

## The space $N_*$ of holomorphic functions on bounded symmetric domains\*

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**Abstract.** Let  $D$  be a bounded symmetric domain in  $C^n$  with Bergman-Šilov boundary  $B$  and  $0 \in D$ , and let  $N_*$  denote the space of holomorphic functions on  $D$  for which the family  $\{\log^+ |f_r| : 0 < r < 1\}$  is uniformly integrable on  $B$ . For  $f, g \in N_*$ , let  $\varrho(f, g) = \int_B \log(1 + |f^* - g^*|) d\mu$ , where  $f^*$  and  $g^*$  denote the boundary values of  $f$  and  $g$ , respectively. In the paper we prove that  $(N_*, \varrho)$  is an  $F$ -algebra, that is, an  $F$ -space with a continuous multiplication. We also prove that if  $\gamma$  is a continuous linear functional on  $N_*$ , then there exists a holomorphic function  $g$  such that

$$\gamma(f) = \lim_{r \rightarrow 1} \int_B f(re^{-1}t) \overline{g(qt)} d\mu(t), \quad 0 < r < q < 1.$$

Some properties of outer functions and the invertible elements in  $N_*$  are also included.

Let  $D$  be a bounded symmetric domain in  $C^n$  with Bergman-Šilov boundary  $B$  and  $0 \in D$ . Using the notation of [3], [14] we denote by  $N_*$  the space of holomorphic functions  $f$  on  $D$  for which the family  $\{\log^+ |f_r| : 0 < r < 1\}$  is uniformly integrable on  $B$ . As in [17], where this space was considered for functions holomorphic in the unit disc, we define a translation invariant metric  $\varrho$  on  $N_*$  and show that  $(N_*, \varrho)$  is an  $F$ -algebra, that is, an  $F$ -space with a continuous multiplication. This is the main result of Section 2.

In Section 3 we extend the concept of an outer function, given for the polydisc in [12], to holomorphic functions on arbitrary bounded symmetric domains in  $C^n$ , and in Section 4 we consider the invertible elements in  $N_*$ . There we prove that a function  $f$  in  $N_*$  is invertible if and only if  $f$  is outer and that inversion is a continuous operation in  $N_*$ . We conclude by giving a characterization of the continuous linear functionals on  $N_*$  in Section 5.

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The present paper was motivated by the results of Yanagihara [17] and it extends some of his results to holomorphic functions on bounded symmetric domains. However, the techniques used in proving the results in general are different. Many of the results depend on properties of plurisubharmonic functions on bounded symmetric domains established in [14] and on properties of functions in  $N_*$  proved in [3], [14].

We are indebted to J. W. Roberts for many interesting and useful discussions of some of these topics.

**1. Definitions and preliminary results.** Let  $D$  be a bounded symmetric domain in  $C^n$  with Bergman-Šilov boundary  $B$ , and assume  $0 \in D$ .  $B$  is the minimal closed set  $\subset \bar{D}$  on which functions holomorphic on  $\bar{D}$  assume their maximum [2], p. 215. By [10]  $D$  is circular and starlike with respect to 0, i.e.,  $tz \in D$  whenever  $z \in D$ ,  $t \in C$ ,  $|t| \leq 1$ .

Let  $G$  denote the connected component of the identity of the group of holomorphic automorphisms of  $D$  and  $K$  the isotropy subgroup of  $G$  at 0. The group  $G$  is transitive on  $D$  and extends continuously to  $\partial D$ , the topological boundary of  $D$ . The Bergman-Šilov boundary  $B$  is circular and invariant under  $G$ . The group  $K$  acts transitively on  $B$  and  $B$  admits a unique  $K$ -invariant normalized regular Borel measure which will be denoted by  $\mu$  and is given by  $d\mu(t) = V^{-1} ds(t)$ ,  $V$  the euclidean volume of  $B$  and  $ds(t)$  the euclidean volume element at  $t \in B$ .

Let  $H(D)$  denote the class of holomorphic functions on  $D$ . As in [3], [14] we define the spaces  $N(D)$  and  $N_*(D)$  of holomorphic functions on  $D$  as follows:

$$N = N(D) = \left\{ f \in H(D) : \sup_{0 < r < 1} \int_B \log^+ |f_r| d\mu < \infty \right\},$$

where  $f_r$ ,  $0 < r < 1$ , is given by  $f_r(z) = f(rz)$ ,  $z \in \bar{D}$ , and

$$N_* = N_*(D) = \left\{ f \in H(D) : \left\{ \log^+ |f_r| : 0 < r < 1 \right\} \right. \\ \left. \text{is uniformly integrable on } B \right\}.$$

The Poisson kernel  $P$  on  $D \times B$  is given by

$$(1.1) \quad P(z, t) = \frac{|S(z, \bar{t})|^2}{S(z, \bar{z})}, \quad (z, t) \in D \times B,$$

where  $S$  is the Szegő kernel of  $D$ . By [7], p. 88,  $S$  is given by

$$(1.2) \quad S(z, \bar{w}) = \sum_{k=0}^{\infty} \sum_{r=1}^{m_k} \varphi_{kr}(z) \overline{\varphi_{kr}(w)},$$

where  $\{\varphi_{kr}\}$  is a complete orthonormal system of homogeneous polynomials on  $D$  orthonormalized with respect to the measure  $\mu$ , and  $m_k$

$= \binom{n+k-1}{k}$ . The series (1.2) converges uniformly on compact subsets of  $D \times \bar{D}$ , where  $\bar{D}$  is the closure of  $D$ , and  $S(z, \bar{w})$  is holomorphic in  $(z, \bar{w})$  on  $D \times D$  and continuous on  $D \times \bar{D}$ .

For a measure  $\nu$  on  $B$ , set

$$(1.3) \quad P_z[d\nu] = \int_B P(z, t) d\nu(t), \quad z \in D.$$

If  $f$  is an integrable function on  $B$ , set  $P_z[f] = P_z[fd\mu]$ .

The following result, which was proved in [14], Theorem 3, will be needed.

LEMMA 1.

(i) Let  $f \in N(D)$ ,  $f \not\equiv 0$ , then there exists a minimal regular Borel measure  $\nu_f$  on  $B$  such that

$$(1.4) \quad \log |f(z)| \leq P_z[d\nu_f], \quad z \in D,$$

and  $d\nu_f = \log |f^*| d\mu + d\sigma_f$ , where

$$(1.5) \quad f^*(t) = \lim_{r \rightarrow 1} f(rt) \quad \text{a.e. on } B,$$

$\log |f^*| \in L^1(B)$  and  $\sigma_f$  is singular (with respect to  $\mu$ ) on  $B$ .

(ii) If  $f \in N_*(D)$ , then  $\sigma_f \leq 0$  and

$$(1.6) \quad \log |f(z)| \leq P_z[\log |f^*|], \quad z \in D.$$

Furthermore

$$(1.7) \quad \lim_{r \rightarrow 1} \int_B |\log^+ |f_r| - \log^+ |f^*|| d\mu = 0.$$

In the above,  $\nu_f$  minimal means that if  $\nu$  is any other regular Borel measure on  $B$  satisfying (1.4), then  $\int g d\nu_f \leq \int g d\nu$  for all non-negative continuous functions  $g$  on  $B$ .

For  $f \in N(D)$ , the minimal measure  $\nu_f$  on  $B$  satisfying (1.4) will be referred to as the *boundary measure* of  $\log |f|$  and in addition to the above it also satisfies

$$(1.8) \quad \int_B d\nu_f = \lim_{r \rightarrow 1} \int_B \log |f_r| d\mu \quad [14].$$

Throughout the paper we will have several occasions in which we need to use a generalization of Lebesgue's dominated convergence theorem given in [11], p. 232. For convenience we include the result as a lemma.

LEMMA 2. Let  $(X, \lambda)$  be a measure space and  $\{f_n\}$  and  $\{g_n\}$  be two sequences of measurable functions which converge a.e. to  $f$  and  $g$ , respectively.

Suppose  $|f_n| \leq g_n$  a.e. for all  $n$  and that  $\lim_{n \rightarrow \infty} \int g_n d\lambda = \int g d\lambda < \infty$ . Then

$$(1.9) \quad \lim_{n \rightarrow \infty} \int f_n d\lambda = \int f d\lambda.$$

**2. The space  $N_*$  as an  $F$ -algebra.** As in [17], where the space  $N_*$  was considered for the case of the unit disc in  $C$ , we define a metric  $\rho$  on  $N_*(D)$  by

$$(2.1) \quad \rho(f, g) = \int_B \log(1 + |f^*(t) - g^*(t)|) d\mu(t),$$

where  $f^*$  and  $g^*$  are given by (1.5). By the inequalities

$$(2.2) \quad \log^+ x \leq \log(1 + x) \leq \log^+ x + \log 2, \quad x \geq 0,$$

$$(2.3) \quad \log(1 + |x + y|) \leq \log(1 + |x|) + \log(1 + |y|),$$

it is clear that  $\rho(f, g)$  is finite and satisfies the triangle inequality.

Suppose  $\rho(f, 0) = 0$ . By (2.1) it follows that  $f^* = 0$  a.e. on  $B$  and from (1.6) we obtain  $f(z) = 0$  for all  $z \in D$ . Hence  $\rho$  is a metric on  $N_*(D)$ .

We now proceed to show that  $N_*$  with the metric  $\rho$  is not only an  $F$ -space in the sense of Banach [1], p. 51, but also an  $F$ -algebra in the sense that multiplication is defined and is a continuous operation.

From the inequality

$$(2.4) \quad \log^+ xy \leq \log^+ x + \log^+ y, \quad x, y \geq 0,$$

it is clear that if  $f, g \in N_*$ , then  $fg \in N_*$  and

$$(2.5) \quad \rho(fg, 0) \leq \rho(f, 0) + \rho(g, 0).$$

Inequality (2.5) follows from the inequality

$$(2.6) \quad \log(1 + |xy|) \leq \log[(1 + |x|)(1 + |y|)] = \log(1 + |x|) + \log(1 + |y|).$$

The following lemmas will be needed in the proof of Theorem 1.

**LEMMA 3.** *Let  $f \in N_*(D)$  and  $C$  a compact subset of  $D$ . Then there exists a constant  $a_c$ , depending only on  $C$ , such that*

$$(2.7) \quad |f(z)| \leq [\exp(a_c \rho(f, 0)) - 1]^{1/2}$$

for all  $z \in C$ .

*Proof.* Since  $f$  is holomorphic in  $D$ ,  $\log(1 + |f(z)|^2)$  is plurisubharmonic in  $D$ . Furthermore by (2.2) and (2.6),

$$\log(1 + |f_r|^2) \leq 2 \ln 2 + 2 \log^+ |f_r|, \quad 0 < r < 1.$$

Therefore, since  $f \in N_*$ ,  $\log(1 + |f_r|^2)$  is uniformly integrable on  $B$  and hence by [14], Theorem 1,  $\log(1 + |f(z)|^2) \leq P_z[\log(1 + |f^*|^2)]$ ,  $z \in D$ . Since  $P(z, t)$  is continuous on  $D \times B$ , given a compact subset  $C$  of  $D$  there

exists a constant  $\beta_c$ , depending only on  $C$ , such that  $\log(1 + |f(z)|^2) \leq \beta_c \int_B \log(1 + |f^*|^2) d\mu$  for all  $z \in C$ . The result now follows by (2.6) with  $a_c = 2\beta_c$ .

LEMMA 4. Let  $f \in N_*(D)$ . Then

$$(2.8) \quad M(r) = \int_B \log(1 + |f_r(t)|^2) d\mu(t)$$

is a non-decreasing function of  $r$  on  $[0, 1)$  and

$$(2.9) \quad \lim_{r \rightarrow 1} \int_B \log(1 + |f_r(t)|^2) d\mu(t) = \int_B \log(1 + |f^*(t)|^2) d\mu(t),$$

where  $f^*$  is given by (1.5).

Proof. (2.8) follows by [4], Lemma 2; [14], Lemma 2; and (2.9) follows by the corollary to Theorem 1 in [14].

THEOREM 1. The space  $(N_*, \varrho)$  is an  $F$ -algebra, that is,  $N_*$  and  $\varrho$  satisfy the following:

$$(i) \quad \varrho(f, g) = \varrho(f - g, 0).$$

(ii) Suppose  $f, f_n \in N_*$  and  $\varrho(f_n, f) \rightarrow 0$ , then for each  $a \in C$

$$(2.10) \quad \varrho(af_n, af) \rightarrow 0.$$

(iii) Suppose  $a, a_n \in C$  and  $a_n \rightarrow a$ , then for each  $f \in N_*$

$$(2.11) \quad \varrho(a_n f, af) \rightarrow 0.$$

(iv) Suppose  $f, g, f_n, g_n \in N_*$  and  $\varrho(f_n, f) \rightarrow 0, \varrho(g_n, g) \rightarrow 0$ , then

$$(2.12) \quad \varrho(f_n g_n, fg) \rightarrow 0.$$

(v)  $N_*$  is complete with respect to the metric  $\varrho$ .

Proof. (i) is obvious and (ii) and (iii) are just special cases of (iv). Suppose  $f_n, f, g_n, g \in N_*$  and that  $f_n$  and  $g_n$  converge to  $f$  and  $g$  respectively in the metric  $\varrho$ . Since

$$(f_n g_n - fg) = (f_n - f)(g_n - g) + (fg_n - fg) + (gf_n - gf),$$

by the triangle inequality and (2.5)

$$\varrho(f_n g_n, fg) \leq \varrho(f_n, f) + \varrho(g_n, g) + \varrho(fg_n, fg) + \varrho(gf_n, gf).$$

Therefore it suffices to prove that if  $f_n$  converges to  $f$  in  $N_*$ ,  $\varrho(gf_n, gf) \rightarrow 0$  for all  $g \in N_*$ .

Fix  $g \in N_*$  and let  $a = \limsup_{n \rightarrow \infty} \varrho(gf_n, gf)$ . Then there exists a subsequence of  $\{f_n\}$ , which without loss of generality we denote by  $\{f_n\}$ , such that  $\varrho(gf_n, gf) \rightarrow a$ . From the inequality

$$(2.13) \quad \int_B |\log(1 + |f_n^*|) - \log(1 + |f^*|)| d\mu \leq \int_B \log(1 + |f_n^* - f^*|) d\mu = \varrho(f_n, f),$$

it follows that there exists a subsequence  $\{f_{n_k}^*\}$  of  $\{f_n^*\}$  such that  $f_{n_k}^* \rightarrow f^*$  a.e. on  $B$ . Therefore  $\log(1 + |g^* f_{n_k}^* - g^* f^*|) \rightarrow 0$  a.e. on  $B$ . Furthermore, by (2.6)

$$0 \leq \log(1 + |g^* f_{n_k}^* - g^* f^*|) \leq \log(1 + |f_{n_k}^* - f^*|) + \log(1 + |g^*|).$$

Since the terms in the right of the above inequality converge a.e. to  $\log(1 + |g^*|)$  and the integrals also converge to the integral of  $\log(1 + |g^*|)$ , by Lemma 2,

$$\begin{aligned} \alpha &= \lim_{n \rightarrow \infty} \varrho(gf_n, gf) = \lim_{k \rightarrow \infty} \int_B \log(1 + |g^* f_{n_k}^* - g^* f^*|) d\mu \\ &= \int_B \lim_{k \rightarrow \infty} (1 + |g^* f_{n_k}^* - g^* f^*|) d\mu = 0. \end{aligned}$$

Consequently  $\lim_{n \rightarrow \infty} \varrho(gf_n, gf) = 0$ , which proves (iv).

(v). Suppose  $\{f_n\}$  is a Cauchy sequence in  $N_*$ , i.e.,  $\varrho(f_n, f_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ . By (2.7)  $\{f_n(z)\}$  converges uniformly on every compact subset of  $D$  to a holomorphic function. Let  $f(z) = \lim_{n \rightarrow \infty} f_n(z)$ ,  $z \in D$ .

We first show that  $f \in N_*$ . By the inequality

$$\int |\log(1 + |f_n^*|) - \log(1 + |f_m^*|)| d\mu \leq \varrho(f_n, f_m),$$

$\log(1 + |f_n^*|)$  is a Cauchy sequence in  $L^1(B)$ . Hence there exists  $g^* \in L^1(B)$ ,  $g^* \geq 0$  a.e., such that  $\log(1 + |f_n^*|) \rightarrow g^*$  in  $L^1(B)$ . Therefore, by (1.6), (2.2), and the above,

$$\begin{aligned} \log^+ |f(z)| &= \lim_{n \rightarrow \infty} \log^+ |f_n(z)| \leq \lim_{n \rightarrow \infty} P_z[\log^+ |f_n^*|] \\ &\leq \lim_{n \rightarrow \infty} P_z[\log(1 + |f_n^*|)] = P_z[g^*]. \end{aligned}$$

Since  $g^* \in L^1(B)$ , by [3], Theorem 4,  $f \in N_*$ .

It still remains to be shown that  $\varrho(f_n, f) \rightarrow 0$ . Let  $\varepsilon > 0$  be arbitrary and choose an integer  $I$  such that  $\varrho(f_n, f_m) < \varepsilon/2$  for all  $n, m \geq I$ . Consider the functions  $\log(1 + |f_n(z) - f(z)|^2)$  and  $\log(1 + |f_n(z) - f_m(z)|^2)$  which are plurisubharmonic on  $D$ . Since  $f_m \rightarrow f$  uniformly on compact subsets of  $D$ , for all  $r$ ,  $0 < r < 1$ ,

$$\int_B \log(1 + |f_n(rt) - f(rt)|^2) d\mu = \lim_{m \rightarrow \infty} \int_B \log(1 + |f_n(rt) - f_m(rt)|^2) d\mu.$$

By Lemma 4 and (2.6),

$$\int_B \log(1 + |f_n(rt) - f_m(rt)|^2) d\mu \leq \int_B \log(1 + |f_n^* - f_m^*|^2) d\mu \leq 2\varrho(f_n, f_m).$$

Therefore for  $n \geq I$  and all  $r \in (0, 1)$ ,  $\int_B \log(1 + |f_n(rt) - f(rt)|^2) d\mu \leq \varepsilon$ .

By Lemma 4,  $\int_B (1 + |f_n^* - f^*|^2) d\mu \leq \varepsilon$ . Hence  $\lim_{n \rightarrow \infty} \int_B \log(1 + |f_n^* - f^*|^2) d\mu = 0$ ,

and consequently one can choose a subsequence  $\{f_{n_k}^*\}$  of  $\{f_n^*\}$  such that  $f_{n_k}^* \rightarrow f^*$  a.e. on  $B$ . By Fatou's lemma, for all  $n \geq I$ ,

$$\varrho(f_n, f) = \int_B \log(1 + |f_n^* - f^*|) d\mu \leq \lim_{k \rightarrow \infty} \int_B \log(1 + |f_n^* - f_{n_k}^*|) d\mu \leq \varepsilon/2.$$

Since  $\varepsilon > 0$  was arbitrary,  $\lim_{n \rightarrow \infty} \varrho(f_n, f) = 0$ , which proves the result.

**THEOREM 2.** *Let  $f \in N_*(D)$ . Then*

$$(2.14) \quad \varrho(f, 0) = \lim_{r \rightarrow 1} \int_B \log(1 + |f_r(t)|) d\mu(t),$$

and

$$(2.15) \quad \lim_{r \rightarrow 1} \varrho(f_r, f) = 0,$$

where  $f_r(z) = f(rz)$ ,  $z \in D$ .

**Proof.** By Lemma 1 (i),  $f_r \rightarrow f^*$  a.e. on  $B$  and by (1.7)  $\int_B \log^+ |f_r| d\mu \rightarrow \int_B \log^+ |f^*| d\mu$  as  $r \rightarrow 1$ . By (2.2),  $\log(1 + |f_r|) \leq \ln 2 + \ln^+ |f_r|$ . Thus by Lemma 2,

$$\varrho(f, 0) = \int_B \log(1 + |f^*|) d\mu = \lim_{r \rightarrow 1} \int_B \log(1 + |f_r|) d\mu,$$

which proves (2.14). The same technique applied to  $\log(1 + |f_r - f^*|)$  gives (2.15).

**3. Outer functions in  $N_*$ .** As in [12], p. 72, a function  $f \in N_*(D)$ ,  $f \neq 0$ , is *outer* if

$$(3.1) \quad \log |f(0)| = \int_B \log |f^*| d\mu,$$

where  $f^*$  is given by (1.5).

**THEOREM 3.** *Let  $f \in N_*(D)$ ,  $f \neq 0$ . Then  $f$  is an outer function on  $D$  if and only if*

$$(3.2) \quad \log |f(z)| = P_z[\log |f^*|], \quad z \in D,$$

where  $P_z[ ]$  is given by (1.3).

**Proof.** Assume  $f \in N_*$ ,  $f \neq 0$ . If  $f$  satisfies (3.2) it is clear that  $f$  is outer.

Suppose  $f$  is outer. Set  $H(z) = \log |f(z)| - P_z[\log |f^*|]$ . By (1.6)  $H(z) \leq 0$  for all  $z \in D$  and by (3.1)  $H(0) = 0$ . The function  $H$  is "strongly" subharmonic in the sense that

$$(3.3) \quad H(g \cdot 0) \leq \int_K H(gkq \cdot 0) dk$$

for all  $g, q \in G$ , where  $dk$  denotes the normalized Haar measure on the compact group  $K$ . Here  $g \cdot 0$  denotes the action of  $G$  on  $D$ . Inequality (3.3) follows since  $\log |f(z)|$  satisfies (3.3) [9] and, for every integrable function  $g^*$  on  $B$ , equality in (3.3) holds for  $P_z[g^*]$  [8]. It is well known that every uppersemicontinuous function on  $D$  satisfying (3.3) satisfies the maximum principle on  $D$ . Therefore  $H(z) = 0$  for all  $z \in D$ , which proves (3.2).

**THEOREM 4.** *If  $f \in N_*(D)$  is outer, then*

$$(3.4) \quad \lim_{r \rightarrow 1} \int_B |\log |f_r| - \log |f^*|| d\mu = 0.$$

*Conversely, if  $f \in N_*(D)$ ,  $f(z) \neq 0$  for all  $z \in D$ , satisfies (3.4), then  $f$  is outer.*

**Proof.** Suppose  $f \in N_*$  is outer. Then, since  $\log |f^*| \in L^1(B)$ , by (3.2)  $f(z) \neq 0$  for all  $z \in D$ . Let  $d\nu_f = \log |f^*| d\mu + d\sigma_f$ ,  $\sigma_f \leq 0$ , be the boundary measure of  $\log |f|$  given by Lemma 1. By (1.4) and (3.1) it follows that  $\sigma_f = 0$ . By the corollary to Theorem 1 in [14], for  $f \in N_*$ ,  $f \neq 0$ ,

$$(3.5) \quad \limsup_{r \rightarrow 1} \int_B |\log |f_r| - \log |f^*|| d\mu \leq \int_B d|\sigma_f|.$$

(3.4) now follows from (3.5).

Conversely, suppose  $f \in N_*$ ,  $f(z) \neq 0$  for all  $z \in D$ , satisfies (3.4). Since  $D$  is simply connected and  $f(z) \neq 0$  for all  $z \in D$ ,  $\log |f(z)|$  is pluriharmonic on  $D$ . By [15], Theorem 2 (iii),  $\log |f(z)| = P_z[\log |f^*|]$ . Therefore  $f$  is outer.

Let  $U = \{z \in C: |z| < 1\}$ . For a function  $f$  defined on  $D$  and  $t \in B$ , set

$$(3.6) \quad f_t(z) = f(zt), \quad z \in U.$$

If  $f$  is holomorphic (plurisubharmonic) in  $D$ ,  $f_t$  is holomorphic (subharmonic) in  $U$ . For a plurisubharmonic function  $f$  defined on  $D$  and  $t \in B$  let

$$(3.7) \quad M(f; t, r) = \frac{1}{2\pi} \int_0^{2\pi} f_t(re^{i\theta}) d\theta, \quad 0 < r < 1.$$

Since  $f_t$  is subharmonic in  $U$ ,  $M(f; t, r)$  is a non-decreasing function of  $r$  on  $(0, 1)$ . Let

$$(3.8) \quad M(f; t) = \lim_{r \rightarrow 1} M(f; t, r).$$

**THEOREM 5.** *If  $f \in N(D)$  ( $N_*(D)$ ), then  $f_t \in N(U)$  ( $N_*(U)$ ), for almost all  $t \in B$ .*



**Proof.** Suppose  $f \in N(D)$ ,  $f \neq 0$ . Consider  $M(\log^+ |f|; t, r)$ . Since  $d\mu(t') = d\mu(t)$  under the change of variable  $t' = te^{i\theta}$ , by Fubini's theorem

$$(3.9) \quad \int_B M(\log^+ |f|; t, r) d\mu(t) = \int_B \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta}t)| d\theta d\mu(t) = \int_B \log^+ |f_r| d\mu,$$

and by the Monotone convergence theorem,

$$(3.10) \quad \int_B M(\log^+ |f|; t) d\mu(t) = \lim_{r \rightarrow 1} \int_B \log^+ |f_r| d\mu < \infty.$$

Consequently  $M(\log^+ |f|; t) < \infty$  for almost all  $t \in B$  and therefore  $f_t \in N(U)$  for almost all  $t \in B$ .

Suppose  $f \in N_*(D)$ . By (1.7) and (3.10)

$$(3.11) \quad \int_B M(\log^+ |f|; t) d\mu(t) = \int_B \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f^*(e^{i\theta}t)| d\theta d\mu(t).$$

However, since  $f_t \in N(U)$  for almost all  $t \in B$ , by Fatou's lemma,

$$(3.12) \quad \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f^*(e^{i\theta}t)| d\theta \leq \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_t(re^{i\theta})| d\theta = M(\log^+ |f|; t).$$

Combining (3.11) and (3.12),

$$(3.13) \quad \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_t(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_t^*(e^{i\theta})| d\theta$$

for almost all  $t \in B$ . Therefore by [18], Theorem 7.5.3,  $f_t \in N_*(U)$  ( $N'$  in the notation of [18]) for almost all  $t \in B$ , which proves the assertion.

**THEOREM 6.** Let  $f \in N_*(D)$ ,  $f \neq 0$ , and let  $\nu_f$  be the boundary measure of  $\log |f|$ . Then the following are equivalent:

(i)  $d\nu_f = \log |f^*| d\mu$ , i.e.,  $\sigma_f = 0$ .

(ii) For almost all  $t \in B$ , the least harmonic majorant  $U[f_t]$  of  $\log |f|$  in  $U$  is given by

$$(3.14) \quad U[f_t](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) \log |f_t^*(e^{i\theta})| d\theta, \quad z \in U,$$

where here  $P(z, e^{i\theta}) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$  is the Poisson kernel for the unit disc.

Proof. Suppose  $dv_f = \log |f^*| d\mu$ . Hence by (1.8),  $\lim_{r \rightarrow 1} \int_B \log |f_r| d\mu = \int_B \log |f^*| d\mu$ . As in (3.9)–(3.11),

$$(3.15) \quad \lim_{r \rightarrow 1} \int_B M(\log |f|; t, r) d\mu(t) = \int_B \frac{1}{2\pi} \int_0^{2\pi} \log |f_t^*(e^{i\theta})| d\theta d\mu(t).$$

Since  $f_t \in N_*(U)$  almost all  $t \in B$ , by (1.6)

$$\log |f_t(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) \log |f_t^*(e^{i\theta})| d\theta, \quad z \in U.$$

Therefore,

$$\lim_{r \rightarrow 1} M(\log |f|; t, r) \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f_t^*(e^{i\theta})| d\theta \quad \text{for almost all } t \in B.$$

Hence by (3.15),

$$(3.16) \quad \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log |f_t(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f_t^*(e^{i\theta})| d\theta \quad \text{a.e.}$$

Consequently if  $t \in B$  is such that (3.16) is satisfied, then by Lemma 1 (ii) and (1.8), the singular part of the boundary measure  $v_{f_t}$  of  $\log |f_t|$  is zero. For the unit disc the harmonic function

$$U[f_t](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) dv_{f_t}(e^{i\theta}), \quad z \in U$$

is the least harmonic majorant of  $\log |f_t(z)|$  [12], which proves (ii).

Conversely, if (ii) holds, then

$$\lim_{r \rightarrow 1} M(\log |f|; t, r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f_t^*(e^{i\theta})| d\theta \quad \text{for almost all } t \in B.$$

By (3.9) and (3.10), with  $\log^+$  replaced by  $\log$ ,

$$\lim_{r \rightarrow 1} \int_B \log |f_r| d\mu = \int_B \log |f^*| d\mu.$$

By (1.8) it now follows that  $\sigma_f = 0$ , proving (i).

**COROLLARY.** *Let  $f \in N_*(D)$ . Then  $f$  is outer if and only if  $f_t$  is outer for almost all  $t \in B$ , where  $f_t$  is given by (3.6).*

**Remark.** For the special case of the polydisc, many of the results of this section were proved by W. Rudin in [12].

**4. Invertible elements in  $N_*$ .** In this section we apply the results of the previous section to the invertible elements in  $N_*$ .

Let  $I_* = I_*(D)$  denote the functions  $f \in N_*(D)$  such that  $\frac{1}{f} \in N_*(D)$ .

Clearly, in order that  $\frac{1}{f} \in N_*(D)$ , the function  $f(z)$  has to be non-zero for all  $z \in D$ .

**THEOREM 7.** *For  $f \in N_*(D)$ ,  $f \in I_*(D)$  if and only if  $f$  is an outer function.*

**Proof.** Suppose  $f \in N_*$  is invertible. Since  $f(z) \neq 0$  for all  $z \in D$ ,  $\log|f(z)|$  is pluriharmonic on  $D$ . Let  $dv_f = \log|f^*|d\mu + d\sigma_f$ ,  $\sigma_f \leq 0$  be the boundary measure of  $\log|f|$ . Since  $f \in N_*$ , it is easily shown that  $\sup_{0 < r < 1} \int_B |\log|f_r||d\mu < \infty$ . Therefore, by [15], Theorem 2(ii),

$$(4.1) \quad \log|f(z)| = P_z[\log|f^*|] + P_z[d\sigma_f], \quad z \in D.$$

However, since  $\frac{1}{f} \in N_*$  and  $\sup_{0 < r < 1} \int_B \left| \log \left| \frac{1}{f_r} \right| \right| d\mu < \infty$ , by Lemma 1 and [15], Theorem 2(ii), we also have

$$(4.2) \quad -\log|f(z)| = -P_z[\log|f^*|] + P_z[d\sigma_{1/f}], \quad z \in D,$$

where  $\sigma_{1/f}$  is the singular part of  $v_{1/f}$ , the boundary measure of  $\log|1/f|$ . Therefore, by (4.1) and (4.2),  $-P_z[d\sigma_{1/f}] = P_z[d\sigma_f]$  for all  $z \in D$ , from which it follows that  $\sigma_{1/f} = -\sigma_f$ . But  $\sigma_f \leq 0$  and  $\sigma_{1/f} \leq 0$  implies  $\sigma_f = 0$ . Therefore, by (4.1)  $f$  satisfies (3.2) and hence is outer.

Conversely, if  $f \in N_*$  is outer, by (3.2) it is clear that  $\frac{1}{f} \in N_*$ .

**THEOREM 8.** *Suppose  $f_n, f \in I_*$  and  $\varrho(f_n, f) \rightarrow 0$  as  $n \rightarrow \infty$ . Then*

$$(4.3) \quad \lim_{n \rightarrow \infty} \varrho\left(\frac{1}{f_n}, \frac{1}{f}\right) = 0.$$

Consequently, the mapping  $\tau: I_* \rightarrow I_*$  given by

$$(4.4) \quad \tau(f) = 1/f$$

is continuous and  $I_*$  is a topological group under multiplication.

**Proof.** We first show that  $\varrho\left(\frac{1}{f_n}, 0\right) \rightarrow \varrho\left(\frac{1}{f}, 0\right)$ . For any  $f \in I_*$ , by Theorem 7 and (3.1)

$$(4.5) \quad \varrho\left(\frac{1}{f}, 0\right) = \varrho(f, 0) - \log|f(0)|.$$

Since  $\varrho(f_n, f) \rightarrow 0$ , by the inequality  $|\varrho(f, 0) - \varrho(f_n, 0)| \leq \varrho(f_n, f)$ ,  $\varrho(f_n, 0) \rightarrow \varrho(f, 0)$ , and by Lemma 3,  $f_n(z) \rightarrow f(z)$  for all  $z \in D$ . Therefore,

by (4.5)

$$(4.6) \quad \lim_{n \rightarrow \infty} e\left(\frac{1}{f_n}, 0\right) = e\left(\frac{1}{f}, 0\right).$$

Let  $a = \overline{\lim}_{n \rightarrow \infty} e\left(\frac{1}{f_n}, \frac{1}{f}\right)$ . Since we can choose a subsequence  $\{f_{n_k}\}$  such that  $e\left(\frac{1}{f_{n_k}}, \frac{1}{f}\right) \rightarrow a$  as  $k \rightarrow \infty$ , we will assume without loss of generality that  $e\left(\frac{1}{f_n}, \frac{1}{f}\right) \rightarrow a$ .

Since  $\lim_{n \rightarrow \infty} \int_B \log(1 + |f_n^* - f^*|) d\mu = 0$ , there exists a subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  such that  $f_{n_k}^* \rightarrow f^*$  a.e. Therefore, since

$$\begin{aligned} e\left(\frac{1}{f_{n_k}}, \frac{1}{f}\right) &= \int_B \log\left(1 + \left|\frac{1}{f_{n_k}^*} - \frac{1}{f^*}\right|\right) d\mu \\ &\leq \int_B \left[ \log\left(1 + \left|\frac{1}{f_{n_k}^*}\right|\right) + \log\left(1 + \left|\frac{1}{f^*}\right|\right) \right] d\mu, \end{aligned}$$

by (4.6) and Lemma 2,

$$\lim_{k \rightarrow \infty} e\left(\frac{1}{f_{n_k}}, \frac{1}{f}\right) = \int_B \lim_{k \rightarrow \infty} \log\left(1 + \left|\frac{1}{f_{n_k}^*} - \frac{1}{f^*}\right|\right) d\mu = 0.$$

Therefore  $a = 0$  and  $\lim_{n \rightarrow \infty} e\left(\frac{1}{f_n}, \frac{1}{f}\right) = 0$ .

**THEOREM 9.** Suppose  $f_n \in I_*$ ,  $n = 1, 2, \dots$  and that  $f_n \rightarrow f \in N_*$ . If  $f \neq 0$ , then  $f(z) \neq 0$  for all  $z \in D$ . Furthermore,  $f \in I_*$  if and only if

$$(4.7) \quad \int_B \log |f^*| d\mu \leq \liminf_{n \rightarrow \infty} \int_B \log |f_n^*| d\mu,$$

where  $f_n^*$ ,  $f^*$  are given by (1.5).

**Proof.** Suppose  $f_n \rightarrow f$ ,  $f_n \in I_*$ ,  $f \in N_*$ , and  $f$  is not identically zero. By Lemma 3,  $f_n \rightarrow f$  uniformly on compact subsets of  $D$ . Since  $f_n(z) \neq 0$  for all  $z \in D$  and all  $n$  and  $f$  is not identically zero, by [2], p. 274,  $f(z) \neq 0$  for all  $z \in D$ .

Clearly, if  $f \in I_*$ , by (3.1) and Theorem 7,  $\int_B \log |f^*| d\mu = \lim_{n \rightarrow \infty} \int_B \log |f_n^*| d\mu$ , which proves (4.7).

Conversely, suppose (4.7) is valid. Then by (1.6),

$$\log |f(0)| \leq \int_B \log |f^*| d\mu \leq \liminf_{n \rightarrow \infty} \int_B \log |f_n^*| d\mu.$$

However, since  $f_n \in I^*$  for all  $n$ ,  $f_n$  is outer and by (3.1),

$$\liminf_{n \rightarrow \infty} \int_B \log |f_n^*| d\mu = \lim_{n \rightarrow \infty} \log |f_n(0)| = \log |f(0)|.$$

Therefore,  $f$  satisfies (3.1) and hence by Theorem 7 is invertible.

Remark. Theorem 9 also shows that the closure of  $I_*$  in  $N_*$ ,  $\bar{I}_*$ , is given by

$$\bar{I}_* = \{f \in N_* \mid f(z) \neq 0 \text{ for all } z \in D\} \cup \{0\}.$$

Since if  $f_n \in I_*$ ,  $f_n \rightarrow f \in N_*$ , then by Theorem 9,  $f \equiv 0$  or  $f(z) \neq 0$  for all  $z \in D$ . Conversely, if  $f(z) \neq 0$  for all  $z \in D$  and  $f \in N_*$ , then for each  $r$ ,  $0 < r < 1$ ,  $f_r \in I_*$  and by (2.15),  $f_r \rightarrow f$  in  $N_*$ .

**5. Representation of continuous linear functionals on  $N_*$ .** Let  $\gamma$  be a continuous linear functional on  $N_*$ , i.e.,  $\gamma \in (N_*)^*$ . Since  $N_*$  is metrizable,  $\gamma$  is continuous if and only if  $\gamma$  is bounded, i.e.,  $\gamma(E)$  is bounded for every bounded subset  $E$  of  $N_*$  [13], Theorem 1.32.

As in Section 1, let  $\Phi = \{\varphi_{kv}(z)\}$ ,  $k = 1, 2, \dots, v = 1, 2, \dots, m_k$  be the complete orthonormal system of homogeneous polynomials. Using the inequality,  $\log(1+x) \leq x$ ,  $x \geq 0$ , and the orthonormality of  $\Phi$ , one can easily show that for any scalar  $a \in C$ ,

$$(5.1) \quad \rho(a\varphi, 0) \leq |a|, \quad \varphi \in \Phi,$$

from which it follows by [13], Theorem 1.30, that  $\Phi$  is a bounded subset of  $N_*$ . Since the result is not needed, we omit the details.

For  $f$  holomorphic on  $D$ ,  $0 < r < 1$ , set

$$(5.2) \quad (f_r, \varphi_{kv}) = \int_B f_r(t) \overline{\varphi_{kv}(t)} d\mu(t).$$

We state the following result, proved in [5], as a lemma.

LEMMA 5. Any holomorphic function  $f$  on  $D$  has a Fourier series expansion

$$(5.3) \quad f(z) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k} a_{kv}(f) \varphi_{kv}(z), \quad a_{kv}(f) = \lim_{r \rightarrow 1} (f_r, \varphi_{kv}),$$

which converges uniformly on compact subsets of  $D$ .

THEOREM 10. Let  $\gamma$  be a continuous linear functional on  $N_*(D)$ . Then there exists a unique holomorphic function  $g$  on  $D$  such that

$$(5.4) \quad \gamma(f) = \lim_{r \rightarrow 1} \int_B f(re^{-1}t) \overline{g(et)} d\mu(t), \quad 0 < r < e < 1,$$

for all  $f \in N_*$ . Conversely, given any  $g$  holomorphic on  $D$  for which  $\lim_{r \rightarrow 1} \int_B f(re^{-1}t) \overline{g(et)} d\mu(t)$  exists for all  $f \in N^*$ , then (5.4) defines a continuous linear functional on  $N_*$ .

Proof. Given  $\gamma \in (N_*)^*$ , let  $b_{k\nu} = \gamma(\varphi_{k\nu})$ ,  $\varphi_{k\nu} \in \Phi$ .  $\{b_{k\nu}\}$  is a bounded subset of  $C$  since  $\Phi$  is bounded. Let  $f \in N_*$  be arbitrary and for each positive integer  $n$  set

$$(5.5) \quad F^n(z) = \sum_{k=0}^n \sum_{\nu=1}^{m_k} a_{k\nu}(f) \varphi_{k\nu}(z),$$

where  $a_{k\nu}(f)$  is given by (5.3). Then  $F^n(z) \rightarrow f(z)$  for all  $z \in D$  and for  $0 < r < 1$ ,  $F_r^n(z) \rightarrow f_r(z)$  uniformly on  $\bar{D}$ . Consequently,

$$(5.6) \quad \lim_{n \rightarrow \infty} \varrho(F_r^n, f_r) = \lim_{n \rightarrow \infty} \int_B \log(1 + |F^n(rt) - f(rt)|) d\mu(t) = 0.$$

Therefore, by the continuity of  $\gamma$  and (5.6)

$$(5.7) \quad \gamma(f_r) = \lim_{n \rightarrow \infty} \gamma(F_r^n) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{\nu=1}^{m_k} a_{k\nu}(f) b_{k\nu} r^k = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} a_{k\nu}(f) b_{k\nu} r^k.$$

In (5.7) we used the homogeneity of  $\varphi_{k\nu}$  to obtain  $\varphi_{k\nu}(rz) = r^k \varphi_{k\nu}(z)f$ . By (2.15),  $\varrho(f_r, f) \rightarrow 0$  as  $r \rightarrow 1$ . Therefore

$$(5.8) \quad \gamma(f) = \lim_{r \rightarrow 1} \gamma(f_r) = \lim_{r \rightarrow 1} \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} a_{k\nu}(f) b_{k\nu} r^k.$$

For each positive integer  $n$ , let

$$(5.9) \quad S_w^n(z) = S^n(z, \bar{w}) = \sum_{k=0}^n \sum_{\nu=1}^{m_k} \varphi_{k\nu}(z) \overline{\varphi_{k\nu}(w)}$$

and

$$(5.10) \quad g_n(z) = \sum_{k=0}^n \sum_{\nu=1}^{m_k} \bar{b}_{k\nu} \varphi_{k\nu}(z).$$

By (1.2),

$$(5.11) \quad \lim_{n \rightarrow \infty} S^n(z, \bar{w}) = S(z, \bar{w}) = S_w(z),$$

where the convergence is uniform on compact subsets of  $\bar{D} \times D$ . Consequently, for fixed  $w \in D$ ,  $S_w$  (given by (5.11)) is holomorphic in  $D$  and continuous on  $\bar{D}$ , hence in  $N_*$ . Furthermore, given any compact subset  $C$  of  $D$  and  $\delta > 0$ , there exists an integer  $I = I(\delta, C)$  such that

$$(5.12) \quad \varrho(S_w^n, S_w) = \int_B \log(1 + |S_w^n(t) - S_w(t)|) d\mu(t) < \delta$$

for all  $n \geq I$  and all  $w \in C$ . (5.12) follows since  $S^n(z, w) \rightarrow S(z, w)$  uniformly for  $(z, w) \in \bar{D} \times C$ .

Let  $\varepsilon > 0$  be arbitrary and let  $C$  be an arbitrary compact subset of  $D$ . By continuity of  $\gamma$ , there exists a  $\delta > 0$  such that  $|\gamma(f)| < \delta$  for all

$f \in N_*$  with  $\varrho(f, 0) < \delta$ . By (5.12), there exists an integer  $I$  such that  $\varrho(S_w^n, S_w) < \delta$  for all  $w \in C$  and all  $n \geq I$ . Therefore

$$(5.13) \quad |\bar{\gamma}(S_w^n) - \bar{\gamma}(S_w)| < \varepsilon$$

for all  $n \geq I$  and all  $w \in C$ . However, by (5.9) and (5.10),  $\bar{\gamma}(S_w^n) = g_n(w)$ . Therefore  $g_n$  converges uniformly on compact subsets to a holomorphic function  $g$  on  $D$ . Let

$$(5.14) \quad g(z) = \lim_{n \rightarrow \infty} g_n(z) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} \bar{b}_{k\nu} \varphi_{k\nu}(z), \quad z \in D.$$

Consider  $\int_B f(r\varrho^{-1}t) \overline{g(\varrho t)} d\mu(t)$ ,  $0 < r < \varrho < 1$ . A routine computation, using the orthonormality of  $\{\varphi_{k\nu}\}$  and the uniform convergence of the series expansions of  $f$  and  $g$  ((5.3) and (5.14) respectively) on compact subsets of  $D$ , gives

$$(5.15) \quad \int_B f(r\varrho^{-1}t) \overline{g(\varrho t)} d\mu(t) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} a_{k\nu}(f) b_{k\nu} r^k.$$

Combining (5.7) with (5.15) gives (5.4). Equation (5.15) also shows that (5.4) is independent of  $\varrho$ ,  $0 < r < \varrho < 1$ . Since

$$\int_B \overline{g(\varrho t)} \varphi_{k\nu}(r\varrho^{-1}t) d\mu(t) = b_{k\nu} \quad \text{for all } k = 0, 1, 2, \dots, \nu = 1, \dots, m_k,$$

$g$  is unique.

Conversely, suppose there exists a holomorphic function  $g$  on  $D$  such that

$$(5.16) \quad \lim_{r \rightarrow 1} \int_B f(r\varrho^{-1}t) \overline{g(\varrho t)} d\mu(t), \quad 0 < r < \varrho < 1,$$

exists for all  $f \in N_*$ . For  $0 < r < \varrho < 1$ , set

$$(5.17) \quad \gamma_r(f) = \int_B f(r\varrho^{-1}t) \overline{g(\varrho t)} d\mu(t).$$

For each  $r \in (0, 1)$  and  $\varrho$ ,  $r < \varrho < 1$ ,  $\gamma_r$  is continuous. Since, if  $f_n \rightarrow f$  in  $N_*$ ,  $f_n \rightarrow f$  uniformly on compact subsets of  $D$  and hence  $\gamma_r(f_n) \rightarrow \gamma_r(f)$ . By (5.16),  $\lim_{r \rightarrow 1} \gamma_r(f)$  exists for every  $f \in N_*$ , and by [13], Theorem 2.8,

$$\gamma(f) = \lim_{r \rightarrow 1} \gamma_r(f)$$

defines a continuous linear functional on  $N_*$ , which completes the proof.

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