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Existence of a minimal solution and a maximal solution of a nonlinear elliptic boundary value problem of the fourth order

by JAN BOCHENEK (Kraków)

Franciszek Leja in memoriam

Abstract. This paper is devoted to the existence of a minimal solution and a maximal solution of a class of nonlinear boundary value problems of the fourth order. The main results of this paper are contained in Theorems 1 and 2. The method used in this paper is based on the method of paper [5].

Introduction. In this paper we are concerned with the existence of a minimal and a maximal solution of a class of nonlinear boundary value problems of the fourth order. A minimal or maximal solution of a given boundary value problem may be investigated in two aspects:

1° as a local extremal solution, i.e., a minimal or maximal solution in a fixed domain (e.g., an interval) of a function space;

2° as a global extremal solution, i.e., a minimal or maximal solution to all solutions.

There are many articles which guarantee a minimal or maximal solution in aspect 1° for boundary value problems of the second order (cf. [1], [2], [4]). The problem of the existence of a minimal and a maximal solution in sense 2° for nonlinear boundary value problems of second order is studied in paper [5].

In this paper we shall consider the existence of a minimal and a maximal solution of some class of nonlinear boundary value problems of the fourth order in two above aspects.

Consider a nonlinear elliptic boundary value problem (BVP) of the form

$$(1)\qquad \begin{aligned} (Lu)(x) &= f(x, u(x)) && \text{for } x \in \Omega, \\ (Bu)(x) &= 0 && \text{for } x \in \partial\Omega, \end{aligned}$$

where $L = L_0 L_1$, $B = (B_0, B_1)$ and $f\colon \bar{\Omega} \times R \to R$.

We assume that $\Omega \subset R^N$ is a nonempty bounded domain with boundary of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$, that the differential operators

$$(2) \qquad L_i \varphi = -\sum_{j,k=1}^{N} a^i_{jk} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} + \sum_{k=1}^{N} a^i_k \frac{\partial \varphi}{\partial x_k} + c^i \varphi, \qquad i = 0, 1$$

uniformly elliptic on $\bar{\Omega}$ and that the coefficients a^i_{jk}, a^i_k, $c^i \in C^{2-2i+\alpha}(\bar{\Omega})$ $(i = 0, 1)$ with c^i $(i = 0, 1)$ are nonnegative. We also require that

$$(3) \qquad B_i \varphi = b^i_0 \varphi + b^i_1 \frac{\partial \varphi}{\partial \beta}, \qquad i = 0, 1,$$

where $\partial/\partial\beta$ denotes the directional derivative with respect to an outward pointing, nowhere tangent vector field β on $\partial\Omega$ of class $C^{1+\alpha}$.

Furthermore we suppose that $f \in C^1(\bar{\Omega} \times R)$ and $Bu = 0$ on $\partial\Omega$ means $(B_0 u)(x) = 0$ and $(B_1 L_0 u)(x) = 0$ for $x \in \partial\Omega$. A similar BVP was considered by the author in [3].

1. Notation and preliminary results. In the sequel the following hypothesis is important for our investigations:

HYPOTHESIS H_1. We assume that the coefficients in (3) satisfy the following conditions: b^i_0, $b^i_1 \in C^{1+\alpha}(\partial\Omega)$, $b^i_0(x) \geqslant 0$, $b^i_1(x) \geqslant 0$ and $b^i_0(x) + b^i_1(x) > 0$ for $x \in \partial\Omega$ $(i = 0, 1)$. Moreover, in the case of $b^i_0 \equiv 0$ on $\partial\Omega$ for $i = 0, 1$ (Neumann boundary condition) we suppose that $c^i > 0$ in $\bar{\Omega}$ for $i = 0, 1$.

Hypothesis H_1 ensures that the maximum principle is valid for (L_i, B_i) on Ω, $i = 0, 1$.

As an important consequence of H_1 we obtain the following lemmas:

LEMMA 1 ([2], Theorem 4.3). *The eigenvalue problem* (EVP)

$$(4) \qquad L_i u = \lambda u \quad \text{in } \Omega, \qquad B_i u = 0 \quad \text{on } \partial\Omega,$$

has the smallest eigenvalue $\lambda^i_0 > 0$ *for* $i = 0, 1$.

LEMMA 2 ([2], Theorem 4.4, and [5], Lemma 2). *Let* $v \in C^\alpha(\bar{\Omega})$. *Then for every* $q \in R$ *with* $q < \lambda^i_0$ *the linear* BVP

$$(5) \qquad (L_i - q) u = v \quad \text{in } \Omega, \qquad B_i u = 0 \quad \text{on } \partial\Omega,$$

has exactly one solution u *in* $C^{2+\alpha}(\bar{\Omega})$. *Finally,* $v \geqslant 0$ *implies* $u \geqslant 0$ *for* $i = 0, 1$.

In the sequel we denote by $u = Kv$ the unique solution BVP(5) for $q = 0$ and $i = 1$.

The Schauder a priori estimates imply that the operator K defined above is a continuous linear operator from $C^\alpha(\bar{\Omega})$ to $C^{2+\alpha}(\bar{\Omega})$. This operator K can be extended to a compact linear operator from $C(\bar{\Omega})$ to $C^1(\bar{\Omega})$.

We shall now prove the following

LEMMA 3. *If the function $f \in C^1(\bar{\Omega} \times R)$ and $a < b$ are fixed numbers, then there exists a number $\varrho > 0$ such that*

(6) $$Kf(x, u_1)+\varrho u_1 \leqslant Kf(x, u_2)+\varrho u_2 \quad \text{for all } x \in \bar{\Omega} \text{ and } a \leqslant u_1 \leqslant u_2 \leqslant b.$$

Proof. Since $f \in C^1(\bar{\Omega} \times R)$, there exists an $m > 0$ such that

(7) $$(\partial f/\partial u)(x, u) > -m \quad \text{for all } x \in \bar{\Omega} \text{ and } u \in [a, b].$$

Let us take arbitrary u_1, u_2 such that $a \leqslant u_1 \leqslant u_2 \leqslant b$. From (7) we get

$$f(x, u_1)-f(x, u_2) \leqslant -m(u_1-u_2) \quad \text{for all } x \in \bar{\Omega}.$$

From this by Lemma 2 follows

(8) $$K[f(x, u_1)-f(x, u_2)] \leqslant mK(u_2-u_1) \quad \text{for all } x \in \bar{\Omega}.$$

Let $\varrho > 0$ be a number to be appropriately selected later. From (8) we get

$$K[f(x, u_1)-f(x, u_2)]+\varrho(u_1-u_2) \leqslant mK(u_2-u_1)-\varrho(u_2-u_1) \quad \text{for } x \in \bar{\Omega}.$$

Since the operator K is bounded and $u_2-u_1 \geqslant 0$, then it is readily observed that

$$mK(u_2-u_1)-\varrho(u_2-u_1) \leqslant 0 \quad \text{for } u_1 \leqslant u_2 \text{ if } \varrho \geqslant m\|K\|.$$

Therefore, if we select $\varrho \geqslant m\|K\|$, then inequality (6) holds. The proof of Lemma 3 is complete.

In the sequel by a solution of the BVP(1) we always mean a function $u \in C^{4+\alpha}(\bar{\Omega})$ which satisfies (1) identically.

We shall say that a function Φ is a lower solution of the BVP(1) if $\Phi \in C^{4+\alpha}(\bar{\Omega})$ and

$$(L\Phi)(x) \leqslant f(x, \Phi(x)) \quad \text{for } x \in \Omega,$$
$$(B\Phi)(x) \leqslant 0 \quad \text{for } x \in \partial\Omega,$$

where $(Bu)(x) \leqslant 0$ for $x \in \partial\Omega$ means $(B_0 u)(x) \leqslant 0$ and $(B_1 L_0 u)(x) \leqslant 0$ for $x \in \partial\Omega$. An upper solution Ψ of the BVP(1) is defined similarly but with the inequality signs reversed.

2. Existence of local extremal solutions of the BVP(1). A solution u of the BVP(1) is said to be a *local minimal* (or *local maximal*) solution in a domain $D \subset C^{4+\alpha}(\bar{\Omega})$ if $u \in D$ and for every solution u^* of the BVP(1) in D the inequality $u \leqslant u^*$ (respectively $u^* \leqslant u$) is satisfied.

We shall prove the following

THEOREM 1. *Let Ψ be an upper solution and Φ a lower solution of the* BVP(1), *with $\Phi \leqslant \Psi$ on $\bar{\Omega}$, such that $(B\Phi)(x) = (B\Psi)(x) = 0$ for $x \in \partial\Omega$. Then there exist a minimal solution u and a maximal solution v of the* BVP(1) *in the interval $[\Phi, \Psi] \subset C^{4+\alpha}(\bar{\Omega})$.*

Proof. Choose a number ϱ such that inequality (6) holds where

$a = \min\{\Phi(x): x \in \bar{\Omega}\}$ and $b = \max\{\Psi(x): x \in \bar{\Omega}\}$. Using the definition of the operator K, we write the BVP(1) in the form

$$(9) \qquad \begin{aligned} (L_0 u)(x) &= Kf(x, u(x)) && \text{for } x \in \Omega, \\ (B_0 u)(x) &= 0 && \text{for } x \in \partial\Omega. \end{aligned}$$

Let us observe that each upper solution (lower solution) of the BVP(1) satisfying $(Bu)(x) = 0$ for $x \in \partial\Omega$ is an upper solution (lower solution) of the BVP(9) satisfying $(B_0 u)(x) = 0$ for $x \in \partial\Omega$, but not conversely.

We define a nonlinear operator T by $v = Tu$ if

$$(10) \qquad (L_0 + \varrho)v = Kf(x, u) + \varrho u \quad \text{in } \Omega, \qquad B_0 v = 0 \quad \text{on } \partial\Omega.$$

Since the function F defined by $F(x, u) = Kf(x, u) + \varrho u$ is increasing in u by choosing the number ϱ, it is easy to show that the operator T is monotone, i.e., if $u_1 \leqslant u_2$, then $Tu_1 < Tu_2$ (see [4]).

Now we define two sequences,

$$u_0 = \Phi \quad \text{and} \quad u_n = Tu_{n-1}, \quad n \in N,$$

and

$$v_0 = \Psi \quad \text{and} \quad v_n = Tv_{n-1}, \quad n \in N.$$

Using the monotonicity of the operator T and the maximum principle for (L_0, B_0) in Ω, we can easily prove that

$$(11) \qquad \Phi \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant v_2 \leqslant v_1 \leqslant \Psi.$$

Moreover, the sequences $\{u_n\}$ and $\{v_n\}$ both converge uniformly in $\bar{\Omega}$ to the functions u and v, respectively. The functions u and v belong to $C^{4+\alpha}(\bar{\Omega})$ and they are solutions of BVP(9) (see [4]). Since each solution of BVP(9) belonging to $C^{4+\alpha}(\bar{\Omega})$ is a solution of BVP(1) and vice versa, then the functions u and v defined above are solutions of BVP(1).

We shall prove that u is the minimal solution and v is the maximal solution of BVP(1) in the interval $[\Phi, \Psi]$. Indeed, let u^* be a solution of BVP(1) such that $\Phi \leqslant u^* \leqslant \Psi$. So, u^* is a solution of BVP(9). We shall prove that $u \leqslant u^*$. The proof of the inequality $u^* \leqslant v$ is quite similar. By assumption we have $\Phi = u_0 \leqslant u^*$. Let us suppose $u_{n-1} \leqslant u^*$. Since

$$\begin{aligned} (L_0 + \varrho)u_n &= Kf(x, u_{n-1}) + \varrho u_{n-1} && \text{in } \Omega, \\ (L_0 + \varrho)u^* &= Kf(x, u^*) + \varrho u^* && \text{in } \Omega, \end{aligned}$$

and

$$B_0 u_n = B_0 u^* = 0 \quad \text{on } \partial\Omega,$$

we have

$$(L_0 + \varrho)(u_n - u^*) = K[f(x, u_{n-1}) - f(x, u^*)] + \varrho(u_{n-1} - u^*) \quad \text{in } \Omega$$

and

$$B_0(u_n - u^*) = 0 \quad \text{on } \partial\Omega.$$

Hence by (6) (where $a = \min \Phi$ and $b = \max \Psi$ on $\bar{\Omega}$) and by the maximum principle we have $u_n \leqslant u^*$. From this by induction we get $u_n \leqslant u^*$ for each $n \in N$. Therefore $\lim u_n = u \leqslant u^*$.

3. Existence of global extremal solutions of the BVP(1). In this section we shall need the following

HYPOTHESIS H_2 (see [5]). $\exists p, s \in R$ with $p < \lambda_0^0 \lambda_0^1$ and $s \geqslant 0$: $\forall \alpha, \beta \in R$ with $\alpha \leqslant \beta$, $\forall x \in \bar{\Omega}$: $f(x, \beta) - f(x, \alpha) \leqslant p(\beta - \alpha) + s$.

LEMMA 4. *If the operator K is defined as in Section 2 and $\mu \in R$ with $\mu < \lambda_0^1$, then*

$$\forall v \in C^{\alpha}(\bar{\Omega}):\ v \geqslant 0, \quad v - \mu K v \geqslant 0. \tag{12}$$

Proof. Suppose that there exists a $v_0 \geqslant 0$ such that $v_0 - \mu K v_0 < 0$. Let $K v_0 = u_0$. We have, by Lemma 2, $u_0 \geqslant 0$. By the definition of the operator K we get

$$v_0 = L_1 u_0 \quad \text{and} \quad B_1 u_0 = 0,$$

and so

$$(L_1 - \mu) u_0 < 0 \quad \text{and} \quad B_1 u_0 = 0.$$

From this, by Lemma 2, we get $u_0 < 0$, which contradicts $u_0 \geqslant 0$.

After these preparations we shall prove the following

THEOREM 2. *Suppose that the* BVP(1) *satisfies hypotheses* H_1 *and* H_2. *Then the* BVP(1) *possesses a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ with respect to the whole space $C^{4+\alpha}(\bar{\Omega})$ (i.e.,* BVP(1) *possesses global extremal solutions $\underline{u}$ and $\bar{u}$).*

Proof. First we construct a lower solution Φ and an upper solution Ψ with $\Phi \leqslant \Psi$ for the BVP(9) by solving two linear BVP's of form (5) for $i = 0$. For arbitrary $\tilde{u} \in C^{2+\alpha}(\bar{\Omega})$ we define $\Phi = \Phi(\tilde{u}) \in C^{2+\alpha}(\bar{\Omega})$ by

$$\begin{aligned} (L_0 - q)(\tilde{u} - \Phi) &= (L_0 \tilde{u} - K f(x, \tilde{u}))^+ + Ks && \text{in } \Omega, \\ B_0(\tilde{u} - \Phi) &= (B_0 \tilde{u})^+ && \text{on } \partial\Omega, \end{aligned} \tag{13}$$

and $\Psi = \Psi(\tilde{u}) \in C^{2+\alpha}(\bar{\Omega})$ by

$$\begin{aligned} (L_0 - q)(\Psi - \tilde{u}) &= (-L_0 \tilde{u} + K f(x, \tilde{u}))^+ + Ks && \text{in } \Omega, \\ B_0(\Psi - \tilde{u}) &= (-B_0 \tilde{u})^+ && \text{on } \partial\Omega, \end{aligned} \tag{14}$$

where $q \in R$ and $p/\lambda_0^1 < q < \lambda_0^0$, and p is from H_2. From Lemma 2 it follows that these definitions are correct and that $\Phi(\tilde{u}) \leqslant \tilde{u} \leqslant \Psi(\tilde{u})$ is true. We shall prove that $\Phi(\tilde{u})$ is a lower solution and $\Psi(\tilde{u})$ is an upper solution of BVP(9). Hypothesis H_2 implies that

$$f(x, \tilde{u}) - f(x, \Phi) \leqslant p(\tilde{u} - \Phi) + s \quad \text{for } x \in \Omega$$

and

$$f(x, \Psi) - f(x, \tilde{u}) \leqslant p(\Psi - \tilde{u}) + s \quad \text{for } x \in \Omega.$$

It follows that

$$-L_0\Phi+Kf(x,\Phi)\geqslant L_0\tilde{u}-Kf(x,\tilde{u})-L_0\Phi+Kf(x,\Phi)-\big(L_0\tilde{u}-Kf(x,\tilde{u})\big)^+$$
$$\geqslant (L_0-q)(\tilde{u}-\Phi)-\big(L_0\tilde{u}-Kf(x,\tilde{u})\big)^+-Ks+(q-pK)(\tilde{u}-\Phi)$$
$$=(q-pK)(u-\Phi)\geqslant 0 \quad \text{in } \Omega$$

and

$$B_0\Phi=B_0u-(B_0u)^+\leqslant 0 \quad \text{on } \partial\Omega.$$

Analogously, we prove that

$$L_0\Psi\geqslant Kf(x,\Psi) \quad \text{in } \Omega, \qquad B_0\Psi\geqslant 0 \quad \text{on } \partial\Omega.$$

By Theorem 2.3.1 of [4] the BVP(9) possesses a solution $u^*\in[\Phi,\Psi]$.

We now prove that every solution u of BVP(9) lies in the interval $[\Phi(u^*),\Psi(u^*)]$. Indeed, let $v\in C^{2+\alpha}(\bar{\Omega})$ be an arbitrary solution of (9). Then we shall show that $v\in[\Phi(u^*),\Psi(u^*)]$. First we define an operator

$$A\colon C_0^{2+\alpha}(\bar{\Omega})\to C^{\alpha}(\bar{\Omega}) \quad \text{by } Au=(L_0-q)u,$$

where

$$C_0^{2+\alpha}(\bar{\Omega}):=\{u\in C^{2+\alpha}(\bar{\Omega})\colon B_0u=0\}.$$

Lemma 2 implies that A^{-1} exists and is positive (i.e., $A^{-1}w\geqslant 0$ if $w\geqslant 0$). We define

$$w_1:=v-A^{-1}[A(v-u^*)]^+-A^{-1}Ks,$$
$$w_2:=u^*+A^{-1}[A(v-u^*)]^++A^{-1}Ks.$$

From this and from the inequality $v-u^*\leqslant A^{-1}[A(v-u^*)]^+$ it follows that $v,u^*\in[w_1,w_2]$.

That w_1 is a lower solution of BVP(9) follows from

$$-L_0w_1+Kf(x,w_1)=L_0v-Kf(x,v)-L_0w_1+Kf(x,w_1)$$
$$\geqslant (L_0-q)(v-w_1)+(q-pK)(v-w_1)-Ks$$
$$\geqslant A(v-w_1)-Ks=[A(v-u^*)]^+\geqslant 0 \quad \text{in } \Omega.$$

Analogously, we prove that

$$L_0w_2\geqslant Kf(x,w_2) \quad \text{in } \Omega.$$

Since $B_0w_1=B_0w_2=0$ on $\partial\Omega$, it follows that w_1 is a lower solution and w_2 is an upper solution of BVP(9).

Theorem 1 implies that there exist solutions u_1 and v_1 with $u_1\leqslant u^*$, $u_1\leqslant v$ and $v_1\geqslant u^*$, $v_1\geqslant v$. Moreover,

$$0=A^{-1}[L_0u^*-Kf(x,u^*)-L_0u_1+Kf(x,u_1)]\geqslant A^{-1}A(u^*-u_1)-A^{-1}Ks,$$
$$0=A^{-1}[L_0v_1-Kf(x,v_1)-L_0u^*+Kf(x,u^*)]\geqslant A^{-1}A(v_1-u^*)-A^{-1}Ks.$$

Let us observe that if $u \in C^{2+\alpha}(\bar{\Omega})$ is a solution of BVP(9), then the lower solution $\Phi(u)$ and the upper solution $\Psi(u)$ defined by (13) and (14), respectively, satisfy the inequalities

$$\Phi(u) \leqslant u - A^{-1}Ks \quad \text{and} \quad \Psi(u) \geqslant u - A^{-1}Ks.$$

Hence, we finally get

$$\Phi(u^*) \leqslant u^* - A^{-1}Ks \leqslant u_1 \leqslant v \leqslant v_1 \leqslant u^* + A^{-1}Ks \leqslant \Psi(u^*).$$

This inequality implies that every solution v of BVP(9) belongs to the interval $[\Phi(u^*), \Psi(u^*)]$. In this case the minimal solution $\underline{u}$ and the maximal solution $\bar{u}$ in $[\Phi(u^*), \Psi(u^*)]$, which exist by Theorem 1, are, respectively, the minimal and the maximal solutions with respect to the whole space $C^{2+\alpha}(\bar{\Omega})$.

Theorem 1 implies that the minimal solution $\underline{u}$ and the maximal solution $\bar{u}$ of the BVP(9) belong to the space $C^{4+\alpha}(\bar{\Omega})$ and are solutions of the BVP(1). It is easy to see that $\underline{u}$ is the minimal solution and $\bar{u}$ is the maximal solution of the BVP(1) with respect to the whole space $C^{4+\alpha}(\bar{\Omega})$. The proof of Theorem 2 is complete.

Remark 1. The proof of Theorem 2 is based on the proof of the analogous Theorem 1 from paper [5], concerning the BVP of the second order.

Also the next theorem generalizes an analogous theorem for the BVP of the second order (see [5], Theorem 2).

Let us denote by BVP(1_i) ($i = 1, 2$) a boundary value problem of the form

$$(1_i) \qquad \begin{aligned} (Lu)(x) &= f_i(x, u(x)) && \text{for } x \in \Omega, \\ (Bu)(x) &= 0 && \text{for } x \in \partial\Omega. \end{aligned}$$

THEOREM 3. *Suppose that the* BVP's (1_1) *and* (1_2) *satisfy Hypothesis* H_1. *We denote by* $\underline{u}_i$ *and* $\bar{u}_i$ *respectively, the minimal and the maximal solutions of the* BVP(1_i) *if Hypothesis* H_2 *is satisfied for* $i = 1$ *or* $i = 2$. *If* H_2 *is satisfied for the* BVP(1_1) *and*

$$(15) \qquad \forall x \in \bar{\Omega}\ \forall w \in C^{4+\alpha}(\bar{\Omega})\ f_1(x, w) \leqslant f_2(x, w),$$

then $\underline{u}_1 \leqslant u$ *for all solutions* u *of the* BVP(1_2).

If H_2 *is satisfied for the* BVP(1_2) *and* (15) *holds, then* $u \leqslant \bar{u}_2$ *for all solutions* u *of the* BVP(1_1).

Proof. We only consider the case where H_2 is satisfied for the BVP(1_1). The case where H_2 is satisfied for the BVP(1_2) is analogous. Let u be an arbitrary solution of the BVP(1_2). Then u is a solution of the BVP(9_2). We construct a lower solution Φ and an upper solution Ψ for the BVP(9_1) as in the proof of Theorem 2 with $\tilde{u} := u$. We have $\Phi \leqslant u \leqslant \Psi$. Since

$$L_0 u - Kf_1(x, u) \geqslant L_0 u - Kf_2(x, u) = 0 \quad \text{in } \Omega$$

and

$$B_0 u = 0 \quad \text{on } \partial\Omega,$$

u is an upper solution of BVP(9_1). Then it follows from Theorem 2.3.1 of [4] that there is a solution u^* of BVP(9_1) in $[\Phi, u]$. Since u^* is also solution of the BVP(1_1), we obtain $\underline{u}_1 \leqslant u^* \leqslant u$. This yields Theorem 3.

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