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On some properties of polynomial functions

by R. Ger (Katowice)

1. Let f be a real-valued function defined on the set R of all real numbers, and let $\Delta_h^p f(x)$ denote the p-th difference of the function f with increment h.

Every solution of the functional equation

$$(1) \qquad \Delta_h^{n+1} f(x) \equiv 0$$

(the identity with respect to x and h) is called a *polynomial function of n-th order* (see [4]).

It is well known that if f fulfilling (1) is continuous, then it is a polynomial of at most n-th degree. It is also known that if f is bounded on a set of positive Lebesgue measure, then it is continuous (see [2]).

McKiernan [3] has proved that every polynomial function of n-th order must be of the form

$$(2) \qquad f(x) = A^0 + A^1(x) + A^2(x) + \ldots + A^n(x),$$

where $A^0 = \text{const}$ and A^i $(i = 1, 2, \ldots, n)$ are the diagonalizations at x of real-valued, symmetric, multi-additive functions $A_i(x_1, x_2, \ldots, x_i)$ of i arguments, i.e.

$$A^i(x) = A_i(x, x, \ldots, x),$$

where $A_i(x_1, x_2, \ldots, x_i)$ are additive in each argument and symmetric with respect to every permutation of the arguments $x_1, x_2, \ldots, x_i$.

Let us notice that for $\lambda \in Q$, where Q denotes the set of all rational numbers, we have $A^i(\lambda x) = \lambda^i \cdot A^i(x)$. This follows from the fact that, for an additive function g and $\lambda \in Q$, $g(\lambda x) = \lambda g(x)$ holds, and from the additivity of A^i in each argument.

It is well known that a functions g defined (in an arbitrary manner) on a Hamel base of the set R has a unique extension to an additive function defined on R.

In general this is not true in the case of polynomial functions. To show this, let us fix a Hamel base of the set R, and take a polynomial function of 2-nd order

$$f(x) = A^1(x)+A^2(x).$$

Let $x = \sum_{\alpha} \lambda_\alpha h_\alpha$, $\lambda_\alpha \in Q$, $h_\alpha \in H$. So we have:

$$A^1(x) = \sum_{\alpha} \lambda_\alpha A_1(h_\alpha); \quad A^2(x) = \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta A_2(h_\alpha, h_\beta),$$

where A_1, A_2 may be defined in an arbitrary manner on H and $H \times H$ respectively, with the only restriction that A_2 is symmetric. Let us put

$$A_2(h_\alpha, h_\beta) \overset{\mathrm{df}}{=} \begin{cases} 1 & \text{for } h_\alpha = h_\beta = h_0, \\ 0 & \text{besides on } H \times H; \end{cases}$$

$$A_1(h_\alpha) \overset{\mathrm{df}}{=} \begin{cases} -1 & \text{for } h_\alpha = h_0, \\ 0 & \text{besides on } H, \end{cases}$$

where h_0 denotes a fixed element of H. Thus $f(h_\alpha) = 0$ for each $h_\alpha \in H$, but if $x = \lambda h_0$, $\lambda \in Q \setminus \{0, 1\}$, then $f(x) = f(\lambda h_0) = \lambda^2 - \lambda \neq 0$.

On the other hand, $f \equiv 0$ is also a polynomial function vanishing on H, which shows that the equality of two polynomial functions on H does not imply the equality of these functions on R.

The following general problem arises: if we have a certain functional equation with the unknown function $\varphi(x)$, which we can shortly note in the form

$$(*) \qquad E(\varphi) = 0$$

and which has to be fulfilled in a certain set X, then we can put the following two questions. What must the set $Z \subset X$ be like in order to ensure that if φ is arbitrarily given on Z, then

(i) there exists an extension of φ, fulfilling $(*)$ on the whole X;

(ii) there exists a unique such extension.

It is obvious that if $Z^* \subset Z$, and the answer to (i) is positive for Z, then it is also positive for Z^*. From this point of view, when we are concerned with (i), it is interesting to find the largest possible set Z. However, if we are interested in (ii), then it may happen that for Z we have a unique extension, but for Z^* we obtain a greater number of extensions, and thus it is interesting to find a possibly "thin" set Z.

Moreover, let us note that, in general, if we know the values of φ at certain points, then equation $(*)$ determines the values of φ at certain new points. Thus the set Z cannot be any set and for the most part it cannot be too "slim".

In the present paper we are concerned with (ii) in connection with equation (1). Namely, for a fixed positive integer n, we construct a set

$Z = Z(n) \subset R$ such that every function defined in an arbitrary manner on Z has a unique extension to a polynomial function of n-th order defined on R.

2. In the sequel n denotes a fixed positive integer. Moreover, we shall use the following notation:

1° if i, k, $i \leqslant k \leqslant n$, are fixed positive integers, then k_i denotes the number of decompositions of k into a sum of i natural summands (two decompositions differing with a succession of different summands are treated as two different decompositions);

2° $s_{mq} = s_{mq}(i, k)$ denotes the q-th summand of m-th decomposition of the number k into the sum of i natural summands, i.e.

$$s_{m1}+s_{m2}+ \quad +s_{mi} = k$$

for $m = 1, 2, \ldots, k_i$;

3° $s_i \stackrel{\mathrm{df}}{=} i_i+(i+1)_i+ \quad +n_i$.

In the sequel we shall consider the following sequences:

$$s_{1q}(i, i); s_{1q}(i, i+1), s_{2q}(i, i+1), \ldots, s_{(i+1)iq}(i, i+1); \ldots; \\ s_{1q}(i, n), s_{2q}(i, n), \ldots, s_{n_iq}(i, n),$$

where $1 \leqslant q \leqslant i$. To simplify the notation we make the following overnumbering:

$$s_{1q}(i, i) \stackrel{\mathrm{df}}{=} t_{1q}, s_{1q}(i, i+1) \stackrel{\mathrm{df}}{=} t_{2q}, s_{2q}(i, i+1) \stackrel{\mathrm{df}}{=} t_{3q}, \ldots, s_{n_iq}(i, n) \stackrel{\mathrm{df}}{=} t_{s_iq}.$$

With the aid of this notation we define (for i henceforth fixed) the following determinant of degree s_i:

$$I_i \stackrel{\mathrm{df}}{=} \det[\lambda_{ip1}^{t_{j1}} \cdot \lambda_{ip2}^{t_{j2}} \cdot \quad \cdot \lambda_{ipi}^{t_{ji}}],$$

where $p = 1, 2, \ldots, s_i$, $j = 1, 2, \ldots, s_i$, whereas λ_{ipq} $(p = 1, 2, \ldots, s_i$, $q = 1, 2, \ldots, i)$ denote rational numbers.

LEMMA 1. *If i is fixed, then the numbers $\lambda_{ipq} \in Q \setminus \{0\}$ may be chosen so that $I_i \neq 0$.*

Proof. We shall give an effective construction of such a determinant. Let $\{2, 3, \ldots, r_i\}$ be the sequence of i successive prime numbers. Let us define the numbers x_j as follows:

$$x_j \stackrel{\mathrm{df}}{=} 2^{t_{j1}} \cdot 3^{t_{j2}} \cdot \ldots \cdot r_i^{t_{ji}}.$$

For $j_1 \neq j_2$, $1 \leqslant j_1 \leqslant s_i$, $1 \leqslant j_2 \leqslant s_i$, we have $x_{j_1} \neq x_{j_2}$. Indeed otherwise we would have

$$\frac{x_{j_1}}{x_{j_2}} = 1 = 2^{t_{j_11}-t_{j_21}} 3^{t_{j_12}-t_{j_22}} \ldots r_i^{t_{j_1i}-t_{j_2i}}$$

and thus

$$t_{j_11} = t_{j_21}, t_{j_2i} = t_{j_22}, \ldots, t_{j_1i} = t_{j_2i}. \tag{3}$$

Let $t_{j_1q} = s_{mq}(i, k)$, $t_{j_2q} = s_{pq}(i, l)$, $q = 1, 2, \ldots, i$, and let us consider two cases:

1° $k \neq l$.

Then $\sum_{q=1}^{i} t_{j_1q} = k \neq l = \sum_{q=1}^{i} t_{j_2q}$ and equalities (3) are impossible;

2° $k = l$.

Then $m \neq p$, since $j_1 \neq j_2$, and we have two different decompositions of the same number k. Thus they must differ at least in the succession of summands, and equalities (3) are not possible, either.

We construct the determinant I_i as follows:

$$I_i \overset{\text{df}}{=} \det[(2^p)^{t_{j1}} \cdot (3^p)^{t_{j2}} \cdot \ldots \cdot (r_i^p)^{t_{ji}}],$$

where $p = 0, 1, 2, \ldots, s_i - 1$; $j = 1, 2, \ldots, s_i$.

Thus I_i is of the form

$$\begin{vmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & & x_{s_i} \\ x_1^2 & x_2^2 & & x_{s_i}^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{s_i-1} & x_2^{s_i-1} & & x_{s_i}^{s_i-1} \end{vmatrix}$$

and from the fact that the numbers x_j, $1 \leqslant j \leqslant s_i$, are pairwise different we infer that $I_i \neq 0$, which completes the proof.

3. For a fixed positive integer i and a fixed Hamel base H of the reals let $\mathscr{H}_i \overset{\text{df}}{=} \bigtimes_{m=1}^{i} H_m$, where $H_m = H$ for $m = 1, 2, \ldots, i$.

In the set $\mathscr{H}_i$ we introduce an equivalence relation r as follows: $(h_1, h_2, \ldots, h_i)\, r\, (h_1', h_2', \ldots, h_i') \Leftrightarrow (h_1, h_2, \ldots, h_i)$ is a permutation of $(h_1', h_2', \ldots, h_i')$. Let $\mathscr{H}_i/r$ denote the set of all equivalence classes.

Now, we choose one element from each equivalence class. Let w_i: $\mathscr{H}_i/r \to \mathscr{H}_i$ denote a function of choice. It easily follows from the definition of A_i that a function φ arbitrarily given on the set $w_i(\mathscr{H}_i/r)$ has a unique extension to a function A_i defined on R^i.

For a linearly independent (over Q) [1] set $B \subset R$ and numbers $\lambda_1, \lambda_2, \ldots, \lambda_l \in Q \setminus \{0\}$ we define the following set:

$$(4) \quad \lambda_1 B + \lambda_2 B + \quad + \lambda_l B$$
$$\overset{\text{df}}{=} \{x \colon x = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_l b_l,\ b_j \in B,\ b_m \neq b_j \text{ for } m \neq j,\ m, j = 1, 2, \ldots, l\}.$$

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[1] Let L be a linear space over a field K. We say that a set $A \subset L$ is linearly independent over K iff every finite subset of A is linearly independent.

In this set we introduce an equivalence relation ρ as follows:

$$(\lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_l b_l)\rho(\lambda_1 b_1' + \lambda_2 b_2' + \ldots + \lambda_l b_l') \Leftrightarrow (b_1, b_2, \ldots, b_l) \text{ is a permutation of } (b_1', b_2', \ldots, b_l').$$

Let $(\lambda_1 B + \lambda_2 B + \quad + \lambda_l B)/\rho$ denote the set of all equivalence classes.

Now, we choose one element from each equivalence class. Let

$$\omega : (\lambda_1 B + \lambda_2 B + \quad + \lambda_l B)/\rho \to (\lambda_1 B + \lambda_2 B + \quad + \lambda_l B)$$

denote a function of choice.

Let us define the set $Z = Z(n, H, \lambda_{ipq}, \omega_{ip})$ as follows:

$$(5) \qquad Z \stackrel{\text{df}}{=} \bigcup_{i=1}^{n} \bigcup_{p=1}^{s_i} \omega_{ip}\{(\lambda_{ip1}H + \lambda_{ip2}H + \quad + \lambda_{ipi}H)/\rho\} \cup \{0\} \ [2]$$

with the following meaning of the symbols:

n — a fixed positive integer;

H — a fixed Hamel base of the set of real numbers;

λ_{ipq} — fixed non-zero rationals, chosen so that the suitable determinant is $I_i \neq 0$ for $i = 1, 2, \ldots, n$, $p = 1, 2, \ldots, s_i$, $q = 1, 2, \ldots, i$ (the possibility of such a choice is guaranteed by Lemma 1);

ω_{ip} — fixed functions of choice (from each equivalence class belonging to $(\lambda_{ip1}H + \lambda_{ip2}H + \quad + \lambda_{ipi}H)/\rho$ one element is chosen), $i = 1, 2, \ldots, n$, $p = 1, 2, \ldots, s_i$.

THEOREM 1. *A function f arbitrarily given on the set Z defined by* (5) *has a unique extension to a polynomial function of n-th order defined on the whole space R of the reals.*

Proof. An arbitrary polynomial function of n-th order is of the form

$$f(x) = A^0 + A^1(x) + \quad + A^n(x),$$

where $A^0 \equiv \text{const}$.

It is sufficient to prove that we can determine the function A_i on the set $w_i(\mathscr{H}_i/r)$ for $i = 1, 2, \ldots, n$.

Since we always have $A^i(0) = 0$ for $i = 1, 2, \ldots, n$, we put $A^0 \stackrel{\text{df}}{=} f(0)$.

Let us take $h_a \in H$. Making $s_1 = n$ expansions of the form $f(\lambda_{1p1}h_a)$, $p = 1, 2, \ldots, s_1$, we obtain the following system of linear equations:

$$f(\lambda_{1p1}h_a) - A^0 = \lambda_{1p1}A^1(h_a) + \lambda_{1p1}^2 A^2(h_a) + \ldots + \lambda_{1p1}^n A^n(h_a).$$

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[2] Let us note that all the summands in the sense of "$\cup$" are pairwise disjoint in virtue of the linear independence of H over Q.

The main determinant of this system is $I_1 \neq 0$.

Repeating this procedure for all $h_a \in H$ we shall determine the values of the functions A^i on H for $i = 1, 2, \ldots, n$. The function A^1 is completely determined, since $H = w_1(\mathscr{H}_1/r)$.

Next, making s_2 expansions of the form $f(\lambda_{2p1}h_a + \lambda_{2p2}h_\beta)$ we take into consideration the sets

$$\omega_{2p}\{(\lambda_{2p1}H + \lambda_{2p2}H)/\rho\}, \quad p = 1, 2, \ldots, s_2.$$

To simplify the notation, let us adopt for fixed numbers $a_1, a_2, \ldots, a_m$, $m \leqslant i$, $a_1 + a_2 + \ldots + a_m = i$, the following symbols:

$$A_i(x_1^{a_1}, x_2^{a_2}, \ldots, x_m^{a_m}) \overset{\text{df}}{=} A_i(\underbrace{x_1, x_1, \ldots, x_1}_{a_1}; \underbrace{x_2, x_2, \ldots, x_2}_{a_2}; \ldots; \underbrace{x_m, x_m, \ldots, x_m}_{a_m}).$$

Thus we shall get

$$\begin{aligned} f(\lambda_{2p1}h_a + \lambda_{2p2}h_\beta) - A^0 - \sum_{i=1}^{n} \lambda_{2p1}^i A^i(h_a) - \sum_{i=1}^{n} \lambda_{2p2}^i A^i(h_\beta) \\ = c_1\lambda_{2p1}\lambda_{2p2}A_2(h_a, h_\beta) + c_2\lambda_{2p1}^2\lambda_{2p2}A_3(h_a^2, h_\beta) + c_3\lambda_{2p1}\lambda_{2p2}^2A_3(h_a, h_\beta^2) + \\ + \ldots + c_{s_2-n+1}\lambda_{2p1}^{n-1}\lambda_{2p2}A_n(h_a^{n-1}, h_\beta) + c_{s_2-n+2}\lambda_{2p1}^{n-1}\lambda_{2p2}^2A_n(h_a^{n-2}, h_\beta^2) + \\ + \ldots + c_{s_2}\lambda_{2p1}\lambda_{2p2}^{n-1}A_n(h_a, h_\beta^{n-1}), \end{aligned}$$

where $p = 1, 2, \ldots, s_2$, $c_1, c_2, \ldots, c_{s_2}$ are positive integers.

The main determinant of this system is $c_1 \cdot c_2 \cdot \ldots \cdot c_{s_2} \cdot I_2 \neq 0$.

Proceeding thus further for all $h_a, h_\beta \in H$, $h_a \neq h_\beta$, we shall determine the function A_2 completely (i.e. on the set of the form $w_2(\mathscr{H}_2/r)$). Moreover, the values of the functions $A_3, A_4, \ldots, A_n$ will be known on certain subsets of the sets of the form $w_3(\mathscr{H}_3/r)$, $w_4(\mathscr{H}_4/r)$, $\ldots$, $w_n(\mathscr{H}_n/r)$ (namely, on those elements of these sets which have at most two different "coordinates").

In the third step the function A_3 is completely determined and the values of $A_4, A_5, \ldots, A_n$ will be known on those elements of the sets $w_4(\mathscr{H}_4/r)$, $w_5(\mathscr{H}_5/r)$, $\ldots$, $w_n(\mathscr{H}_n/r)$ which have at most three different "coordinates".

In the n-th step the situation looks as follows: we already have the functions $A_1, A_2, \ldots, A_{n-1}$ completely determined (i.e. on the sets of the form $w_1(\mathscr{H}_1/r)$, $w_2(\mathscr{H}_2/r)$, $\ldots$, $w_{n-1}(\mathscr{H}_{n-1}/r)$, respectively); however, the values of the function A_n are not yet known on those elements of the set $w_n(\mathscr{H}_n/r)$ which have the "coordinates" pairwise different. We take into consideration the set

$$\omega_{n1}\{(\lambda_{n11}H + \lambda_{n12}H + \ldots + \lambda_{n1n}H)/\rho\}$$

($s_n = n_n = 1$). Taking a fixed element $\lambda_{n11}h_{a_1} + \lambda_{n12}h_{a_2} + \ldots + \lambda_{n1n}h_{a_n}$ (then $(h_{a_1}, h_{a_2}, \ldots, h_{a_n})$ belongs to the above mentioned set $w_n(\mathscr{H}_n/r)$)

we have to consider only one linear equation. The main determinant of this "system" is $I_n = \lambda_{n11}\cdot\lambda_{n12}\cdot\ldots\cdot\lambda_{n1n} \neq 0$.

Continuing this procedure for all these elements we shall determine the function A_n completely.

Thus finally the function f is given completely, which ends the proof.

4. The set Z seems to be "large". However, it depends on the choice of the Hamel base H. Our further considerations will be concerned with the construction of such a base that the set Z generated by it is of measure zero.

At first we shall prove the following

LEMMA 2. *If $k_1, k_2, \ldots, k_m$ are fixed non-zero integers, then there exists a Hamel base H such that the set*

$$X = k_1H+k_2H+\ldots+k_mH\ (^3)$$

is measurable of measure zero.

Proof. Let $N \stackrel{\mathrm{df}}{=} 2(|k_1|+|k_2|+\ldots+|k_m|)+4$ be the base of counting. Let us consider two sets:

$A \stackrel{\mathrm{df}}{=} \{x\in R\colon N\text{-adic expansion of } x \text{ contains the digits 0 and 1 only}\}$,

$B \stackrel{\mathrm{df}}{=} \{x\in R\colon N\text{-adic expansion of } x \text{ does not contain the digit } N/2\}$.

Both A and B are measurable and of measure zero (a certain generalization of the construction of the Cantor set; see [1]).

Moreover, let us note that the set A spans the space R, i.e. the set of all linear combinations of elements of A with rational coefficients yields the whole space R.

Thus, the set A must contain a Hamel base H. It is easily seen that such a base H has the property that

$$X = k_1H+k_2H+\ldots+k_mH \subset B.$$

So X is measurable and of measure zero, which completes the proof of the lemma.

LEMMA 3. *For fixed numbers $\varrho_1, \varrho_2, \ldots, \varrho_m \in Q\setminus\{0\}$ there exists a Hamel base H such that the set*

$$Y \stackrel{\mathrm{df}}{=} \varrho_1H+\varrho_2H+\ldots+\varrho_mH$$

is measurable of measure zero.

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(3) For real numbers α, β and sets $A, B\subset R$ we put

$$\alpha A+\beta B \stackrel{\mathrm{df}}{=} \{x\colon x = \alpha a+\beta b,\ a\in A,\ b\in B\}.$$

Proof. Let $\varrho_i = k_i/M$, where k_i are integers and M is a positive integer, $i = 1, 2, \ldots, m$. Let $Y = \frac{1}{M}X$, where $X = k_1H + k_2H + \ldots + k_mH$ and H is the Hamel base constructed in Lemma 2. The measurability of X is equivalent to the measurability of Y. Moreover, we have $m(Y) = \frac{1}{M}m(X) = 0$ ($m(A)$ denotes here the Lebesgue measure of the set A).

LEMMA 4. *If*

$$A = \varrho_1H + \varrho_2H + \quad + \varrho_mH, \quad B = \varrho_{m_1}H + \varrho_{m_2}H + \ldots + \varrho_{m_k}H,$$
$$\{\varrho_{m_1}, \varrho_{m_2}, \ldots, \varrho_{m_k}\} \subset \{\varrho_1, \varrho_2, \ldots, \varrho_m\} \quad \textit{and} \quad m(A) = 0,$$

then $m(B) = 0$.

Proof. Let

$$h_0 \in H, \quad H^* \overset{\mathrm{df}}{=} H - h_0, \quad a \overset{\mathrm{df}}{=} \sum_{i=1}^{m} \varrho_i h_0, \quad b \overset{\mathrm{df}}{=} \sum_{i=1}^{m_k} \varrho_{m_i} h_0.$$

We have $A - a = \varrho_1H^* + \varrho_2H^* + \ldots + \varrho_mH^*$, $m(A-a) = 0$. Since $0 \in H^*$, we have

$$\varrho_{m_1}H^* + \varrho_{m_2}H^* + \ldots + \varrho_{m_k}H^* \subset \varrho_1H^* + \varrho_2H^* + \ldots + \varrho_mH^*,$$

and thus

$$0 = m(\varrho_{m_1}H^* + \varrho_{m_2}H^* + \quad + \varrho_{m_k}H^*) = m(B-b) = m(B).$$

The construction of the set Z (see (5)) is based upon the choice of $m = \sum_{i=1}^{n} is_i$ rational numbers λ_{ipq}. Let $\{\varrho_1, \varrho_2, \ldots, \varrho_m\}$ denote the set of all rational coefficients which appear in the construction of Z, and take a base H such that the set

$$\varrho_1H + \varrho_2H + \quad + \varrho_mH$$

is measurable of measure zero.

In virtue of Lemma 4 each summand in the sense of "$\cup$" which appears in Z is of measure zero and thus the set Z is of measure zero.

Thus we have proved the following

THEOREM 2. *There exists a set Z of measure zero and such that a function f arbitrarily given on Z has a unique extension to a polynomial function of n-th order defined on the whole space R of the reals.*

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