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Boundedness theorems for some second order differential equations I

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Abstract. The boundedness as $t\to\infty$ of solutions of Liénard's equation

$$\ddot{x}+f(x)\dot{x}+g(x) = p(t), \tag{1}$$

and other equations, are considered by replacing (1) by

$$\dot{x} = y-F(x,t), \quad \dot{y} = -g(x). \tag{2}$$

For (1),

$$F(x) = \int_0^x f(\xi)d\xi, \quad P(t) = \int_0^t p(\tau)d\tau, \quad F(x,t) = F(x)-P(t).$$

The results for (1) are similar to those of Graef, but the method is also applied to

$$\ddot{x}+kf(x,k)\dot{x}+g(x,k) = kp(t,k), \quad k\geqslant 1, \tag{3}$$

to obtain results involving constants independent of k. Similar results for k small are obtained by a modification of the method, and also for (3) with k small and $g(x)$ replaced by x, $kp(t,k)$ by $E\cos\omega t$, provided that $|\omega^2-1|\geqslant c>0$. Some special results are obtained for (1) with $f(x,\dot{x},t)$ in place of $f(x)$.

The principal hypotheses for (2) are

$$F(x,t)\operatorname{sign}x > b > 0 \quad \text{for } |x|\geqslant 1,$$

$$g(x)\operatorname{sign}x > 0 \quad \text{for } |x|\geqslant 1,$$

$$F(x,t)+G(x)\to\infty \quad \text{as } |x|\to\infty, \text{ where } G(x) = \int_0^x g(\xi)d\xi.$$

1. Introduction. The object of this paper is to establish the boundedness as $t\to+\infty$ of the solutions of some differential equations of the form

$$\ddot{x}+f(x,\dot{x},t)\dot{x}+g(x,t) = 0$$

subject to certain very general conditions. From the point of view of physical interpretation it is natural to write this equation in the slightly less general form

$$\ddot{x}+f(x,\dot{x},t)\dot{x}+g(x) = p(t) \tag{1}$$

in which the restoring term and the forcing term have been separated. Important special cases of (1) are Liénard's equation

$$\ddot{x}+f(x)\dot{x}+g(x) = p(t) \tag{2}$$

and the corresponding equation without a forcing term; these already possess a considerable literature, which is comprehensively surveyed in Graef [7]. It is usual to write

$$F(x) = \int_0^x f(\xi)d\xi, \quad P(t) = \int_0^t p(\tau)d\tau, \quad F(x,t) = F(x)-P(t) \tag{3}$$

and to replace (2) by the pair of first order equations

$$\dot{x} = y-F(x,t), \quad \dot{y} = -g(x). \tag{4}$$

Here $F(x,t)$ has a very special form. In some cases, however, it is desirable to obtain boundedness theorems with constants independent of some parameter k or ε which occurs in f, g or p; and some of these cases can only be handled by reducing (2) to a more general equation of type (4).

We assume that $F(x,t)$ is continuous for all x and t, and that $g(x)$ is continuous for all x. Of course, for a trajectory starting when $t = t_0$ we are only concerned with the behaviour of $F(x,t)$ for $t \geqslant t_0$; but it is convenient to state our hypotheses in a form independent of t_0. Further we assume the existence and uniqueness of solutions for assigned initial conditions, not only for (4) but for all the second order differential equations and pairs of first order equations considered in this paper.

The additional difficulties imposed by considering a general $F(x,t)$ in (4) instead of that given by (3) are slight; indeed our methods and results for (4), although discovered independently, are very similar to those of Graef [7] for (2). In (2) and (3) it is usual to assume that $P(t)$ is bounded, and this with the continuity of $F(x)$ implies that there is a continuous function $F_M(x)$ such that

$$|F(x,t)| \leqslant F_M(x). \tag{5}$$

In particular there is a constant c_1 such that

$$|F(x,t)| < c_1 \quad \text{for } |x| \leqslant 1. \tag{6}$$

Again, since $g(x)$ is continuous there is a constant c_2 such that

$$|g(x)| < c_2 \quad \text{for } |x| \leqslant 1. \tag{7}$$

By analogy with (3) we write

$$G(x) = \int_0^x g(\xi)d\xi.$$

THEOREM 1. *Suppose that the above conditions hold and that there is a constant $b > 0$ such that*

$$F(x, t)\operatorname{sign}x > b > 0 \quad \text{for } |x| \geqslant 1, \tag{8}$$

$$g(x)\operatorname{sign}x > 0 \quad \text{for } |x| \geqslant 1, \tag{9}$$

$$|F(x, t)| + G(x) \to \infty \quad \text{uniformly in } t \text{ as } |x| \to \infty. \tag{10}$$

Then there are constants B_1, B_2 and a function $t_1(x_0, y_0)$, all depending on F and g, with the following property. If $x(t)$, $y(t)$ is the solution of (4) *satisfying*

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

for some given t_0, then $x(t)$ and $y(t)$ are defined for all $t \geqslant t_0$ and satisfy

$$|x(t)| \leqslant B_1, \quad |y(t)| \leqslant B_2 \quad \text{for } t \geqslant t_0 + t_1(x_0, y_0).$$

This may be compared with Graef's Theorem 3.5. Our proof gives explicit values for B_1 and B_2 in terms of F and g. We can thereby deduce the results of Cartwright and Littlewood [4] giving bounds for the solutions of

$$\ddot{x} + kf(x, k)\dot{x} + g(x, k) = kp(t, k) \quad (k \geqslant 1)$$

with constants independent of k under much more general conditions than theirs on f and g. An alternative proof of Theorem 1, giving different formulae for B_1 and B_2, implies their results for a small positive parameter $k = \varepsilon$. Moreover, the use of the system (4) makes it possible to show that all solutions of equations such as

$$\ddot{z} + \varepsilon(z^2 - 1)\dot{z} + z = E\cos\omega t, \quad |\omega^2 - 1| \geqslant c > 0,$$

where $0 < \varepsilon < 1$, are bounded by constants which depend on E and c but not on ε or ω.

A similar transformation can be applied to the equation

$$\ddot{x} + f(x, \dot{x}, t)\dot{x} + g(x) = 0, \tag{11}$$

which is (1) without a forcing term. For this we write

$$h(x, \dot{x}, t) = f(x, \dot{x}, t) - f(x), \quad F(x) = \int_0^x f(\xi)\,d\xi$$

for some continuous function $f(x)$ which is at our disposal; then (11) is equivalent to the pair of first order equations

$$\dot{x} = y - F(x), \quad \dot{y} = -\dot{x}h(x, \dot{x}, t) - g(x). \tag{12}$$

THEOREM 2. *Suppose that there is a continuous function $h_M(x, \dot{x})$ such that*

$$h_M(x, \dot{x}) \geqslant h(x, \dot{x}, t) \geqslant 0 \tag{13}$$

and there is a constant $b > 0$ such that

(14) $$F(x)\operatorname{sign} x > b > 0 \quad \text{for } |x| \geqslant 1,$$

(15) $$g(x)\operatorname{sign} x > 0 \quad \text{for } |x| \geqslant 1,$$

16) $$|F(x)| + G(x) \to \infty \quad \text{as } |x| \to \infty.$$

Then there are constants B_3, B_4 and a function $t_2(x_0, y_0)$, all depending on F and g (and t_2 depending also on h), with the following property. If $x(t)$, $y(t)$ is the solution of (12) *satisfying*

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

for some given t_0, then $x(t)$ and $y(t)$ are defined for all $t \geqslant t_0$ and satisfy

$$|x(t)| \leqslant B_3, \quad |y(t)| \leqslant B_4 \quad \text{for } t \geqslant t_0 + t_2(x_0, y_0).$$

Conditions (14) to (16) are the analogues of (8) to (10) in Theorem 1, so we may interpret this theorem as saying that for an equation without a forcing term, increasing the damping cannot destroy boundedness. Theorem 2 may be compared with the boundedness theorem of Levinson and Smith [9] for equations of the form

$$\ddot{x} + f(x, \dot{x})x + g(x) = 0.$$

Their boundedness conditions are closely tailored to their method of proof, so much so that their paper has not attracted the recognition it deserves. We believe that neither result can be deduced from the other, but that for the kind of equations which occur in practice our theorem will prove as applicable as theirs, and in some cases more applicable.

A serious weakness of Theorem 2 is that it does not allow for a forcing term in the underlying second order equation, which therefore has to be of type (11) rather than (1). The obvious way of treating the forcing term is to incorporate it in the first equation (12), which would be analogous to the treatment of the forcing term in the reduction of (2) to (4). The equations thus obtained, however, cannot be treated by the methods of the present paper; and in the light of the following general counterexample we believe that the transformation involved is the wrong one to choose, and that the proper condition to impose on the forcing term is the boundedness of $p(t)$ rather than of $P(t)$.

THEOREM 3. *Given any continuous functions $F(x)$ and $g(x)$ there exist continuous functions $P(t)$ and $h(x, \dot{x})$ satisfying*

$$|P(t)| \leqslant 1, \quad h(x, \dot{x}) \geqslant 0$$

such that every solution of

(17) $$\dot{x} = y - F(x) + P(t), \quad \dot{y} = -\dot{x}h(x, \dot{x}) - g(x)$$

satisfies $x(t) \to \infty$ as $t \to \infty$.

If we assume that $F(x)$ and $P(t)$ have continuous first derivatives $f(x)$ and $p(t)$, then (17) reduces to the second order equation

$$\ddot{x} + \{f(x) + h(x, \dot{x})\}\dot{x} + g(x) = p(t);$$

so Theorem 3 effectively says that for a suitable forcing term we can make all solutions of (2) diverge to infinity merely by increasing the damping (in a way that depends on $\dot{x}$ as well as x). Because the details are tedious, we postpone the proof of Theorem 3 and some related results to a further paper; but we hope that the following physical analogy will make it plausible. We make $h(x, \dot{x})$ infinite for $\dot{x} < 0$; this means that movement to the left is prohibited. We take $P(t) = \frac{1}{2}\operatorname{sign}(\sin \pi t)$, so that $p(t)$ corresponds to alternate unit impulses to the right and to the left. Assume for convenience of description that $f(x)$ and $g(x)$ are positive for all x, which is the condition least favourable to rightward motion. An impulse to the left merely ensures that $x(t)$ remains constant until the next impulse to the right. A unit impulse to the right then sets $\dot{x} = 1$, and in the unit time before the next leftward impulse $x(t)$ increases by a non-zero amount which depends continuously on its value at the time of the rightward impulse. It is now easy to see that $x(t)$ tends to infinity in a series of jerks. The rigorous proof of Theorem 3 consists of showing that we can approximate well enough to this situation within the conditions of the theorem.

However, there is an alternative treatment of the forcing term by incorporating it into the second equation (12), so that we replace (1) by the pair of first order equations

$$\dot{x} = y - F(x), \qquad \dot{y} = -\dot{x}h(x, \dot{x}, t) - g(x) + p(t) \tag{18}$$

with the same conventions as in (12). We can apply our methods, with some modification, to this system; but it costs us nothing to consider the more general system

$$\dot{x} = y - F(x), \qquad \dot{y} = \dot{x}h(x, \dot{x}, t) - g(x, t) \tag{19}$$

which corresponds to the second order equation

$$\ddot{x} + \{f(x) + h(x, \dot{x}, t)\}\dot{x} + g(x, t) = 0.$$

THEOREM 4. *Suppose that $F(x)$, $g(x, t)$ and $h(x, \dot{x}, t)$ are continuous functions of their arguments and that there are a continuous function $h_M(x, \dot{x})$ and constants b, c_2 such that*

$$h_M(x, \dot{x}) \geqslant h(x, \dot{x}, t) \geqslant 0,$$

$$|g(x, t)| < c_2 \qquad \textit{for } |x| \leqslant 1, \tag{20}$$

$$F(x)\operatorname{sign}x > b > 0 \qquad \textit{for } |x| \geqslant 1. \tag{21}$$

Suppose also that there are continuous functions $g_1(x), g_2(x)$ *such that*

$$g_1(x) \geqslant g(x, t)\,\mathrm{sign}\, x \geqslant g_2(x),$$

$$g_2(x) > 0 \quad \textit{for } |x| \geqslant 1, \tag{22}$$

$$|F(x)| + \int_0^x g_2(\xi)\,\mathrm{sign}\,\xi\, d\xi \to \infty \quad \textit{as } |x| \to \infty; \tag{23}$$

and suppose, moreover, that

$$\big(F(x) - b\big)^2 \geqslant 2 \int_1^x \big(g_1(\xi) - g_2(\xi)\big)\, d\xi \quad \textit{for } x \geqslant 1, \tag{24}$$

$$\big(F(x) + b\big)^2 \geqslant 2 \int_x^{-1} \big(g_1(\xi) - g_2(\xi)\big)\, d\xi \quad \textit{for } x \leqslant -1. \tag{25}$$

Then there are constants B_5, B_6 *and a function* $t_3(x_0, y_0)$, *all depending on* F *and* g (*and* t_3 *depending also on* h), *with the following property. If* $x(t)$, $y(t)$ *is the solution of* (19) *satisfying*

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

for some given t_0, *then* $x(t)$ *and* $y(t)$ *are defined for all* $t \geqslant t_0$ *and satisfy*

$$|x(t)| \leqslant B_5, \quad |y(t)| \leqslant B_6 \quad \textit{for } t \geqslant t_0 + t_3(x_0, y_0).$$

As it stands this theorem does not yield a satisfactory result for equations containing either a large or a small parameter; for (22) will fail for the natural type of equation with a large parameter, and (24) and (25) will fail for the natural type of equation with a small parameter. In the large parameter case this seems inevitable; for examples similar to that of Theorem 3 show that there can be no analogue of the results of Cartwright and Littlewood [4], or of our Theorem 7. We believe that a satisfactory small parameter result could be obtained by modifying the proof of Theorem 4 on lines similar to those of sections 10 to 14; but we have not examined this question in detail.

In this paper we have confined ourselves to the case when the damping term $\dot{x}f(x, \dot{x}, t)$ acts against motion everywhere outside the critical strip $|x| \leqslant 1$. Most previous authors — such as Langenhop, Levinson, Opial, and Reuter — who have considered such equations have considered more general critical regions, though with conditions more restrictive in other ways. Our methods can be applied to equations with more general critical regions, though the details become more complicated. We intend to consider this case in a further paper, and therefore postpone a comparison of our results with previous ones. However, we feel bound to mention one historically important equation (1) which does not appear to fall within the hypotheses of Theorem 2 or 4, but which can be brought within them by a suitable change of variables. This is Rayleigh's equation

$$\ddot{z}+k(\tfrac{1}{3}\dot{z}^3-\dot{z})+z = 0, \tag{26}$$

which can be reduced to the form (4) by writing $x = -\dot{z}$, $y = z$ so that

$$\dot{x} = y-k(\tfrac{1}{3}x^3-x), \quad \dot{y} = -x.$$

However, no such device would be available if the restoring term z in (26) were replaced by $g(z)$, though Reuter has shown that the solutions are bounded under quite moderate conditions on $g(z)$.

Various authors have considered equation (2) under so-called 'one-sided' conditions. Graef has pointed out that results of this type can be trivially deduced from theorems with two-sided conditions by increasing y in (4) by a constant. It is also possible to obtain theorems analogous to our Theorems 2 and 4 but with 'one-sided' conditions.

We are grateful to Dr N. G. Lloyd for a number of valuable suggestions.

2. Standardization. Other types of condition in Theorem 1, such as

$$\liminf_{|x|\to\infty}\{\min_t F(x, t)\operatorname{sign}x\} > 0 \tag{27}$$

can be reduced to the form (8) for some b. For (27) implies that there exist $x_0 > 0$ and $b > 0$ such that $F(x, t)\operatorname{sign}x \geqslant b$ for $|x| \geqslant x_0$. Putting $x = x_0 x'$ we obtain a system in x', y for which $F(x', t)$ satisfies (8). Similarly if

$$g(x)\operatorname{sign}x > 0 \quad \text{for all large } |x|, \tag{28}$$

there exists an $x_0 > 0$ such that $g(x)\operatorname{sign}x > 0$ for $|x| \geqslant x_0$; and again (28) can be reduced to (9) by writing $x = x_0 x'$. For the conditions of Theorem 1 as a set, the larger of the two values of x_0 must of course be taken. Condition (5) is imposed solely in order to ensure that (6) remains valid even after this normalization.

Similar remarks apply to the other theorems.

3. The nested domains for Theorem 1. Theorem 1 is an obvious consequence of the following theorem, which is also essential for the application of Brouwer's fixed point theorem in order to show the existence of a periodic solution of (4) when $F(x, t)$ is periodic.

THEOREM 5. *Suppose that the conditions of Theorem 1 hold, and let $x(t)$, $y(t)$ be as in Theorem 1. Then there is a sequence of domains $D_0 \subset D_1 \subset D_2 \subset \ldots$ such that each D_n is bounded by a Jordan curve J_n and the union of the D_n is the entire (x, y)-plane. If the point (x_0, y_0) is in $D_{n+1}-D_n$, then $\big(x(t), y(t)\big)$ is in D_{n+1} for all $t \geqslant t_0$; as t increases, $\big(x(t), y(t)\big)$ crosses J_n*

into D_n and thereafter remains in D_n. Moreover, there is a $t_1(n)$ depending on n but not on x_0, y_0 such that $(x(t), y(t))$ is in D_n for $t \geqslant t_0 + t_1(n)$.

Corollary 1. *Theorem 1 holds.*

Corollary 2. *Under the conditions of Theorem 1, each solution of (4) lies in D_0 from a certain time onwards.*

Corollary 3. *Suppose that the conditions of Theorem 1 hold and that $F(x, t)$ is periodic in t with period $2\pi/\omega$ independent of x. Then there is a solution of (4) which is periodic with period $2\pi/\omega$, and it lies entirely in D_0.*

It is easy to see that the Corollaries follow from the Theorem, together with Brouwer's fixed point theorem in the case of Corollary 3. In the next three sections we give a proof of Theorem 5. In Sections 11 and 12 we give an alternative sequence of domains with associated Jordan curves, for which the Theorem again holds.

4. Preliminary lemmas. Write

$$F_m(x) = \inf_t [F(x, t)\operatorname{sign} x], \tag{29}$$

$$\Gamma = \{(x, y):\ y = F_m(x)\operatorname{sign} x\}. \tag{30}$$

If the conditions of Theorem 1 hold, $F_m(x)$ is at any rate upper semicontinuous and

$$F_m(x) \geqslant b > 0 \quad \text{for } |x| \geqslant 1;$$

in most cases of interest $F_m(x)$ will be continuous, as in Fig 1. Since (9) implies that $G(x)$ is continuous and increasing with x in $x \geqslant 1$, the point set

$$\mathscr{A}(C) = \{(x, y);\ (y-b)^2 = C - 2G(x),\ x \geqslant 1,\ y \geqslant b\} \tag{31}$$

is a finite or infinite Jordan arc provided that $C > 2G(1)$, a condition which will always be satisfied in what follows. Through any point $(1, y)$ with $Y > b$ there is just one such arc. We wish to prove that, with some abuse of language, $\mathscr{A}(C)$ crosses Γ if (10) holds and C is large enough.

Lemma 1. *Suppose that the conditions of Theorem 1 hold and $G(x) \to \infty$ as $x \to \infty$. Then for every $Y > \max(F_m(1), b)$ there is an $x(Y) > 1$ with the following property: if $\mathscr{A}(C)$ is the arc (31) through $(1, Y)$ and (x, y) is on $\mathscr{A}(C)$, then $x(Y)$ is the least value of x such that $y \leqslant F_m(x)$. Moreover, for any fixed $X > 1$ there exists $Y_0(X)$ such that $x(Y) > X$ for $Y > Y_0(X)$.*

Since $G(x)$ increases with x on $\mathscr{A}(C)$, y decreases; and $2G(x) \leqslant C$ for all points of $\mathscr{A}(C)$. Since $G(x) \to \infty$ as $x \to \infty$, this implies that $\mathscr{A}(C)$ ends at some point $(x_0(C), b)$ and $1 \leqslant x \leqslant x_0(C)$ on $\mathscr{A}(C)$. Now on $\mathscr{A}(C)$ the function $F_m(x) - y$ is negative at $x = 1$, non-negative at $x = x_0(C)$, and upper semi-continuous; so there is a least x, say $x(Y)$, at which it is non-negative. Moreover, if Y is large so is C; and since

$$\{F_m(x(Y))-b\}^2+2G(x(Y))\geqslant C,$$

at least one of $F_m(x(Y))$ and $G(x(Y))$ is large. Since $G(x)$ is continuous and $F_m(x)$ is bounded above by the continuous function $F(x,0)$ in $x\geqslant 1$, this shows that $x(Y)$ is large which gives the result required.

LEMMA 2. *Suppose that the conditions of Theorem* 1 *hold and* $F(x,t)\to\infty$ *uniformly in* t *as* $x\to\infty$. *Then the result of Lemma* 1 *holds.*

Since $F(x,t)\to\infty$ uniformly in t, $F_m(x)\to\infty$ as $x\to\infty$. Hence there is an $x_1(C)$ such that $F_m(x_1(C))\geqslant Y$; and so $F_m(x)-y$ is non-negative at $x=x_1(C)$. The rest of the proof is the same as for Lemma 1.

Since $G(x)$ is increasing in $x\geqslant 1$, (10) implies that the extra hypothesis in at least one of Lemmas 1 and 2 is valid; so in either case the conditions of Theorem 1 imply the conclusions of Lemma 1.

5. The construction of J_n for Theorem 5. Let b, c_1, c_2 be the constants in (6) to (8), and let

$$Y_n = c_1+2c_2b^{-1}+(n+1)b, \qquad X_n = x(Y_n),$$

where $x(Y_n)$ is as in Lemmas 1 and 2; this is legitimate since $Y_n>c_1 \geqslant F_m(1)$ by (6). Let

$$P_n = (-1,\, Y_n-b), \qquad Q_n = (1,\, Y_n);$$

let $\mathscr{A}(C_n)$ be the arc (31) through Q_n and let

$$R_n = (X_n, Z_n)$$

be that point of $\mathscr{A}(C_n)$ at which $x=X_n$. Let $\mathscr{A}_n$ be the subarc of $\mathscr{A}(C_n)$ from Q_n to R_n and let $\bar{\mathscr{A}}_n$ be its reflection in $y=b$. Then $\bar{\mathscr{A}}_n$ joins

$$S_n = (X_n,\, 2b-Z_n) \quad \text{to } T_n = (1,\, 2b-Y_n).$$

Let $P'_n=(1,\, b-Y_n)$ and let Q'_n, R'_n, S'_n, T'_n be obtained from P'_n in the same way (with appropriate reversals of sign) as Q_n, R_n, S_n, T_n were obtained from P_n. The curve J_n consists of the straight line joining P_n to Q_n, the arc $\mathscr{A}_n$ joining Q_n to R_n, the vertical line $x=X_n$ from R_n to S_n, the arc $\bar{\mathscr{A}}_n$ from S_n to T_n, the vertical line $x=1$ from T_n to P'_n and the corresponding arc $P'_nQ'_nR'_nS'_nT'_nP_n$; it is illustrated in Fig 1.

It is easy to verify that J_n is a simple continuous closed curve, and that if D_n is its interior then $D_0\subset D_1\subset D_2\subset\ldots$ and the union of the D_n is the entire (x,y)-plane. Moreover, $T'_n=P_{n-1}$ and $T_n=P'_{n-1}$, so any trajectory in D_{n+1} which crosses either of the lines $x=\pm 1$ in the sense of decreasing $|x|$ thereby enters D_n.

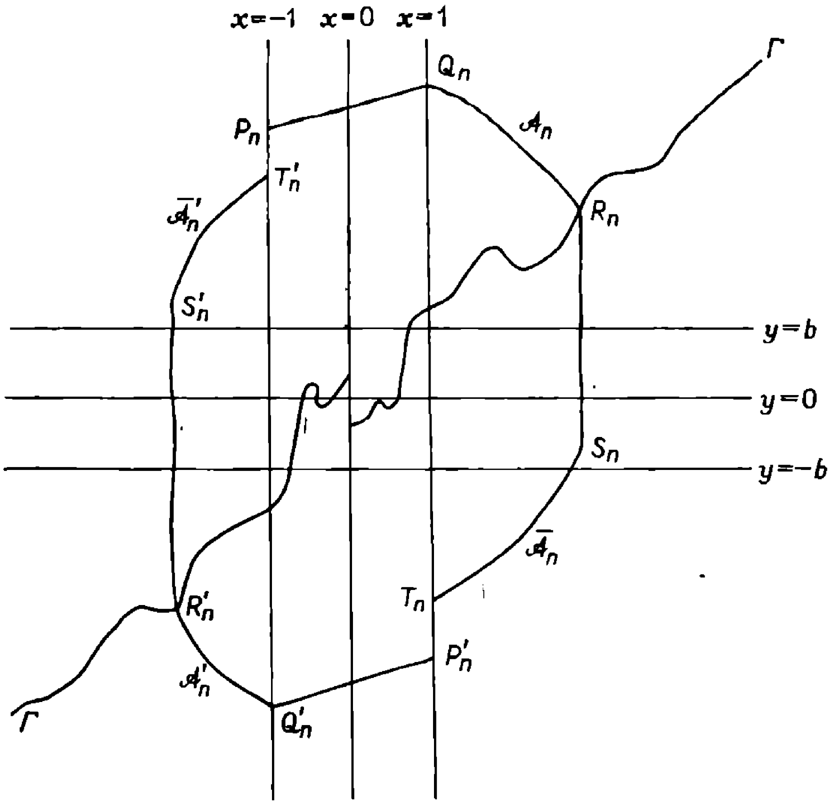

Fig. 1.

6. Proof of Theorem 5. To prove Theorem 5 it only remains to show that no trajectory can cross J_n outwards, that so long as a trajectory remains outside D_0 it spirals clockwise round D_0, and that a trajectory can only remain in $D_{n+1}-D_n$ for a bounded time.

On or above P_nQ_n in $|x|\leqslant 1$ we have $\dot{x} = y-F(x,t)\geqslant 2c_2b^{-1}>0$ and so a trajectory of (4) in this region satisfies $\dot{x}>0$ and

$$\left|\frac{dy}{dx}\right| = \left|\frac{\dot{y}}{\dot{x}}\right| = \left|\frac{g(x)}{y-F(x,t)}\right| < \frac{b}{2}$$

while the slope of P_nQ_n is $\frac{1}{2}b$. Hence a trajectory can only cross P_nQ_n downwards into D_n, and it only remains for a time at most bc_2^{-1} in $|x|\leqslant 1$ above P_nQ_n.

On $\mathscr{A}_n$ and $\bar{\mathscr{A}}_n$,

$$\frac{d}{dt}\{(y-b)^2+2G(x)\} = 2\dot{y}(y-b)+2\dot{x}g(x) = 2\{b-F(x,t)\}g(x)<0$$

so that a trajectory can only cross these arcs into D_n. On R_nS_n and T_nP_n' a trajectory lies below Γ, and hence $\dot{x}<0$ and the trajectory can only cross these arcs into D_n. Moreover, in $x\geqslant 1$ we have $\dot{y}<0$, so that all trajectories are moving downwards. Since a trajectory cannot cross $Q_{n+1}Q_n$

to the left, and by crossing $T_n T_{n+1}$ to the left it enters D_n since $P'_n = T_{n+1}$, we need to show that a trajectory can only remain in the part of $D_{n+1} - D_n$ to the right of $x = 1$ for a bounded time. Since $g(x)$ is continuous and strictly positive in $x \geqslant 1$, there is a $g_n > 0$ such that

$$g(x) \geqslant g_n > 0 \quad \text{for } 1 \leqslant x \leqslant X_n.$$

Thus $\dot{y} = -g(x) \leqslant -g_{n+1}$ in $(D_{n+1} - D_n) \cap (x \geqslant 1)$, and so no trajectory can remain in this region for a time greater than $2(Y_{n+1} - b) g_{n+1}^{-1}$.

These results, and the corresponding ones for the symmetrically placed arcs and regions, complete the proof of Theorem 5.

7. The constants B_1, B_2 in Theorem 1. It is easy to see that the extreme values of y on J_0 occur at $Q_0 = (1, c_1 + 2c_2 b^{-1} + b)$ and at Q'_0 which is the image of Q_0 in the origin; thus we can take

$$B_2 = c_1 + 2c_2 b^{-1} + b. \tag{32}$$

The maximum of x on J_0 occurs on $R_0 S_0$, where $x = X_0$; and X_0 is the least value of x greater than 1 such that

$$\{F_m(x) - b\}^2 + 2G(x) \geqslant (c_1 + 2c_2 b^{-1})^2 + 2G(1). \tag{33}$$

A similar result holds for the minimum of x on J_0; it is the greatest value of x less than -1 such that

$$\{F_m(x) - b\}^2 + 2G(x) \geqslant (c_1 + 2c_2 b^{-1})^2 + 2G(-1).$$

It follows from (9) that $G(x) \geqslant G(1)$ in $x \geqslant 1$ and $G(x) \geqslant G(-1)$ in $x \leqslant -1$. Hence in particular, if there is an $x > 1$ such that $F_m(x) \geqslant B_2$ and $F_m(-x) \geqslant B_2$, we can certainly take that x as B_1; this remark will be used in the proof of Theorem 7.

8. Second order equations. From Theorem 1 we can immediately deduce Graef's boundedness theorem for solutions of the second order equation (2). We state it explicitly, largely in order to define the constants B'_1 and B'_2.

THEOREM 6. *Suppose that $f(x)$, $g(x)$ and $p(t)$ are continuous and that there are positive constants b_1, b_2 such that*

$$\begin{aligned} &|P(t)| \leqslant b_1, && \\ &F(x)\operatorname{sign}x > b_1 + b_2 && \textit{for } |x| \geqslant 1, \\ &g(x)\operatorname{sign}x > 0 && \textit{for } |x| \geqslant 1, \\ &|F(x)| + G(x) \to \infty && \textit{as } |x| \to \infty. \end{aligned} \tag{34}$$

Then there are constants B_1', B_2' such that each solution of (2) *satisfies*

$$|x(t)| \leqslant B_1', \qquad |\dot{x}(t)| \leqslant B_2'$$

for all large enough positive t.

To prove this it is enough to verify that $F(x, t) = F(x) - P(t)$ and $g(x)$ satisfy the conditions of Theorem 1, which is obvious. To obtain formulae for B_1' and B_2' we define c_2 by (7) as before and b_3 by

$$|F(x)| < b_3 \quad \text{for } |x| \leqslant 1.$$

Thus we can take $b = b_2$ and $c_1 = b_3 + b_1$; and we have

$$F_m(x) \geqslant F(x)\operatorname{sign} x - b_1.$$

In the open domain D_0 we have $F_m(x) \leqslant B_2$ because $\mathscr{A}_0$ lies above Γ except at its right-hand end point; thus $|F(x, t)| \leqslant B_2 + 2b_1$ and we can take

$$B_2' = 2B_2 + 2b_1 = 4c_2 b_2^{-1} + 4b_1 + 2b_2 + 2b_3. \tag{35}$$

Also we can take $B_1' = B_1$.

9. Large parameters. Consider the equation

$$\ddot{x} + kf(x, k)\dot{x} + g(x, k) = kp(t, k), \tag{36}$$

where $k \geqslant 1$ is a parameter which should be thought of as large; for each value of k we assume that $f(x, k), g(x, k), p(t, k)$ are continuous functions of x or t respectively. Let

$$F(x, k) = \int_0^x f(\xi, k)d\xi, \qquad G(x, k) = \int_0^x g(\xi, k)d\xi, \qquad P(t, k) = \int_0^t p(\tau, k)d\tau$$

and suppose that there are constants c_{02}, b_{03} independent of k such that

$$|g(x, k)| < c_{02} \quad \text{for } |x| \leqslant 1,$$
$$|F(x, k)| < b_{03} \quad \text{for } |x| \leqslant 1.$$

THEOREM 7. *Suppose, in addition to the conditions just stated, that there are positive constants b_{01}, b_{02} independent of k such that for all $k \geqslant 1$,*

$$|P(t, k)| \leqslant b_{01},$$
$$F(x, k)\operatorname{sign} x > b_{01} + b_{02} \quad \textit{for } |x| \geqslant 1,$$
$$g(x, k)\operatorname{sign} x > 0 \quad \textit{for } |x| \geqslant 1,$$
$$|F(x, k)| \to \infty \quad \textit{uniformly in } k \textit{ as } |x| \to \infty. \tag{37}$$

Then there are constants B'_{01}, B'_{02} independent of k such that each solution of (36) *satisfies*

$$(38) \qquad |x(t)| \leqslant B'_{01}, \qquad |\dot{x}(t)| \leqslant kB'_{02}$$

for all large enough positive t.

Note that (37) here is not the natural analogue of (34) in Theorem 6; but Theorem 7 would cease to be true if we replaced (37) by the weaker condition

$$|F(x, k)| + G(x, k) \to \infty \quad \text{uniformly in } k \text{ as } |x| \to \infty.$$

The hypotheses of Theorem 7 imply those of Theorem 6 with

$$b_1 = kb_{01}, \quad b_2 = kb_{02}, \quad b_3 = kb_{03}, \quad c_2 = c_{02};$$

so we can use the formulae for B'_1 and B'_2 in Section 8. In particular (35) gives

$$B'_2 = 4k^{-1}c_{02}b_{02}^{-1} + k(4b_{01} + 2b_{02} + 2b_{03}),$$

and since $k \geqslant 1$ this gives the second inequality (38) with

$$B'_{02} = 4c_{02}b_{02}^{-1} + 4b_{01} + 2b_{02} + 2b_{03}.$$

To prove the first inequality (38) for some B'_{01} we note that in this case

$$F_m(x) \geqslant kF(x, k)\operatorname{sign} x - kb_{01};$$

and by (32) and the last sentence of Section 7 we have only to choose B'_{01} so that both $F_m(B'_{01})$ and $F_m(-B'_{01})$ exceed

$$2k^{-1}c_{02}b_{02}^{-1} + k(b_{01} + b_{02} + b_{03}).$$

For this it is enough to choose B'_{01} independent of k so that

$$|F(\pm B'_{01}, k)| \geqslant 2c_{02}b_{02}^{-1} + 2b_{01} + b_{02} + b_{03}$$

for both choices of sign; and this is possible by (37).

A result of this type is a necessary preliminary for work on van der Pol's equation

$$(39) \qquad \ddot{x} + k(x^2 - 1)\dot{x} + x = kb\cos\omega t$$

with large parameter k and forcing term. Cartwright and Littlewood [10] showed that for certain values of b between 0 and $\frac{2}{3}$ there are two stable periodic solutions with long periods $2(2n+1)\pi/\omega$ and $2(2n-1)\pi/\omega$ besides a large number of unstable solutions with various periods, giving rise to a topological mapping of great complexity. Levinson [8] showed that an equation of somewhat similar type but piecewise linear has similar properties, and the work of Cartwright and Littlewood holds for more general equations. Lloyd [11] has shown that if $b > \frac{2}{3}$, then for large enough k all solutions of (39) tend to a single solution with period $2\pi/\omega$.

10. The case of small $F(x, t)$. If $b = \varepsilon b_1$ in (8), where $\varepsilon > 0$ is a small parameter appearing in the differential equations (4) and b_1 is independent of ε, then the value of B_2 given by (32) is large; indeed $B_2 > 2c_2 b_1^{-1}\varepsilon^{-1}$. The same then happens for B_1. However, if (10) is replaced by the stronger condition that $G(x) \to \infty$ as $|x| \to \infty$, a modification of the proof of Theorem 5 gives constants independent of ε for $0 < \varepsilon < 1$. As in Theorem 7 the method allows us to consider a system in which the functions $F(x, t)$ and $g(x)$ depend on ε in a very general way, provided that certain inequalities are satisfied uniformly for $0 < \varepsilon < 1$.

Instead of (4) we consider the equations

$$\dot{x} = y - \varepsilon F(x, t, \varepsilon), \qquad \dot{y} = -g(x, \varepsilon), \tag{40}$$

where $F(x, t, \varepsilon)$ is continuous in x and t and $g(x, \varepsilon)$ is continuous in x for each ε. Here ε is a parameter which satisfies $0 < \varepsilon < 1$ and is to be thought of as small. As usual we write

$$G(x, \varepsilon) = \int_0^x g(\xi, \varepsilon)\, d\xi;$$

and we suppose that there are constants c_{11}, c_{12} independent of ε such that

$$\begin{aligned} |F(x, t, \varepsilon)| &< c_{11} \quad \text{for } |x| \leqslant 1, \\ |g(x, \varepsilon)| &< c_{12} \quad \text{for } |x| \leqslant 1, \end{aligned} \tag{41}$$

THEOREM 8. *Suppose, in addition to the conditions just stated, that there is a constant b_2 independent of ε such that*

$$\begin{aligned} &F(x, t, \varepsilon)\,\mathrm{sign}\,x > b_2 > 0 \qquad \textit{for } |x| \geqslant 1, \\ &g(x, \varepsilon)\,\mathrm{sign}\,x > 0 \qquad \textit{for } |x| \geqslant 1, \\ &G(x, \varepsilon) \to \infty \quad \textit{uniformly in } \varepsilon \textit{ as } |x| \to \infty. \end{aligned} \tag{42}$$

Then there are constants B_{11}, B_{12} independent of ε such that each solution $x(t), y(t)$ of (40) satisfies

$$|x(t)| \leqslant B_{11}, \qquad |y(t)| \leqslant B_{12} \tag{43}$$

for all large enough positive t.

We shall first give a modified proof of Theorem 5, in which more care is taken to balance out the effect of the restoring force $g(x, \varepsilon)$ in the critical strip $|x| \leqslant 1$. For this purpose we construct a new sequence of domains D_n^* bounded by closed Jordan curves J_n^*. When ε is small, this will give better constants B_1, B_2 in Theorem 1 than those obtained in Section 7, and Theorem 8 will follow immediately.

11. The construction of J_n^*. Write

$$\gamma = (2c_1c_2b^{-1})^{\frac{1}{2}}.$$

We replace the straight lines P_nQ_n and $P_n'Q_n'$ of J_n by arcs $\mathscr{B}_n^*$, $\mathscr{B}_n^{*\prime}$ of

$$y^2+2G(x)-2xc_1c_2\gamma^{-1} = (c_1+\gamma)^2+2c_2(1+c_1\gamma^{-1})+2n\gamma b, \tag{44}$$

$$y^2+2G(x)+2xc_1c_2\gamma^{-1} = (c_1+\gamma)^2+2c_2(1+c_1\gamma^{-1})+2n\gamma b \tag{45}$$

respectively, with y positive on $\mathscr{B}_n^*$ and negative on $\mathscr{B}_n^{*\prime}$. The arc $\mathscr{B}_n^*$ joins P_n^* on $x = -1$ to Q_n^* on $x = 1$, and $\mathscr{B}_n^{*\prime}$ joins $P_n^{*\prime}$ on $x=1$ to $Q_n^{*\prime}$ on $x = -1$. By (6) and (7),

$$y \geqslant c_1+\gamma \quad \text{on } \mathscr{B}_n^*, \qquad y \leqslant -c_1-\gamma \quad \text{on } \mathscr{B}_n^{*\prime}. \tag{46}$$

The rest of the construction proceeds as for J_n. We write $Q_n^* = (1, Y_n^*)$ and let $\mathscr{A}(C_n^*)$ be that arc (31) which goes through Q_n^*. Let $R_n^* = (X_n^*, Z_n^*)$ be the point of $\mathscr{A}(C_n^*)$ at which $x = X_n^* = x(Y_n^*)$, where $x(Y)$ is the function in Lemmas 1 and 2, and let $\mathscr{A}_n^*$ be the sub-arc of $\mathscr{A}(C_n^*)$ from Q_n^* to R_n^* and $\bar{\mathscr{A}}_n^*$ be its reflection in $y = b$. Then $\bar{\mathscr{A}}_n^{*\prime}$ joins

$$S_n^* = (X_n^*, 2b-Z_n^*) \quad \text{ to } T_n^* = (1, 2b-Y_n^*).$$

We obtain $R_n^{*\prime}$, $S_n^{*\prime}$ and $T_n^{*\prime}$ analogously from $Q_n^{*\prime}$. The curve J_n^* consists of the arc $\mathscr{B}_n^*$ from P_n^* to Q_n^*, the arc $\mathscr{A}_n^*$ from Q_n^* to R_n^*, the vertical line $x = X_n^*$ from R_n^* to S_n^*, the arc $\bar{\mathscr{A}}_n^*$ from S_n^* to T_n^*, the vertical line $x=1$ from T_n^* to $P_n^{*\prime}$, and the corresponding arc $P_n^{*\prime}Q_n^{*\prime}R_n^{*\prime}S_n^{*\prime}T_n^{*\prime}P_n^*$; it is sufficiently similar to the curve J_n of Fig. 1 for a further diagram to be unnecessary. The only point of ambiguity in the diagram is resolved by the following lemma.

LEMMA 3. *T_n^* is above $P_{n-1}^{*\prime}$ on $x = 1$, and $T_n^{*\prime}$ is below P_{n-1}^* on $x = -1$, for $n \geqslant 1$.*

It is enough to prove the first statement. Let η_{n-1} be the value of y at $P_{n-1}^{*\prime}$; then by comparing (44) and (45) at $x=1$ we have

$$Y_n^{*2}-\eta_{n-1}^2 = 4c_1c_2\gamma^{-1}+2\gamma b = 4\gamma b.$$

Again by (46) we have

$$Y_n^*-\eta_{n-1} \geqslant 2(c_1+\gamma) > 2\gamma.$$

Division now gives

$$Y_n^*+\eta_{n-1} < 2b$$

so that in an obvious notation

$$y(T_n^*)-y(P_{n-1}^{*\prime}) = 2b-Y_n^*-\eta_{n-1} > 0.$$

12. The domains D_n^*. We have to prove that Theorem 5 remains true if the domains D_n and curves J_n are replaced by D_n^* and J_n^* respectively. It is easy to check that J_n^* is a simple continuous closed curve, that the D_n^* form an increasing sequence of domains and that their union is the entire (x, y)-plane.

The first step is to prove that if a trajectory crosses J_n^*, it crosses it inwards and thereby enters D_n^*. At any point of $\mathscr{B}_n^*$ a trajectory of (4) satisfies

$$\frac{d}{dt}\{y^2+2G(x)-2xc_1c_2\gamma^{-1}\} = -2F(x,t)g(x)-2c_1c_2\gamma^{-1}\big(y-F(x,t)\big) < 0$$

by (6), (7) and (46); so a trajectory can only cross $\mathscr{B}_n^*$ downwards. A similar argument shows that a trajectory can only cross $\mathscr{B}_n^{*\prime}$ upwards. For the remaining arcs of J_n^* we need only repeat the arguments used in Section 6 for the corresponding arcs of J_n.

It remains to show that a trajectory can only remain in $D_{n+1}^*-D_n^*$ for a bounded time, and here again we imitate the argument of Section 6. Above $\mathscr{B}_n^*$ in $|x|\leqslant 1$ we have

$$\dot{x} = y-F(x,t) \geqslant \gamma$$

by (6) and (46), so a trajectory can remain for a time at most $2\gamma^{-1}$ in $|x|\leqslant 1$ above $\mathscr{B}_n^*$. It leaves this region either by crossing $\mathscr{B}_n^*$ into D_n^* or by crossing the line $x = 1$; and as in Section 6, the trajectory can only stay in $x\geqslant 1$ for a bounded time. It next crosses $x = 1$ to the left, and this crossing must be below Γ and above T_{n+1}^*. By Lemma 3, the trajectory enters D_n^* not later than this crossing. A similar argument works for the other half of $D_{n+1}^*-D_n^*$.

13. Proof of Theorem 8. It follows from the results of the last section that each trajectory of (4) enters and thereafter remains in D_0^*. We now apply this to equations (40) of Theorem 8. The maximum of y on J_0^* is attained at some point of $\mathscr{B}_0^*$, and since every point of $\mathscr{B}_0^*$ is in $|x|\leqslant 1$ we have

$$y^2 \leqslant (c_1+\gamma)^2+4c_2(1+c_1\gamma^{-1}) \tag{47}$$

on it, by (7) and (44). In our case

$$c_1 = \varepsilon c_{11}, \quad c_2 = c_{12}, \quad \gamma = (2c_{11}c_{12}b_2^{-1})^{1/2}$$

and combining (47) with the corresponding result on $\mathscr{B}_0^{*\prime}$ we see that we can take

$$B_{12}^2 = (c_{11}+\gamma)^2+4c_{12}(1+c_{11}\gamma^{-1}) \tag{48}$$

which is independent of ε.

For the first inequality (43) we note that the maximum of x on J_0^* is $x = X_0^*$, which is the least value of x greater than 1 such that

$$\{F_m(x)-b\}^2+2G(x) \geqslant (Y_0^*-b)^2+2G(1); \tag{49}$$

here $G(1) \leqslant c_2$ and Y_0^* is given by

$$Y_0^{*2} = (c_1+\gamma)^2+2c_2(1+2c_1\gamma^{-1})-2G(1).$$

Thus in our case the right-hand side of (49) is bounded above by a constant independent of ε. Taking account also of the similar argument for the minimum of x on J_0^*, we see that to ensure the first inequality (43) it is sufficient to choose B_{11} so that both $G(B_{11})$ and $G(-B_{11})$ exceed certain constants which are independent of ε. That this can be done, with B_{11} independent of ε, follows from (42).

14. Second order equations with small parameter. Consider the equation

$$\ddot{x}+\varepsilon f(x, \varepsilon)\dot{x}+g(x, \varepsilon) = \varepsilon p(t, \varepsilon), \tag{50}$$

where $0<\varepsilon<1$ and $f(x, \varepsilon)$, $g(x, \varepsilon)$ and $p(t, \varepsilon)$ are continuous functions of x or t respectively for each ε. Let

$$F(x, \varepsilon) = \int_0^x f(\xi, \varepsilon)d\xi, \quad G(x, \varepsilon) = \int_0^x g(\xi, \varepsilon)d\xi, \quad P(t, \varepsilon) = \int_0^t p(\tau, \varepsilon)d\tau;$$

then for $F(x, t, \varepsilon) = F(x, \varepsilon)-P(t, \varepsilon)$ we can apply Theorem 8 to obtain bounds for the solutions of (50). We assume that there are constants c_{12}, b_{13} independent of ε such that

$$|g(x, \varepsilon)| < c_{12} \quad \text{for } |x| \leqslant 1, \tag{51}$$

$$|F(x, \varepsilon)| < b_{13} \quad \text{for } |x| \leqslant 1. \tag{52}$$

It should be noted that in some important examples of (50) the function $f(x, \varepsilon)$ and $g(x, \varepsilon)$ are independent of ε, and then (51) and (52) are satisfied in virtue of continuity. However, in the case of Duffing's equation and problems about hard and soft springs $g(x, \varepsilon) = \omega^2(x+\varepsilon a x^3)$ occurs, so that it is worth preserving a generality which involves no additional arguments.

THEOREM 9. *Suppose that, in addition to the conditions just stated, there are positive constants b_{11}, b_{12} independent of ε such that for $0<\varepsilon<1$,*

$$\begin{aligned} &|P(t, \varepsilon)| \leqslant b_{11}, && \\ &F(x, \varepsilon)\operatorname{sign} x > b_{11}+b_{12} && \textit{for } |x| \geqslant 1, \\ &g(x, \varepsilon)\operatorname{sign} x > 0 && \textit{for } |x| \geqslant 1, \\ &G(x, \varepsilon) \to \infty && \textit{uniformly in } \varepsilon \textit{ as } |x| \to \infty. \end{aligned} \tag{53}$$

Then there are constants B'_{11}, B'_{12} independent of ε such that each solution of (50) *satisfies*

(54) $$|x(t)| \leqslant B'_{11}, \qquad |\dot{x}(t)| \leqslant B'_{12}$$

for all large enough positive t.•

The hypotheses of Theorem 8 are all satisfied, so the first inequality (54) with $B'_{11} = B_{11}$ follows from the first inequality (43). For the second inequality we argue as in Section 8. We have, in the notation of (29),

$$F_m(x) \geqslant \varepsilon F(x, \varepsilon)\,\mathrm{sign}\,x - \varepsilon b_{11};$$

and in the open domain D_0^* we have $F_m(x) \leqslant B_{12}$ because $\mathscr{A}_0^*$ lies above Γ except at its right-hand end point; thus

$$|F(x, t)| = |\varepsilon F(x, t, \varepsilon)| \leqslant B_{12} + 2\varepsilon b_{11}$$

and we can take $B'_{12} = 2B_{12} + 2b_{11}$.

Under the conditions stated, equation (50) includes various cases of resonance, in particular van der Pol's equation with small damping and forcing term:

(55) $$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = \varepsilon a \cos \omega t$$

with $\omega^2 - 1$ small. This equation was the subject of intensive study not only by van der Pol but also by Andronov and Witt and others in the early days of the theory of non-linear oscillations connected with radio; see for example [1] and [13]. Indeed the Poincaré–Bendixson theory of limit cycles and the Poincaré theory of stable and unstable nodes and foci and of saddle points seem to have been first applied in this area by Andronov and other Soviet authors to the system of two first order equations in A and a obtained by supposing that the solution of (55) was of the form

$$x = A\cos(\omega t + a), \qquad \dot{x} = -A\omega \sin(\omega t + a),$$

where A and a vary slowly. These equations were also studied later by Cartwright and Littlewood [2] and Gillies ([5], [6]) from a more strictly mathematical point of view. For a rigorous treatment it is essential to establish first that $x(t), \dot{x}(t)$ are bounded by constants independent of ε and ω for sufficiently large positive t.

Equation (50) also covers some of the cases which arise in connection with asymmetric resonance in Duffing's equation. Loud [12] has made a detailed study of the equation

(56) $$\ddot{x} + c\dot{x} + g(x) = \varepsilon p(t),$$

where ε is small, $c > 0$ is small and is chosen after ε, and $p(t)$ is periodic. His main interest was in the case in which periodic solutions of (56) reduce for $\varepsilon = c = 0$ to periodic solutions of

$$\ddot{x}+g(x) = 0 \tag{57}$$

with the same period as $p(t)$. It is not clear that this means that (56) is necessarily of the form (50); for c may be small compared with ε, in which case condition (53) will fail.

15. The non-resonant case with small parameter. It is evident from the linear equation

$$\ddot{x}+\varepsilon\dot{x}+x = a\cos\omega t \tag{58}$$

that if the period of $p(t)$ is close to that of solutions of the unforced equation (57), then solutions of

$$\ddot{x}+\varepsilon f(x)\dot{x}+g(x) = p(t)$$

are not necessarily bounded independently of ε for large positive t. For if $\omega = 1$ the solutions of (58) all tend to the particular solution $x(t) = a\varepsilon^{-1}\sin t$, and there is a similar effect for all ω near 1. It is well known that when $g(x)$ is non-linear the period of a periodic solution of (57) depends on its amplitude; and little is known about the solutions of

$$\ddot{x}+g(x) = p(t)$$

with $p(t)$ periodic except in certain very special cases such as those considered by Morris. However, the equation

$$\ddot{z}+\varepsilon f(z, \varepsilon)\dot{z}+z = b_1\cos\omega t \tag{59}$$

can be reduced to a system of the form (40) provided that

$$|\omega^2-1| \geqslant c > 0 \tag{60}$$

for some c independent of ε and ω. For this we write

$$x = z-\frac{b_1\cos\omega t}{1-\omega^2}, \qquad y = \dot{z}+\varepsilon F(z, \varepsilon)+\frac{b_1\omega\sin\omega t}{1-\omega^2}, \tag{61}$$

where as usual $F(z, \varepsilon)$ is the indefinite integral of $f(z, \varepsilon)$. Now (59) becomes

$$\dot{x} = y-\varepsilon F(z, \varepsilon), \qquad \dot{y} = -x. \tag{62}$$

THEOREM 10. *Suppose in addition to* (60) *that there are a constant b_2 and a continuous function $F_M(z)$, both independent of ε and ω, such that*

$$F(z, \varepsilon)\operatorname{sign} z > b_2 > 0 \quad \textit{for } |z| \geqslant 1,$$
$$|F(z, \varepsilon)| = \leqslant F_M(z). \tag{63}$$

Then there are constants B_{11}', B_{12}' independent of ε and ω such that each solution of (59) *satisfies*

$$|z(t)| \leqslant B_{11}', \qquad |\dot{z}(t)| \leqslant B_{12}'$$

for all large enough positive t.

To system (62) we can apply Theorem 8, renormalized so that the critical strip is $|x| \leqslant |b_1 c^{-1}| + 1$ instead of $|x| \leqslant 1$; condition (41) holds because of (63). It follows that $x(t)$ and $y(t)$ are eventually bounded independently of ε and ω. The result for $z(t)$ follows from the first equation (61); and since by (63) this implies also that $F(z, \varepsilon)$ is eventually bounded, the result for $\dot{z}(t)$ follows from the second equation (61).

Similar arguments can be applied to equations like (59) but with a more general right-hand side, provided the right-hand side contains no almost-resonant term.

16. Proof of Theorem 2. Our proof of Theorem 2 is a straightforward modification of our proof of Theorem 1; the main step will be to show that the analogue of Theorem 5 for the system (12) holds with the same J_n and D_n as before. In the notation of Sections 4 and 5, $F(x, t)$ is now simply $F(x)$, so that $F_m(x) = F(x)\operatorname{sign} x$ and Γ is the Jordan curve $y = F(x)$. In particular, $\dot{x} > 0$ above Γ and $\dot{x} < 0$ below Γ.

We now compare trajectories of (12) with trajectories of

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x). \tag{64}$$

Above Γ, a trajectory of (12) has at any point the same horizontal velocity and a greater downward velocity than the trajectory of (64) through that point; below Γ just the opposite is true. By Theorem 5 a trajectory of (64) which crosses J_n must cross it inwards, into D_n. Inspection of Fig. 1 shows that the same must happen *a fortiori* for trajectories of (12).

It remains to show that a trajectory of (12) can only spend a bounded time in $D_{n+1} - D_n$. As before, $\dot{x} > 2c_2 b^{-1}$ in $|x| \leqslant 1$ above $P_n Q_n$; so a trajectory of (12) can spend a time at most bc_2^{-1} in this region, and leaves it either by entering D_n or by entering $x \geqslant 1$. In the part of $D_{n+1} \cap \{x \geqslant 1\}$ above Γ,

$$\dot{y} \leqslant -g(x) \leqslant -g_{n+1} < 0$$

with the g_{n+1} of Section 6; so a trajectory of (12) can spend a time at most Y_{n+1}/g_{n+1} in this region. A trajectory can only leave this region by crossing Γ vertically downwards into the part of $D_{n+1} \cap \{x \geqslant 1\}$ below Γ, and a trajectory can only leave this latter region by crossing $x = 1$ to the left, when it enters D_n if it has not done so already. Now let h_{n+1} be a constant such that

$$h_{n+1} \geqslant h_M(x, \dot{x}) \geqslant h(x, \dot{x}, t) \geqslant 0 \quad \text{in } D_{n+1},$$

where $\dot{x} = y - F(x)$; the existence of h_{n+1} follows from the continuity of $h_M(x, \dot{x})$ as a function of x and y. In $D_{n+1} \cap \{x \geqslant 1\}$ below Γ every trajectory of (12) satisfies

$$h_{n+1}\dot{x} + \dot{y} = \{h_{n+1} - h(x, \dot{x}, t)\}\dot{x} - g(x) \leqslant -g_{n+1} < 0;$$

and since $h_{n+1}x+y$ is bounded in this region, a trajectory can only stay in this region for a bounded time.

This, together with the corresponding arguments for the other half of $D_{n+1}-D_n$, proves the analogue of Theorem 5 and thereby also proves Theorem 2. It is clear that the argument also works for the J_n^* and D_n^* of Sections 10 to 12.

17. Implications of Theorem 2. In Sections 8 to 14 we applied Theorem 1 to prove theorems for the second order equation (2) and for equations (36) and (50) containing a large and a small parameter respectively. We can apply Theorem 2 in exactly the same way; the process is essentially trivial since our earlier bounds were obtained by studying the domains D_0 and D_0^*, which are the same as before. Hence the new theorems are obtained from the old by deleting the forcing term and imposing the condition (13); and in this condition there need be no uniformity with respect to the parameter. We state the resulting theorems without further proof.

THEOREM 11. *Suppose that $f(x, \dot{x}, t)$ and $g(x)$ are continuous and that there are continuous functions $h_M(x, \dot{x}), f(x)$ and a constant b_2 such that*

$$h_M(x, \dot{x})+f(x) \geqslant f(x, \dot{x}, t) \geqslant f(x),$$

$$F(x)\operatorname{sign} x > b_2 > 0 \quad \text{for } |x| \geqslant 1,$$

$$g(x)\operatorname{sign} x > 0 \quad \text{for } |x| \geqslant 1,$$

$$|F(x)|+G(x) \to \infty \quad \text{as } |x| \to \infty,$$

where

$$F(x) = \int_0^x f(\xi)\,d\xi, \quad G(x) = \int_0^x g(\xi)\,d\xi.$$

Then there are constants B_3', B_4' such that each solution of (11) *satisfies*

$$|x(t)| \leqslant B_3', \quad |\dot{x}(t)| \leqslant B_4'$$

for all large enough positive t.

In Theorem 12 we shall write

$$F(x, k) = \int_0^x f(\xi, k)\,d\xi, \quad G(x, k) = \int_0^x g(\xi, k)\,d\xi$$

and similarly in Theorem 13 with ε instead of k.

THEOREM 12. *Suppose that $k \geqslant 1$ and that $f(x, \dot{x}, t, k)$ and $g(x, k)$ are continuous in $x, \dot{x}, t$ for each k. Suppose also that there are functions $f_M(x, \dot{x}, k)$ and $f(x, k)$, continuous in $x, \dot{x}$ for each k, and there are constants*

b_{02}, b_{03}, c_{02} *independent of* k *such that*

$$f_M(x, \dot{x}, k) \geqslant f(x, \dot{x}, t, k) \geqslant f(x, k),$$
$$F(x, k)\operatorname{sign} x > b_{02} > 0 \qquad \text{for } |x| \geqslant 1,$$
$$g(x, k)\operatorname{sign} x > 0 \qquad \text{for } |x| \geqslant 1,$$
$$|F(x, k)| \leqslant b_{03} \qquad \text{for } |x| \leqslant 1,$$
$$|g(x, k)| \leqslant c_{02} \qquad \text{for } |x| \leqslant 1,$$
$$|F(x, k)| \to \infty \qquad \text{uniformly in } k \text{ as } |x| \to \infty.$$

Then there are constants B'_{03}, B'_{04} *independent of* k *such that each solution of*

$$\ddot{x} + kf(x, \dot{x}, t, k)\dot{x} + g(x, k) = 0$$

satisfies

$$|x(t)| \leqslant B'_{03}, \qquad |\dot{x}(t)| \leqslant kB'_{04}$$

for all large enough positive t.

THEOREM 13. *Suppose that* $0 < \varepsilon < 1$ *and that* $f(x, \dot{x}, t, \varepsilon)$ *and* $g(x, \varepsilon)$ *are continuous in* x, $\dot{x}$, t *for each* ε. *Suppose that there are functions* $f_M(x, \dot{x}, \varepsilon)$ *and* $f(x, \varepsilon)$ *continuous in* x, $\dot{x}$ *for each* ε, *and there are constants* b_{12}, b_{13}, c_{12} *independent of* ε *such that*

$$f_M(x, \dot{x}, \varepsilon) \geqslant f(x, \dot{x}, t, \varepsilon) \geqslant f(x, \varepsilon),$$
$$F(x, \varepsilon)\operatorname{sign} x > b_{12} > 0 \qquad \text{for } |x| \geqslant 1,$$
$$g(x, \varepsilon)\operatorname{sign} x > 0 \qquad \text{for } |x| \geqslant 1,$$
$$|F(x, \varepsilon)| \leqslant b_{13} \qquad \text{for } |x| \leqslant 1,$$
$$|g(x, \varepsilon)| \leqslant c_{12} \qquad \text{for } |x| \leqslant 1,$$
$$G(x, \varepsilon) \to \infty \qquad \text{uniformly in } \varepsilon \text{ as } |x| \to \infty.$$

Then there are constants B'_{13}, B'_{14} *independent of* ε *such that each solution of*

$$\ddot{x} + \varepsilon f(x, \dot{x}, t, \varepsilon)\dot{x} + g(x, \varepsilon) = 0$$

satisfies

$$|x(t)| \leqslant B'_{13}, \qquad |\dot{x}(t)| \leqslant B'_{14}$$

for all large enough positive t.

18. The nested domains for Theorem 4. For the rest of this paper, we abandon the notations of Sections 2 to 17 but retain those of Section 1. Theorem 4 is an obvious consequence of the following theorem, which is also essential for the application of Brouwer's fixed point theorem in order to show the existence of a periodic solution of (19) when $h(x, \dot{x}, t)$ and $g(x, t)$ are periodic.

THEOREM 14. *Suppose that the conditions of Theorem 4 hold, and let $x(t), y(t)$ be as in Theorem 4. Then there is a sequence of domains $D_0 \subset D_1 \subset D_2 \subset \ldots$ such that each D_n is bounded by a Jordan curve J_n and the union of the D_n is the entire (x, y)-plane. If the point (x_0, y_0) is in $D_{n+1} - D_n$, then $\big(x(t), y(t)\big)$ is in D_{n+1} for all $t \geqslant t_0$; as t increases, $\big(x(t), y(t)\big)$ crosses J_n into D_n and thereafter remains in D_n. Moreover, there is a $t_3(n)$ depending on n but not on x_0, y_0 such that $\big(x(t), y(t)\big)$ is in D_n for $t \geqslant t_0 + t_3(n)$.*

COROLLARY 1. *Theorem 4 holds.*

COROLLARY 2. *Under the conditions of Theorem 4, each solution of* (19) *lies in D_0 from a certain time onwards.*

COROLLARY 3. *Suppose that the conditions of Theorem 4 hold and that $g(x, t)$ and $h(x, \dot{x}, t)$ are periodic in t with period $2\pi/\omega$ independent of x and $\dot{x}$. Then there is a solution of* (19) *which is periodic with period $2\pi/\omega$, and it lies entirely in D_0.*

It is easy to see that the Corollaries follow from the Theorem together with Brouwer's fixed point theorem in the case of Corollary 3 and the fact that the domains D_n defined below do not depend on $h(x, \dot{x}, t)$ in the case of Corollary 1. In the next two sections, which conclude this paper, we prove Theorem 14.

19. The construction of J_n for Theorem 14. In what follows we shall write

$$G_\nu(x) = \int_0^x g_\nu(\xi)\operatorname{sign}\xi\, d\xi \quad \text{for } \nu = 1, 2$$

and we shall define a constant c_1 such that

$$|F(x)| < c_1 \quad \text{for } |x| \leqslant 1, \tag{65}$$

which is possible by continuity. Let Γ denote the curve $y = F(x)$, so that $\dot{x} > 0$ above Γ and $\dot{x} < 0$ below Γ. For any $C > 2G_2(1)$ the point set

$$\mathscr{A}(C) = \{(x, y); (y-b)^2 = C - 2G_2(x), x \geqslant 1, y \geqslant b\} \tag{66}$$

is a finite or infinite Jordan arc, and through any point $(1, Y)$ with $Y > b$ there passes just one such arc. In virtue of (23), the argument of Section 4 shows that $\mathscr{A}(C)$ crosses Γ, perhaps more than once, provided $Y > \max\big(F(1), b\big)$. For any such Y, we define $x(Y) > 1$ to be the least value of x such that $\big(x, F(x)\big)$ lies on the Jordan arc $\mathscr{A}(C)$ through $(1, Y)$. Similarly for any $C > 2G_1(1)$ the point set

$$\mathscr{C}(C) = \{(x, y); (y-b)^2 = C - 2G_1(x), x \geqslant 1, y \leqslant b\}$$

is a finite or infinite Jordan arc, and through any point $(1, Y)$ with $Y < b$ there passes just one such arc, which lies entirely below Γ by (21).

Let b, c_1, c_2 be the constants in (21), (65) and (20), and let

$$Y_n = c_1 + 2c_2 b^{-1} + (n+1)b, \quad X_n = x(Y_n)$$

with $x(Y)$ as above; this is legitimate since $Y_n > c_1 > F(1)$ by (65). Let

$$P_n = (-1, Y_n - b), \quad Q_n = (1, Y_n), \quad T_n = (1, 2b - Y_n);$$

let $\mathscr{A}(C_n)$ be the arc (66) through Q_n and let $R_n = (X_n, F(X_n))$ be the point of $\mathscr{A}(C_n)$ at which $x = X_n$. Let $\mathscr{A}_n$ be the subarc of $\mathscr{A}(C_n)$ from Q_n to R_n.

LEMMA 4. *Under the conditions of Theorem 4, the arc $\mathscr{C}(C)$ through T_n meets the line $x = X_n$ at a point S_n say.*

By the definition of X_n we have

$$(Y_n - b)^2 + 2G_2(1) = (F(X_n) - b)^2 + 2G_2(X_n). \tag{67}$$

The condition that the arc $\mathscr{C}(C)$ through T_n should meet the line $x = X_n$ is

$$(Y_n - b)^2 + 2G_1(1) \geqslant 2G_1(X_n),$$

and this is satisfied in virtue of (24) and (67); indeed it is for this lemma that condition (24) is included in the statement of the theorem.

Now let $\mathscr{C}_n$ be the sub-arc of $\mathscr{C}(C)$ from S_n to T_n. Also let $P'_n = (1, b - Y_n) = T_{n+1}$, let $Q'_n = (-1, -Y_n)$ and let $T'_n = (-1, Y_n - 2b) = P_{n-1}$; and let R'_n, S'_n, $\mathscr{A}'_n$, $\mathscr{C}'_n$ be obtained from Q'_n and T'_n in the same way that R_n, S_n, $\mathscr{A}_n$, $\mathscr{C}_n$ are obtained from Q_n and T_n. The curve J_n consists of the straight line joining P_n to Q_n, the arc $\mathscr{A}_n$ joining Q_n to R_n, the vertical line $x = X_n$ from R_n to S_n, the arc $\mathscr{C}_n$ from S_n to T_n, the vertical line $x = 1$ from T_n to P'_n and the corresponding arc $P'_n Q'_n R'_n S'_n T'_n P_n$; one again Fig. 1 gives a clear enough picture of the situation.

It is easy to verify that J_n is a simple continuous closed curve, and that if D_n is its interior than $D_0 \subset D_1 \subset D_2 \subset \ldots$ and the union of the D_n is the entire (x, y)-plane.

20. Proof of Theorem 14. To prove Theorem 14 it is enough to show that no trajectory of (19) can cross J_n outwards, that so long as a trajectory remains outside D_0 it spirals clockwise round D_0, and that a trajectory can only remain in $D_{n+1} - D_n$ for a bounded time.

On or above $P_n Q_n$ in $|x| \leqslant 1$ we have $\dot{x} = y - F(x) > 2c_2 b^{-1} > 0$ and $\dot{y} \leqslant -g(x, t) \leqslant c_2$; so a trajectory in this region moves to the right and either moves downwards or moves upwards with slope less than $\frac{1}{2}b$. Since the slope of $P_n Q_n$ is $\frac{1}{2}b$, a trajectory which crosses $P_n Q_n$ can only cross it downwards into D_n, and a trajectory can spend a time at most bc_2^{-1} in $|x| \leqslant 1$ above $P_n Q_n$.

$\mathscr{A}_n$ is above Γ and therefore $\dot{x} \geqslant 0$ on $\mathscr{A}_n$; thus on $\mathscr{A}_n$

$$\frac{d}{dt}\{(y-b)^2+2G_2(x)\} = -2\dot{x}(y-b)h(x,\dot{x},t)+$$
$$+2(y-b)\big(g_2(x)-g(x,t)\big)+2\big(b-F(x)\big)g_2(x)<0$$

so that a trajectory which crosses $\mathscr{A}_n$ can only cross it into D_n. Similarly $\mathscr{C}_n$ is below Γ and indeed below $y = b$, and therefore $\dot{x} \leqslant 0$ on $\mathscr{C}_n$; thus on $\mathscr{C}_n$

$$\frac{d}{dt}\{(y-b)^2+2G_1(x)\} = -2\dot{x}(y-b)h(x,\dot{x},t)+$$
$$+2(y-b)\big(g_1(x)-g(x,t)\big)-2\big(F(x)-b\big)g_1(x)<0$$

so that a trajectory which crosses $\mathscr{C}_n$ can only cross it into D_n. Again the arcs R_nS_n and T_nP_n' are vertical and lie below Γ, so that a trajectory which crosses either of them has $\dot{x}<0$ at the point of crossing and so crosses to the left, into D_n. These arguments, and the corresponding ones for the symmetrically placed arcs of J_n, show that a trajectory which crosses J_n crosses it into D_n.

As in Sections 6 and 16, it only remains to show that a trajectory can only stay in $D_{n+1}\cap\{x\geqslant 1\}$ for a bounded time and that in leaving this region it enters D_n if it has not done so already. The proof of this is the same as the proof of the corresponding statement in Section 16, with $g_2(x)$ in place of $g(x)$; and this completes the proof of Theorem 14.

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