ANNALES
POLONICI MATHEMATICI
XXXV (1977)

General continuous solution of a linear homogeneous functional equation

by MAREK KUCZMA (Katowice)

Abstract. Functional equation (1) is studied under conditions (i)–(iii). The paper gives the general construction of all the continuous solutions $\varphi\colon I \to K$ of equation (1). This extends some previous results [1], [3], where the construction of the continuous solutions φ was given under some additional assumptions about the behaviour of the sequence $G_n(x)$ (cf. (2)).

In the present note we are concerned with the continuous solutions φ of the linear homogeneous functional equation of order one

$$\varphi[f(x)] = g(x)\varphi(x). \tag{1}$$

The given functions f and g will be assumed to fulfil the following conditions:

(i) $f\colon I \to I$, $I = [\xi, a)$, $-\infty \leqslant \xi < a \leqslant \infty$, is continuous and strictly increasing and $\xi < f(x) < x$ for $x \in (\xi, a)$ so that $f(\xi) = \xi$.

(ii) $g\colon I \to K$, where K is the field of the real numbers or that of the complex numbers, is continuous and $g(x) \neq 0$ for $x \in (\xi, a)$.

Write

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \quad x \in I,\ n = 1, 2, \ldots, \tag{2}$$

where f^i denotes the i-th iterate of f. We shall deal with the case where the continuous solution $\varphi\colon I \to K$ of equation (1) depends on an arbitrary function, or, what ammounts to the same (cf. [1] and [2], Chapter 2), the following condition is fulfilled:

(iii) There exists an interval $J \subset I$ such that

$$\lim_{n\to\infty} G_n(x) = 0 \tag{3}$$

almost uniformly in J (i.e., uniformly on every compact contained in J).

As R. Węgrzyk [3] has shown, there exists a maximal open subset U of I such that (3) holds almost uniformly on U. We put

$$m(x) = \sup_n |G_n(x)|, \qquad x \in U. \tag{4}$$

Since G_n tends to zero on U, for every $x \in U$ there is an index n_x such that

$$m(x) = |G_{n_x}(x)|. \tag{5}$$

It follows that $m(x)$ is finite and positive on U.

For an $x_0 \in (\xi, a)$ we write

$$I_k = [f^{k+1}(x_0), f^k(x_0)], \qquad k = 0, 1, 2, \ldots$$

Thus $I_0 = [f(x_0), x_0]$ and $I_k = f^k(I_0)$. Also,

$$\bigcup_{k=0}^{\infty} I_k = (\xi, x_0]. \tag{6}$$

The following theorem has been proved in [1] (cf. also [2], Chapter 2: Theorem 2.2 and the adjoining remark).

THEOREM 1. *Under conditions* (i)–(iii), *if there exists*(¹) *an* $x_0 \in (\xi, a)$ *such that* $I_0 \subset U$, *then every continuous function* $\varphi_0\colon I_0 \to K$ *fulfilling the condition*

$$\varphi_0[f(x_0)] = g(x_0)\varphi_0(x_0) \tag{7}$$

may be uniquely extended onto I *to a continuous solution* $\varphi\colon I \to K$ *of equation* (1); *and all the continuous solutions of* (1) *in* I *may be obtained in this manner.*

This theorem has been generalized by R. Węgrzyk [3] as follows.

THEOREM 2. *Under conditions* (i)–(iii), *if there exists*(¹) *an* $x_0 \in (\xi, a)$ *such that the sequence* $G_n(x)$ *is bounded on* $I_0 \cap U$, *then every continuous function* $\varphi_0\colon I_0 \to K$ *fulfilling condition* (7) *and such that*

$$\varphi_0(x) = 0 \quad \textit{for } x \in I_0 \setminus U \tag{8}$$

may be uniquely extended onto I *to a continuous solution* $\varphi\colon I \to K$ *of equation* (1); *and all the continuous solutions of* (1) *in* I *may be obtained in this manner.*

In the present paper we are going to prove the following generalization of the above theorem.

THEOREM 3. *Under conditions* (i)–(iii) *every continuous function* $\varphi_0\colon I_0 \to K$ *fulfilling conditions* (7), (8) *and*

$$\varphi_0(x) = o\big(1/m(x)\big), \quad x \to u,\ x \in I_0 \cap U, \quad \textit{for every } u \in I_0 \cap (\overline{U} \setminus U), \tag{9}$$

(¹) It follows from the relation $G_{n+1}(x) = g(x)G_n[f(x)]$ that if there exists an $x_0 \in (\xi, a)$ with this property, then every $x_0 \in (\xi, a)$ has this property.

may be uniquely extended onto I to a continuous solution $\varphi\colon I \to K$ of equation (1); *and all the continuous solutions of* (1) *in I may be obtained in this manner.*

Proof. Let $\varphi_0\colon I_0 \to K$ be a continuous function fulfilling conditions (7), (8), (9). It follows from (7) (cf. [2], Theorem 2.1) that there exists a unique continuous extension $\varphi\colon (\xi, a) \to K$ of φ_0 satisfying equation (1) in (ξ, a). We put further

$$\varphi(\xi) = 0. \tag{10}$$

Thus φ is a function from I to K, satisfies equation (1) in the whole I and is continuous in (ξ, a). It remains to prove that

$$\lim_{x\to\xi}\varphi(x) = 0. \tag{11}$$

We obtain from (1) and (2)

$$\varphi[f^n(x)] = G_n(x)\varphi(x), \quad x \in I,\ n = 1, 2, \ldots \tag{12}$$

Let x_p be an arbitrary sequence of points of I tending to ξ. Relation (11) will be proved if we show that

$$\lim_{p\to\infty}\varphi(x_p) = 0. \tag{13}$$

There is no loss of generality if we assume (cf. in particular (10)) that $x_p \in (\xi, x_0)$, $p = 1, 2, \ldots$ By (6) we get

$$x_p \in I_{k_p}, \quad p = 1, 2, \ldots; \quad \lim_{p\to\infty} k_p = \infty. \tag{14}$$

Put

$$y_p = f^{-k_p}(x_p) \in I_0. \tag{15}$$

We obtain from (12), (14) and (15)

$$\varphi(x_p) = G_{k_p}(y_p)\varphi_0(y_p). \tag{16}$$

If (13) did not hold, there would exist a subsequence x_{p_q} of x_p such that 0 would be not a cluster point of $\varphi(x_{p_q})$. Choosing from y_{p_q} a convergent subsequence $y_{p_{q_r}}$ we see that $\lim\varphi(x_{p_{q_r}}) \neq 0$. Thus it is enough to prove (13) under the additional assumption that the sequence y_p converges:

$$\lim_{p\to\infty} y_p = y_0 \in I_0.$$

We shall distinguish three cases.

1. $y_0 \in U$. Then

$$\lim_{p\to\infty} G_{k_p}(y_p) = 0 \tag{17}$$

in view of (14) and of the fact that (3) holds almost uniformly on U. On the other hand, the sequence $\varphi_0(y_p)$ is bounded, since φ_0 is continuous on I_0. Thus (13) results from (16) and (17).

2. $y_0 \in I_0 \setminus \overline{U}$. Then φ_0 vanishes in a neighbourhood of y_0 in virtue of (8). Thus $\varphi_0(y_p) = 0$ for large p and (13) results from (16).

3. $y_0 \in I_0 \cap (\overline{U} \setminus U)$. In view of (8) and (16) we need consider only the case where $y_p \in I_0 \cap U$. Then

$$\lim_{p\to\infty} m(y_p)\,|\varphi_0(y_p)| = 0$$

by (9). Since $|G_{k_p}(y_p)| \leqslant m(y_p)$, we get hence

$$\lim_{p\to\infty} G_{k_p}(y_p)\varphi_0(y_p) = 0$$

and (13) results from (16).

Thus we have proved (13), and hence also (11).

Let us note that the extension is unique, since every continuous solution $\varphi\colon I \to K$ of equation (1) in the case considered must fulfil condition (10) ([1] and [2], Theorem 2.2).

To complete the proof we must show that every continuous solution $\varphi\colon I \to K$ of equation (1) may be obtained in this manner.

Let $\varphi\colon I \to K$ be a continuous solution of equation (1) and put $\varphi_0 = \varphi | I_0$. The thing to show is that φ_0 fulfils conditions (7), (8) and (9).

Condition (7) results from the fact that φ satisfies (1) in I, condition (8) was proved in [3]. Now take an arbitrary $u \in I_0 \cap (\overline{U} \setminus U)$ and an arbitrary $\varepsilon > 0$. Since the sequence f^n converges to ξ uniformly on I_0 and since φ is a continuous solution of (1) in I (and hence fulfils (10)), there is an index N such that

$$|\varphi[f^n(x)]| < \varepsilon \quad \text{for } n > N \text{ and } x \in I_0. \tag{18}$$

Since f and g fulfil (i) and (ii), the number

$$M = \max_{1\leqslant n\leqslant N} \sup_{x\in I_0} |G_n(x)| \tag{19}$$

is finite and positive. Finally, since φ_0 is continuous and fulfils (8), there is a neighbourhood V of u (in I_0) such that

$$|\varphi_0(x)| < \varepsilon/M \quad \text{for } x \in V. \tag{20}$$

Now, we have from (5) and (12) for $x \in I_0 \cap U$

$$m(x)\,|\varphi_0(x)| = |G_{n_x}(x)|\,|\varphi_0(x)| = |\varphi[f^{n_x}(x)]|. \tag{21}$$

If $n_x > N$, then we obtain by (18) and (21)

$$m(x)\,|\varphi_0(x)| < \varepsilon. \tag{22}$$

If $n_x \leqslant N$, then (22) results for $x \in V \cap U$ in virtue of (21), (19) and (20). Thus (22) holds for $x \in V \cap U$, whence (9) follows and the proof of the theorem is complete.

Remarks. 1. Węgrzyk [3] shows by an example that in the case where sequence (2) is not bunded on U (or, what ammounts to the same, function (4) is not bounded on U), then not every continuous function $\varphi_0: I_0 \to K$ fulfilling conditions (7) and (8) may be extended to a continuous solution φ of equation (1) on I. It follows from Theorem 3 that this always is the case; it is enough to take any continuous function φ_0 on I_0 fulfilling (7) and (8), but not (9); and this is always possible whenever m is not bounded on $U \cap I_0$.

2. On the other hand, if m is bounded on $U \cap I_0$, then condition (9) is a consequence of (8) and of the continuity of φ_0. In fact, in such a case

$$\lim_{x\to u} \varphi_0(x) = \varphi_0(u) = 0 \qquad \text{for} \quad u \in I_0 \cap (\overline{U} \setminus U)$$

in virtue of (8) and of the continuity of φ_0 in I_0, whence

$$\varphi_0(x) = o(1) = o\big(1/m(x)\big), \qquad x \to u, \qquad x \in I_0 \cap U.$$

Thus condition (9) disappears and Theorem 3 reduces to Theorem 2.

Similarly, if $I_0 \subset U$, then m is bounded on $I_0 \cap U = I_0$, since G_n tends to zero uniformly on I_0. Thus in this case Theorem 2 reduces to Theorem 1, since condition (8) is void in this case.

This shows that Theorem 3 contains Theorems 1 and 2 as particular cases.

3. Theorem 3 gives the construction of the general continuous solution $\varphi: I \to K$ of equation (1) under the sole conditions (i), (ii), (iii). On the other hand, if (iii) is not fulfilled, the general continuous solution $\varphi: I \to K$ of equation (1) is known (cf. [1], [2]).

References

[1] B. Choczewski and M. Kuczma, *On the indeterminate case in the theory of a linear functional equation,* Fund. Math. 58 (1966), p. 163–175.

[2] M. Kuczma, *Functional equations in a single variable,* Monografie Mat. 46, Warszawa 1968.

[3] R. Węgrzyk, *Continuous solutions of a linear homogeneous functional equation,* this fasc., p. 15–20.

Reçu par la Rédaction le 26. 5. 1975