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On centers of cubic systems

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1. Introduction. Let X, Y be two polynomials in $(x, y) \in \boldsymbol{R}^2$ with real coefficients, relatively prime, one of which has degree $= n$ and the other has degree $\leqslant n$.

A singular point S of the differential system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{1.1}$$

$(\dot{x} = dx/dt,\ \dot{y} = dy/dt)$ is a center if there is a neighbourhood of S entirely covered by cycles surrounding S.

When $n > 1$ there are different types of centers.

In the quadratic case $(n = 2)$, a complete classification is known (cf. Conti [1], Frommer [4]).

The aim of this paper is to contribute to the classification of centers in the cubic case $(n = 3)$, which is still far from being satisfactory.

2. Types of centers. Let S be a center of (1.1) and let G be the family of cycles surrounding S and no other singular point. Let us denote by $\operatorname{int}\gamma$ the region interior to a cycle $\gamma \in G$, by N_S the region defined by

$$N_S = \bigcup_{\gamma \in G} \operatorname{int}\gamma$$

and by ∂N_S the boundary of N_S.

∂N_S is an invariant set and we can divide centers of (1.1) into four types as follows: S is of *type* A when $\partial N_S = \emptyset$, of *type* B when $\partial N_S \neq \emptyset$ does not contain singular points, of *type* C when ∂N_S is unbounded and contains at least one singular point, of *type* D when ∂N_S is bounded.

We shall now examine the four types separately.

3. Centers of type A. In the linear case $(n = 1)$, a center can only be of type A. Further, all solutions $t \to x(t)$, $t \to y(t)$ of (1.1) are periodic with the same period, i.e., the center is isochronous.

For $n = 2$, centers of type A cannot exist (cf. [1]).

For $n = 3$, centers of type A do exist, but they are not necessarily isochronous. For instance, $0 = (0, 0)$ is a center of type A for the two systems

$$\dot{x} = -y+x^2, \quad \dot{y} = x-2xy+2x^3$$

and

$$\dot{x} = -y^3, \quad \dot{y} = x^3;$$

it is isochronous for the first system and non-isochronous for the second.

4. Centers of type B. From now on we assume $\partial N_S \neq \emptyset$.

It can be shown (cf. Conti [2]) that for system (1.1) of arbitrary degree n, the number of connected components of ∂N_S is $\leqslant n+1$ and that if ∂N_S is unbounded then each connected component is unbounded.

Therefore, for a center S of type B, ∂N_S is the union of $k \leqslant n+1$ unbounded trajectories, each dividing the plane into two unbounded regions, one of which contains N_S.

We will say that a center S of type B is of *subtype* B^k if ∂N_S contains k connected components.

For $n = 2$ only subtype B^1 is possible (cf. [1], [4]).

For $n = 3$ subtypes B^1 and B^2 are possible, as the following examples show.

EXAMPLE 4.1. The system

$$\dot{x} = 2y-x^2, \quad \dot{y} = -2x-2xy+x^3$$

has a center of subtype B^1 at 0. The trajectories are

$$(2y-x^2+1)\exp(-2y-x^2) = c$$

and N_0 is the region interior to the parabola $2y-x^2+1 = 0$.

EXAMPLE 4.2. The system

$$\dot{x} = y-x^2y, \quad \dot{y} = -x-xy^2$$

has a center of subtype B^2 at 0. The trajectories are the conics $x^2+(1-r^2)y^2 = r^2$ and N_0 is the open strip $x^2 < 1$.

QUESTION 4.1. Do centers of subtypes B^3, B^4 exist?

5. Centers of type C. The positive (negative) limit set of an open trajectory $\gamma \subset \partial N_S$, if it is not empty, reduces to a singular point $\Sigma \in \partial N_S$ and γ approaches Σ for $t \to +\infty$ (for $t \to -\infty$) along a certain direction (cf. [2]).

Let us now introduce some terminology.

An open trajectory $\gamma \subset \partial N_S$ is *totally unbounded* if both its limit sets are empty, *partially unbounded* if exactly one limit set is empty, *homoclinic* if the two limit sets coincide, *heteroclinic* if the two limit sets are distinct.

A *loop* is a simple closed curve consisting of a homoclinic trajectory and its limit point.

A *chain*, or *k-chain*, is a simple closed curve consisting of $k \geq 2$ singular points $\Sigma_1, \ldots, \Sigma_k$ and k heteroclinic trajectories $\gamma_1, \ldots, \gamma_k$ joining Σ_1 to $\Sigma_2, \ldots, \Sigma_k$ to Σ_1.

Loops and chains are also generically called *separatrix cycles*. If a separatrix cycle is contained in ∂N_S, it can be given the same orientation as the cycles of N_S.

A *train* is what remains by suppressing an open trajectory from a chain.

It can be shown (cf. [2]) that an unbounded connected component of ∂N_S is either totally unbounded or it must contain two partially unbounded orbits and it may contain also separatrix cycles and trains.

Centers of type C can be divided into subtypes $C^k_{\sigma,\omega}$, where k denotes the number of connected components of ∂N_S, σ the number of singular points and ω the number of open trajectories.

For $n = 2$, a center of type C can only be of subtype $C^1_{1,2}$ (cf. [1], [4]).

For $n = 3$, the following examples show the existence of centers of subtypes $C^1_{\sigma,\omega}$, $C^2_{\sigma,\omega}$.

Example 5.1. The trajectories of the system

$$\dot{x} = -2y, \quad \dot{y} = 3x + 6x^2 - 3y^2 + 3x^3$$

are $[y^2 - (x+1)^3] \exp(-3x) = c$. The singular points are $(-1, 0)$, a non-elementary one, and 0, a center of subtype $C^1_{1,2}$. ∂N_0 is the cubic $y^2 = (x+1)^3$.

Example 5.2. The trajectories of the system

$$\dot{x} = 2xy, \quad \dot{y} = 1 - x - y^2 + xy^2$$

are $x(y^2 - 1)\exp(-x) = c$. The singular points are the two saddles $(0, 1)$, $(0, -1)$ and a center of subtype $C^1_{2,3}$ at $S = (1, 0)$. N_S is the half-strip $x > 0$, $y^2 < 1$.

Example 5.3. The system

$$\dot{x} = -y + x^2 y, \quad \dot{y} = x - x^2 + xy^2$$

has a non-elementary singular point at $(1, 0)$ and a center of subtype $C^2_{1,3}$ at 0. N_0 is the strip $x^2 < 1$. The cycles are represented by $(1-x)(1+x)^{-1}$ $\exp[2(y^2 - x + 1)/(x^2 - 1)] = c \geq 0$.

Example 5.4. The system

$$\dot{x} = y - x^2 y, \quad \dot{y} = -x + x^3 - xy^2$$

has a center of subtype $C^2_{2,4}$ at 0 and two non-elementary singular points at $(-1, 0)$, $(1, 0)$. The trajectories are represented by $(1-x^2)\exp[-y^2/(1-x^2)] = c$ and N_0 is the strip $x^2 < 1$.

Several questions about centers of type C of cubic systems remain open.

QUESTION 5.1. Do centers of subtype $C^2_{2,4}$ exist with the two singular points on the same connected component?

QUESTION 5.2. Do centers of subtypes $C^3_{\sigma,\omega}$, $C^4_{\sigma,\omega}$ exist?

QUESTION 5.3. Do centers of type C with $\sigma > 2$ exist?

QUESTION 5.4. Do centers of type C exist with ∂N_S containing a separatrix cycle?

6. Centers of type D. It can be shown (cf. [2]) that when ∂N_S is bounded then either it is a separatrix cycle or a connected set consisting of an outer separatrix cycle Γ and one or more separatrix cycles, mutually exterior, with no more than one singular point in common with Γ.

A center of type D is said to belong to *subtype* $D_{\sigma,\omega}$ if ∂N_S contains σ singular points and ω open trajectories.

For $n = 2$, ∂N_S is a loop, a 2-chain or a 3-chain, so that the subtypes reduce to $D_{1,1}$, $D_{2,2}$, $D_{3,3}$ (cf. [1], [4]).

For $n = 3$, the range of subtypes $D_{\sigma,\omega}$ is wider and not completely known yet.

We shall give a few examples, as usual, and pose some questions.

EXAMPLE 6.1. The system

$$\dot{x} = -2y-2xy+x^2y+y^3, \qquad \dot{y} = 3x^2+y^2-x^3-xy^2$$

has a center at $S = (3, 0)$ and a non-elementary singular point at 0. The trajectories are the quartics $(x^2+y^2)^2-4x(x^2+y^2)-4y^2 = c$ and N_S is a loop corresponding to $c = 0$.

EXAMPLE 6.2. The quartics $(y+1+x^2)(y-1+2x^2) = c$ are the trajectories of

$$\dot{x} = 2y+3x^2, \qquad \dot{y} = -2x-6xy-8x^3.$$

0 is a center and ∂N_0 a 2-chain, corresponding to $c = 0$, which contains the two saddles $(-\sqrt{2}, -3)$, $(\sqrt{2}, -3)$.

EXAMPLE 6.3. The system

$$\dot{x} = 9y+12xy+x^2y+y^3, \qquad \dot{y} = -9x+6x^2-6y^2-x^3-xy^2$$

has four singular points, namely, the origin, which is a center, and $(-3/2, -3\sqrt{3}/2)$, $(-3/2, 3\sqrt{3}/2)$, $(3, 0)$, which are non-elementary. The trajectories are the quartics $(x^2+y^2)^2+8x(3y^2-x^2)+18(x^2+y^2) = c$, and ∂N_0 is a 3-chain corresponding to $c = 27$.

EXAMPLE 6.4. The system

$$\dot{x} = y-5x^2y+4y^3, \qquad \dot{y} = -x-4x^3+5xy^2$$

has five singular points, namely, the origin, which is a center, and four saddles at $(-1, -1)$, $(-1, 1)$, $(1, 1)$, $(1, -1)$. The trajectories are the quartics $(1-2x^2+y^2)(1-2y^2+x^2) = c$ and ∂N_0 is a 4-chain, corresponding to $c = 0$.

QUESTION 6.1. Can ∂N_S be a k-chain with $k > 4$?

EXAMPLE 6.5. The trajectories of

$$\dot{x} = 7y-x^2y-y^3, \qquad \dot{y} = -6-7x+x^3+xy^2$$

are the quartics $(x^2+y^2)^2-14(x^2+y^2)-24x = c$; the singular points are a saddle at $(-2, 0)$ and two centers at $(-1, 0)$, of subtype $D_{1,1}$, and at $S = (3, 0)$, of subtype $D_{1,2}$. ∂N_S is the union of two loops, corresponding to $c = 8$.

Example 6.5 shows that ∂N_S can be the union of an outer loop and an inner loop. It is not difficult to show that, for a cubic system, N_S cannot be the union of two chains or of an outer chain and an inner loop, or of an outer loop and an inner k-chain, $k > 2$.

QUESTION 6.2. Can ∂N_S be the union of an outer loop and an inner 2-chain?

7. Systems with more than one center. A quadratic system can have two centers, both of type B or of type $D_{1,1}$, or of type $D_{2,2}$ (cf. [1], [4], Li [5]).

For $n = 3$, the number of possible combinations is much higher. Several examples are available, but a complete classification would require the answers to all Questions listed in the preceding sections.

So far, very little is known (cf. Ushkho [6], Conti [3]). Here we just limit ourselves to an example and a further question.

EXAMPLE 7.1. The quartics $(x^2+2y^2-1)(2x^2+y^2-1) = c$ are the trajectories of the system

$$\dot{x} = -3y+5x^2y+4y^3, \qquad \dot{y} = 3x-4x^3-5xy^2.$$

The singular points are the four saddles $(\pm\sqrt{3}/3, \pm\sqrt{3}/3)$ and the five centers, all of type D, at 0, $(\pm\sqrt{3}/2, 0)$, $(0, \pm\sqrt{3}/2)$.

QUESTION 7.1. Can a cubic system have more than five centers?

References

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