

Oscillatory solutions of second order ordinary differential equations

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While investigating the behavior of quickly oscillating solutions of second order ordinary differential equations, we construct a counter-example by using the Tietze Extension theorem. This method seems useful for a variety of problems which ask questions like, "Is there an equation having a solution with a specified behavior?" The Tietze extension theorem has been used by other authors in the construction of examples for differential equations. The point of interest in our application is to show that the set on which the right-hand side is originally defined is a closed set.

1. Consider a differential equation

$$(1) \quad x^{(n)} = f(x, x', \dots, x^{(n-1)}),$$

where f is real valued function defined in the whole space R^n . A solution $x(t)$ of equation (1) will be called *quickly oscillating* if there is a sequence of points $\{t_i\}$ such that

$$(2) \quad x(t_i) = 0$$

and

$$(3) \quad \lim_{i \rightarrow \infty} t_i = \infty, \quad t_{i+1} > t_i, \quad \lim_{i \rightarrow \infty} (t_{i+1} - t_i) = 0.$$

In [1], [2] proofs have been given showing that if f satisfies some regularity conditions, for example the Lipschitz condition

$$(4) \quad |f(x_0, x_1, \dots, x_{n-1}) - f(y_0, y_1, \dots, y_{n-1})| \leq \sum_{i=0}^{n-1} L_i |x_i - y_i|,$$

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then every quickly oscillating solution of equation (1) converges to zero as $t \rightarrow \infty$. Using the well-known de la Vallée Poussin theorem [5], it is easy to obtain an estimate for such a solution; namely, setting $C = |f(0, \dots, 0)|$ and $h_k = \sup\{t_{i+n-1} - t^i : i \geq k\}$ we have

$$(5) \quad \left(1 - \sum_{i=0}^{n-1} \frac{L_i}{(n-i)!} h_k^{n-i}\right) |x(t)| \leq \frac{h_k^n}{n!} C \quad \text{for } t \geq t_k.$$

For the second order differential equation

$$(6) \quad x'' = f(x, x')$$

in virtue of Opial's inequalities [2], [4], (5) can be replaced by a more precise

$$(7) \quad \left(1 - \frac{L_0}{4} h_k - \frac{L_1}{\pi^2} h_k^2\right) |x(t)| \leq \frac{h_k^2}{2\pi} C.$$

It is easy to see that estimate (5) and (7) are non-trivial only for h_k sufficiently small and the larger the L_i are, the smaller h_k should be. Therefore, the interesting problem is the behavior of quickly oscillating solutions for non-Lipschitz functions f .

For equation (6) we can prove the following

THEOREM. *If the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, then for every quickly oscillating solution of (6) we have either*

$$(8) \quad \limsup_{t \rightarrow \infty} |x(t)| = \infty,$$

or

$$(9) \quad \limsup_{t \rightarrow \infty} |x(t)| = 0.$$

Proof. Suppose that (8) does not hold. Let

$$K = \sup\{|x(t)| : t \geq 0\}, \quad M = \sup\{|f(x, y)| : |x| \leq K, |y| \leq 1\}.$$

For every i there is a point $v_i \in (t_i, t_{i+1})$ such that

$$|x(v_i)| = \max\{|x(t)| : t \in [t_i, t_{i+1}]\}, \quad x'(v_i) = 0.$$

Setting $u_i^0 = \sup\{t \leq v_i : |x'(t)| > 1\}$ and $u_i = \max\{t_i, u_i^0\}$, we have for $u \in [u_i, v_i]$

$$\begin{aligned} |x'(u)| &= |x'(u) - x'(v_i)| \leq (v_i - u_i) \max\{|x''(t)| : t \in [u_i, v_i]\} \\ &\leq (t_{i+1} - t_i) \max\{|f(x(t), x'(t))| : t \in [u_i, v_i]\} \\ &\leq (t_{i+1} - t_i) M. \end{aligned}$$

If the integer i is sufficiently large ($i \geq i_0$) we have $(t_{i+1} - t_i)M < 1$. Consequently $u_i = t_i$ and $|x'(t)| \leq 1$ for $t \in [t_i, v_i]$. From this it follows immediately that

$$|x(v_i)| = |x(v_i) - x(t_i)| \leq (v_i - t_i) \leq (t_{i+1} - t_i) \quad \text{for } i \geq i_0$$

which completes the proof.

We now show that for the differential equation (1) with $n \geq 3$ the theorem is not true.

2. We ask if there exists an $f: R^3 \rightarrow R$ such that for $n = 3$, equation (1) has a quickly oscillating solution $x(t)$ which satisfies neither (8) or (9). Choose such a specific quickly oscillating, namely $x(t) \stackrel{\text{def}}{=} \sin t^2$ for $t \geq 0$. To define f , we first show f can be defined on a closed set which includes the trajectory $S \subset R^3$, where $S \stackrel{\text{def}}{=} \{(x(t), x'(t), x''(t)) : t \geq 0\}$, i.e.,

Claim 1. S is closed. Let

$$z(t) \stackrel{\text{def}}{=} (x(t), x'(t), x''(t)) = (\sin t^2, 2t \cos t^2, 2 \cos t^2 - 4t^2 \sin t^2).$$

It suffices to show that $\|z(t)\| \rightarrow \infty$ as $t \rightarrow \infty$. If for some n $t^2 \in [n\pi + \pi/4, n\pi + 3\pi/4]$, then $|\sin t^2| \geq \sqrt{2}/2$ and

$$|x''(t)| \geq 2t^2 \sqrt{2} - 2.$$

If $t^2 \in [n\pi - \pi/4, n\pi + \pi/4]$ for some n , $2|\cos t^2| \geq \sqrt{2}$ and $|x'(t)| \geq t\sqrt{2}$. Hence $\|z(t)\|^2 \geq x'(t)^2 + x''(t)^2 \rightarrow \infty$ as $t \rightarrow \infty$, so S is closed.

Claim 2. The function $z: [0, \infty) \rightarrow S$ is a homeomorphism. Since $\|z(t)\| \rightarrow \infty$ as $t \rightarrow \infty$, it is sufficient to show that z is one-to-one. Choose t_1 and $t_2 \geq 0$, $t_1 \neq t_2$. Suppose $z(t_1) = z(t_2)$; hence $\sin t_1^2 = \sin t_2^2 \stackrel{\text{def}}{=} \sigma$. If $\sigma = \sin t_1^2 = 0$, then $|x'(t_1)| = |2t_1| \neq |2t_2| = |x'(t_2)|$. If $\sigma \neq 0$,

$$x''(t_1) - x''(t_2) = [4t_2^2 - 4t_1^2]\sigma \neq 0,$$

contradicting the assumption $z(t_1) = z(t_2)$, proving claim 2. Write the inverse of z as $T: S \rightarrow [0, \infty)$.

Define $f: S \rightarrow R$ as $x^{(3)} \circ T$. Then for $t \geq 0$

$$f(x(t), x'(t), x''(t)) = x^{(3)}(T(x(t), x'(t), x''(t))) = x^{(3)}(t),$$

so x is a solution of (1). To define $f: R^3 \rightarrow R$ apply the following version of the Tietze Extension Theorem [6]: *Let S be a closed subset of a metric space E and let $f: S \rightarrow R$ be continuous. Then f may be defined on $E \setminus S$ so that $f: E \rightarrow R$ is continuous.*

Extend the definition of f to all of R^3 by the above theorem. The construction is now complete.

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