$. *Moreover, if Assumption* H_6 *is fulfilled for any* $f_n(x)$, $n = 0, 1, \ldots$, *and* $\tilde{y}_n$ *are fixed points of* $f_n(x)$, *then* $\tilde{y}_n \to \bar{x}$.

Proof. The first part of the assertion of the theorem is the simple consequence of Theorem 6. It only remains to prove that $\tilde{y}_n \to \bar{x}$. We have

$$r(\tilde{y}_n, \bar{x}) \leqslant r\big(f_n(\tilde{y}_n), f_n(\bar{x})\big) + r\big(f_n(\bar{x}), f_n(y_n)\big) + r(y_{n+1}, \bar{x})$$
$$\leqslant a\big(r(\tilde{y}_n, \bar{x})\big) + a\big(r(y_n, \bar{x})\big) + r(y_{n+1}, \bar{x})$$
$$\leqslant a\big(r(\tilde{y}_n, \bar{x})\big) + a(w_n) + w_{n+1} ,$$

whence, by Lemma 4, we obtain

$$r(\tilde{y}_n, \bar{x}) \leqslant m\big(a(w_n) + w_{n+1}\big) \searrow 0 , \quad \text{i.e. } \tilde{y}_n \to \bar{x} .$$

THEOREM 8. *If Assumption* H_8 *is fulfilled and*

$$r\big(f_n(x, x), f(x)\big) \leqslant \varepsilon_n(x) \searrow 0 ,$$

then there exists a unique solution $\bar{x}$ *of the equation*

$$x = f(x) .$$

The sequence $\{y_n\}$ *is well-defined by the relation* $y_n = f_n(y_n, y_{n-1})$, $y_0 = x^*$, *where* x^* *is an arbitrarily fixed element of* R, *and* $y_n \to \bar{x}$.

Proof. The proof of the existence of $\bar{x}$ and y_n is essentially the same as that of Theorem 5 (see also the first part of the proof of Theorem 6).

Put $q_n = \varepsilon_n(\bar{x})$, $n = 0, 1, \ldots$ Suppose that $q \leqslant q_0$, where $q \geqslant r\big(x^*, f_n(x^*, x^*)\big)$, $n = 0, 1, \ldots$, and $m_n(q_n)$ is defined in Lemma 7. Put $u_n = r(y_n, \bar{x})$, $n = 0, 1, \ldots$

Now we have

$$u_0 = r(y_0, \bar{x}) = r(x^*, \bar{x}) \leqslant m(q) \leqslant m(q_0) = m_0 ,$$

$$r(y_n, \bar{x}) \leqslant r\big(f_n(y_n, y_{n-1}), f_n(\bar{x}, \bar{x})\big) + r\big(f_n(\bar{x}, \bar{x}), f(\bar{x})\big)$$
$$\leqslant A\big(r(y_n, \bar{x}), r(y_{n-1}, \bar{x})\big) + \varepsilon_n(\bar{x}) ,$$

$$u_n \leqslant A(u_n, u_{n-1}) + q_n , \quad n = 1, 2, \ldots$$

From Lemma 7 we obtain $u_n \leqslant m_n(q_n) \searrow 0$, and consequently $y_n \to \bar{x}$.

THEOREM 9. *If Assumption* H_6 *is fulfilled for* $f(x)$ *and* $f_n(x)$, $n = 0, 1, \ldots$, *and if*

1° $r\big(f_{n+1}(\tilde{y}_n), f(\tilde{y}_n)\big) \leqslant \varrho_n$, *where* $\tilde{y}_n$ *is the fixed point of* $f_n(x)$,

2° *there exists a solution* $p_0 \in G$ *of the equation*

$$u = 3a(u) + \varrho_1 + \varrho_0 + r(\tilde{y}_0, \tilde{y}_1) ,$$

3° $u = 0$ *is the only solution of the equation* $u = 3a(u)$,

then there exists the sequence $\{\varepsilon_n\}$, $\varepsilon_n \searrow 0$, *such that* $r\big(\tilde{y}_{n+1}, f(\tilde{y}_n)\big) \leqslant \varepsilon_n$, $n = 0, 1, \ldots$, *and consequently* $\tilde{y}_n \to \bar{x}$.

Proof. Put $p_n = m\big(\varrho_n + \varrho_{n-1} + 2a(p_{n-1})\big)$. From the definition of p_0 we get

$$p_0 \geqslant m\big(\varrho_1 + \varrho_0 + 2a(p_0)\big) = p_1 ,$$

and by induction we obtain $p_n \searrow 0$.

Now (cf. the proof of Theorem 4),

$$r\big(\tilde{y}_{n+1}, f(\tilde{y}_n)\big) \leqslant a\big(r(\tilde{y}_{n+1}, \tilde{y}_n)\big) + \varrho_n ,$$

$$r(\tilde{y}_{n+1}, \tilde{y}_n) \leqslant a\big(r(\tilde{y}_{n+1}, \tilde{y}_n)\big) + \varrho_n + \varrho_{n-1} + 2a\big(r(\tilde{y}_n, \tilde{y}_{n-1})\big) ,$$

whence, by Lemma 4, we find

$$r(\tilde{y}_{n+1}, \tilde{y}_n) \leqslant m\Big(\varrho_n + \varrho_{n-1} + 2a\big(r(\tilde{y}_n, \tilde{y}_{n-1})\big)\Big) .$$

Observe that $r(\tilde{y}_1, \tilde{y}_0) \leqslant p_0$. This is a simple consequence of the definition of p_0. Now by induction we have

$$r(\tilde{y}_{n+1}, \tilde{y}_n) \leqslant p_n , \quad n = 0, 1, \ldots$$

Finally, we conclude

$$r\big(\tilde{y}_{n+1}, f(\tilde{y}_n)\big) \leqslant a(p_n) + \varrho_n .$$

Put $\varepsilon_n = a(p_n) + \varrho_n$. It is easy to see that $\varepsilon_n \searrow 0$.

Now the relation $\tilde{y}_n \to \bar{x}$ follows from Theorem 6.

Remark 10. Theorem 9 is a generalization of the results of papers [2] and [5].

References

[1] M. Kwapisz, *On the approximate solutions of an abstract equation*, Ann. Polon. Math. 19 (1967), pp. 47-60.

[2] — *Uwagi o pewnym algorytmie rozwiązywania równań różniczkowych zwyczajnych*, Zastosow. Mat. 9 (1968), pp. 275-287.

[3] J. M. Ortega and W. C. Rheinboldt, *On a class of approximate iterative processes*, Arch. Rat. Mech. Anal. 23, No. 5 (1967), pp. 352-365.

[4] T. Ważewski, *Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison*, Bull. Acad. Sci., sér. sci. math. astr. et phys., 8, No. 1 (1960), pp. 45-52.

[5] R. Zuber, *O pewnym algorytmie rozwiązywania równań różniczkowych zwyczajnych pierwszego rzędu* (I), Zastosow. Mat. 8 (1966), pp. 351-363.

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