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A contribution to the connection of V. Hlavatý

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Abstract. For a given differentiable manifold M we construct some bundles, e.g. bundles of pseudoscalars, pseudovectors, pseudotensors, etc. We define an angular Weyl metric to be a certain field of pseudotensors. Then we define a Hlavatý connection which is consistent with this angular metric just like a Levi-Civita connection is consistent with a Riemannian metric. Further we introduce a pseudotensor of curvatures and a pseudoscalar curvature which indicate analogies with the corresponding notions of the Riemannian geometry. More analogies are obtained by investigations of submanifolds in conformal spaces. In final remarks there is given a criticism of the traditional approach to fundamental notions related to Weyl structures.

In this note we present an intrinsic approach to the fundamental notions related to Weyl structures on manifolds [11]. Connections introduced by V. Hlavatý [5], [6] are consequently treated as a special case of the classical infinitesimal connections [8]. Further we indicate some analogies to the differential geometry of surfaces in Euclidean spaces.

We shall use the following notations:

R is the additive group of reals,

$\boldsymbol{R}$ is the vector space of reals (thus $\boldsymbol{R}$ is the Lie algebra of R),

$R_{n,n}$ is the ring of all matrices of the form $[a_{kh}]_{k\leqslant n, h\leqslant n}$,

R^n and $\boldsymbol{R}^n$ are respectively the Cartesian products of n copies of the structures indicated above,

$GL(R^n)$ is the real linear group of dimension n,

$GL(\boldsymbol{R}^n)$ is the Lie algebra of $GL(R^n)$.

We respect the Einstein convention on summation.

We take into considerations the direct group product $R\times GL(R^n)$. We define two left actions, λ and Λ, as follows:

$$(1)\quad \lambda\colon \big(R\times GL(R^n)\big)\times R^n\to R^n,\qquad \big((t,[A^i_j]),[v^k]\big)\mapsto [e^t A^k_j v^j];$$

$$(2)\quad \Lambda\colon \big(R\times GL(R^n)\big)\times R_{n,n}\to R_{n,n},\qquad \big((t,[A^i_j]),[a_{kh}]\big)\mapsto [e^{2t}A^i_k A^j_h a_{ij}].$$

Let M be a differentiable manifold of dimension n, $n \geqslant 2$. Let $L(M)$ denote a bundle of linear frames over M. We denote by $W(M)$ a new bundle which is obtained from $L(M)$ by producting each fibre by R. Thus $W(M)$ is a principal bundle over M and the direct product $R \times GL(R^n)$ considered above is a structure group of $W(M)$. The corresponding canonical right action will be noted simply by a dot. In order to define a Weyl structure on the whole of M we have to define fields of pseudovectors and of some pseudotensors. For this purpose we consider the products $W(M) \times R^n$ and $W(M) \times R_{n,n}$ and two relations (equivalences), ϱ and ϱ_2 on these products. We set

$\big((s, r), v\big) \varrho \big((\bar{s}, \bar{r}), \bar{v}\big) \Leftrightarrow$ there exists $(t, A) \in R \times GL(R^n)$ such that $\bar{s} = s + t$, $\bar{r} = r \cdot A$ and $\bar{v} = \lambda(-t, A^{-1} \cdot v)$,

$\big((s, r), a\big) \varrho_2 \big((\bar{s}, \bar{r}), \bar{a}\big) \Leftrightarrow$ there exists $(t, A) \in R \times GL(R^n)$ such that $\bar{s} = s + t$, $\bar{r} = r \cdot A$ and $\bar{a} = \Lambda(-t, A^{-1} \cdot a)$.

Thus the quotient manifolds $W(M) \times R^n / \varrho$ and $W(M) \times R^n / \varrho_2$ have natural structures of fibre bundles which are associated with $W(M)$. We denote these bundles by $Y(M)$ and by $Y_2(M)$, respectively.

Definition 1. Any cross-section $M \to Y(M)$ will be called a *field of pseudovectors*. Any cross-section $M \to Y_2(M)$ will be called a *pseudotensor* of type 0/2. (We do not specificate weights of the quantities in question because we shall make use of those which are just defined.)

Remark. Mr. O. Kowalski in Praha has kindly informed me that there exist isomorfisms of $Y(M)$ onto the common tangent bundle $T(M)$ over m. On the other hand there exists no "natural" way to check and fix such an isomorfism. If j_1 and j_2 are two such isomorfisms from $Y(M)$ onto $T(M)$, then $j_1 \circ j_2^{-1}$ is an automorfism of $T(M)$, namely, there exists a scalar field s such that $j_1 \circ j_2^{-1}(v) = sv$ holds for any cross-section v of $T(M)$. Pseudotensors may be considered analogously.

Proposition 1. *If there is given a field of pseudotensor, b, then there is defined an associated bilinear mapping $Y(M) \times Y(M) \to \boldsymbol{R}$. We shall denote this mapping by the same symbol b.*

Proof. We present this mapping in local coordinates. Given any coordinate neighbourhood, $\mathcal{U}$, we fix some field of linear frames on $\mathcal{U}$. Then we complete this field of frames to a cross-section $\mathcal{U} \to W(M)$ by prefixing any differentiable function, i.e. a cross-section $\mathcal{U} \to U \times R$. If $\underset{1}{a}$ and $\underset{2}{a}$ are two pseudovector fields, we can relate their restrictions $\underset{1}{a} | \mathcal{U}$ and $\underset{2}{a} | \mathcal{U}$ to those frames. If $[\underset{2}{a}^i]$, $[\underset{2}{a}^i]$ and $[b_{ij}]$ are the corresponding matrices of coordinates, then the value

$$b(\underset{1}{a}, \underset{2}{a}) | \mathcal{U} := b_{ij} \underset{1}{a}^i \underset{2}{a}^j \tag{3}$$

is invariant under the action of the structure group. This follows immediately from (1) and (2).

Remark. Formula (3) may be applied in the case when $[\underset{1}{a}^i]$ and $[\underset{2}{a}^i]$ are matrices of coordinates of some vectors. In this case the result is not a scalar field but it is some object which transforms under a change of coordinates as follows

$$b(\underset{1}{a}, \underset{2}{a}) \mapsto e^{2t} b(\underset{1}{a}, \underset{2}{a}). \tag{4}$$

DEFINITION 2. A field of pseudotensors of type 0/2 will be called a *Weylian metric* if for each pair $(\underset{1}{a}, \underset{2}{a})$ of pseudovectors there holds

(i) $g(\underset{1}{a}, \underset{1}{a}) \geqslant 0$ except the case $\underset{1}{a} = 0$,

(ii) $g(\underset{1}{a}, \underset{2}{a}) = g(\underset{2}{a}, \underset{1}{a})$.

PROPOSITION 2. *If there is given a Weylian metric on the manifold M, then there is well defined an angular metric. Namely we may assume*

$$\cos(v, w) = g(v, w)/\sqrt{g(v, v)\, g(w, w)}$$

for arbitrary vectors v and w.

We pass to connections in $W(M)$. We follow the fundamental notions of [8]. We have to deal with holonomic fields of frames. Let $(\mathscr{U}, x)$ be a local chart on M, i.e. $\mathscr{U}$ is open in M and x: $\mathscr{U} \to R^n$. Then an associated holonomic field of frames on $\mathscr{U}$ is a tuple $[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ of vector fields on $\mathscr{U}$, $\boldsymbol{x}_j(p)$ being a vector which sends any differentiable scalar function σ to $(\partial_j \sigma \circ x^{-1}) \circ x(p)$. (Using the traditional notation we should write $(\partial/\partial x^j)|_{x(p)}$ instead of $\boldsymbol{x}_j(p)$.) Then we complete this field of linear frames to a local frame field in $W(M)$ by prefixing a differentiable scalar function θ: $\mathscr{U} \to R$. In such a way we obtain a local field of completed frames

$$p \mapsto [\theta(p), \boldsymbol{x}_1(p), \ldots, \boldsymbol{x}_n(p)].$$

If there is given a connection C in $W(M)$, then the operator of covariant differentiation ∇^C is determined uniquely, cf. [1], [10]. The components Γ of C which are associated with the above local field of frames may be found from the following decompositions:

$$\begin{aligned} \nabla_i^C \theta &= \Gamma_i, \\ \nabla_i^C \boldsymbol{x}_j &= \Gamma_{ij}^h \boldsymbol{x}_h. \end{aligned} \tag{5}$$

Thus $\nabla^C \theta$ is different from $\partial_i(\theta \circ x^{-1})$ in general.

The following proposition follows directly from the general connection theory as well as from formula (5), PROPOSITION 3. *Suppose that we have another field of frames on $\mathcal{U}$, say $[\theta, x_1, \ldots, x_n]$, which is related with the previous one by the formulas*

$$\bar{\theta}(p) = \theta(p) + t(p),$$

$$\bar{x}_i(p) = A_i^j(x(p))\, x_j(p),$$

where $(t, [A_i^j])$ is some mapping from $R\times\mathcal{U}$ to $R\times GL(R^n)$. Then the associated components of the connection C are transformed as follows:

(6) $\quad \Gamma_{\bar{k}} = A_{\bar{k}}^h(\Gamma_h + \partial_h t),$

$[\Gamma_{jk}^i]$ *are transformed according to the rule of transformation of the object of linear connection.*

The general theory of covariant differentiation yields directly the following propositions:

PROPOSITION 4. *A local expression for the covariant derivatives of a pseudovector field a is*

$$\nabla_i^C a^k = \partial_i a^k - \Gamma_i a^k + \Gamma_{ij}^k a^j.$$

PROPOSITION 5. *The local expressions for the covariant derivative of any field b of pseudovector of type $0/2$ is*

$$\nabla_i^C b_{kh} = \partial_i b_{kh} + 2\Gamma_i b_{kh} - \Gamma_{ik}^j b_{jh} - \Gamma_{ih}^j b_{kj}.$$

DEFINITION 3. If v is a vector field, then we denote by $\nabla_v^C a$ and, respectively, by $\nabla_v^C b$, the corresponding directional covariant derivatives.

PROPOSITION 6. *$\nabla_v^C a$ is a pseudovector field and $\nabla_v^C b$ is a pseudotensor field of the same type.*

We distinguish some connections which were been introduced by V. Hlavatý [5]. From now on we assume that there is given a pseudotensor g, which defines a Weylian metric on M. Then we assume the following

DEFINITION 4. A *Hlavatý connection* is an infinitesimal connection, D, in $W(M)$ which satisfies the following axioms:

(i) the identity

$$\partial_v g(\underset{1}{a}, \underset{2}{a}) = g(\nabla_v^D \underset{1}{a}, \underset{2}{a}) + g(\underset{1}{a}, \nabla_v^D \underset{2}{a})$$

holds for any vector field v and any pseudovectors $\underset{1}{a}, \underset{2}{a}$;

(ii) for each local holonomic field of frames x we have

$$\nabla_{x_i}^D x_j = \nabla_{x_j}^D x_i.$$

THEOREM 1. *If a Weyl structure (M, g) is given, then there exists a bundle of Hlavatý connections in $W(M)$. The dimension of this bundle is $n = \dim M$.* (Cf. [9].)

Proof. Propositions 4, 5 and axioms (i), (ii) of Definition 4 imply the following relations between the holonomic components of the connections in question:

$$\Gamma^h_{li} = \{^{\,h}_{li}\} + \Gamma_l \delta^h_i + \Gamma_i \delta^h_e - g^{hj}\Gamma_j g_{li}, \tag{7}$$

where $\{\ldots\}$ are Christoffel symbols related to the local components of g. Thus we have to choose n functions $\Gamma_1, \ldots, \Gamma_n$ which are local components of some form γ. Thus γ is this part of the connection form, which is projected into $\boldsymbol{R}$ in the decomposition: Lie algebra of $R \times GL(R^n) = \boldsymbol{R} \oplus GL(R^n)$ (the direct sum). Since Γ_j may be chosen arbitrarily, the theorem is proved.

COROLLARY. *There exists exactly one Hlavatý connection, D_0, such that $\nabla^{D_0}\mathscr{I} = 0$ identically. $\mathscr{I}$ denotes here the function which is equal to 1 identically on M.*

We are now going to investigate the curvature of a Hlavatý connection. We decompose the connection form, ω, into components:

$$\omega = \boldsymbol{I} \otimes \gamma + \boldsymbol{I}^h_k \otimes \omega^k_h, \tag{8}$$

where $\boldsymbol{I}$ is the unit vector in $\boldsymbol{R}$ and $\boldsymbol{I}^k_h$ are components of the natural frame in the Lie algebra $GL(R^n)$. Denoting the curvature form by Ω we have

$$\Omega = \boldsymbol{I} \otimes d\gamma + \boldsymbol{I}^h_k (d\omega + \omega^k_j \wedge \omega^j_h).$$

The local representation of $d\gamma$ is the following: If we write $\gamma = \Gamma_j dx^j$, then we have

$$d\gamma = (\partial_i \Gamma_j - \partial_j \Gamma_i)\, dx^i \wedge dx^j. \tag{9}$$

We take into consideration a Weylian object of curvature of the connection D, namely we put

$$H_{v,w} a = (\nabla^D_v \circ \nabla^D_w - \nabla^D_w \circ \nabla^D_w - \nabla^D_{[v,w]})\, a, \tag{10}$$

v and w being arbitrary vector fields and a being an arbitraray pseudovector. [;] is here the Poisson bracket. In order to find local expressions for H we take a local field of holonomic frames, $[\theta, x_2, \ldots, x_n]$ and we put $H_{x_i, x_j} a^k = H_{ijl}{}^k a^l$. An elementary computation yields the following formulas:

$$H_{ijl}{}^k = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{ih}\Gamma^h_{jl} - \Gamma^k_{jh}\Gamma^h_{il} - (\partial_i \Gamma_j - \partial_j \Gamma_i)\, \delta^k_l, \tag{11}$$

(Cf. [3], XXX, S. 7.) We notice that the right-hand member is like a Riemannian curvature tensor minus some quantity obtained from $d\gamma$. It follows from Proposition 6, that $H_{v,w}a$ is a pseudovector. Hence H itself is a tensor field. The right-hand member of (11) is composed of a coefficient of the Riemannian curvature tensor and of the coefficients of (9). We perform the contraction of the tensor H with respect to indices k and i. We obtain

$$H_{.jl}^{\cdot} = \partial_k \Gamma^k_{jl} - \partial_j \Gamma^{\cdot}_{\cdot l} + \Gamma^{\cdot}_{\cdot h} \Gamma^h_{jl} - \Gamma^k_{jh} \Gamma^h_{kl} - (\partial_l \Gamma_j - \partial_j \Gamma_l). \tag{12}$$

The tensor K with components $K_{jl}: = H_{ijl}{}^{i}$ may be defined also in an intrinsic way as a mapping $(w, a) \mapsto K_w a$, where

$$K_w a = (\text{trace of the mapping } v \mapsto H_{v,w}a). \tag{13}$$

The value of $K_w a$ is a pseudovector field.

Now we have to give an intrinsic definition of some geometric object which is the analogue of the Gaussian curvature. For this purpose we define the bundle $Y^*(M)$ as follows: We introduce a transformation rule (i.e. a group action) by:

$$\lambda^*: \big(\boldsymbol{R} \times GL(R^n)\big) \times \boldsymbol{R}^n \to \boldsymbol{R}^n,$$
$$\big((t, [A^i_j]), [u_i]\big) \mapsto [e^t A^i_j u_i].$$

Then we consider the following equivalence ϱ^* on $W(M) \times R^n$: $\big((s, r), u\big)\varrho^*\big((\bar{s}, \bar{r}), \bar{u}\big)$ iff there exists $(t, A) \in R \times GL(R^n)$ such that $\bar{s} = s + t$, $\bar{r} = r \cdot A$ and $\bar{u} = \lambda^*(t, A, u)$.

Then the quotient manifold $W(M) \times R^n/\varrho^*$ is a bundle over M. We denote this bundle by $Y^*(M)$.

We define yet another bundle. We take the following transformation rule:

$$\mu: \big(R \times GL(R^n)\big) \times \boldsymbol{R} \to \boldsymbol{R},$$
$$\big((t, [A^i_j]), c\big) \mapsto e^t c.$$

(We observe that the $GL(R^n)$-component acts here trivially.) Then we consider the equivalence σ in $R \times GL(R^n) \times R$ which is defined as follows

$\big((s, r), c\big)\sigma\big((\bar{s}, \bar{r}), \bar{c}\big)$ iff there exists some $(t, A) \in R \times GL(R^n)$ such that $\bar{s} = s + t$, $\bar{r} = r \cdot A$ and $\bar{c} = e^{-t}c$.

The quotient manifold $W(M) \times R/\sigma$ is a bundle $Y_0(M)$ over M.

DEFINITION 5. A cross-section $M \to Y_0(M)$ will be called a *field of pseudocovectors*. Any cross-section $M \to Y_0(M)$ will be called a *field of pseudoscalars*.

PROPOSITION 7. *Each field of pseudovectors is a mapping of pseudovectors into the scalars. The local expression of this mapping is* $([u_i], [v^i]) \mapsto u_i v^i$.

PROPOSITION 8. *A Weyl metric g establishes an isomorphism of the algebra of vector fields on to that of covector fields. The corresponding mapping is*

$$g^{\wedge} v := g(v, -).$$

In view of Definition 2 (i) there exists a mapping reciprocal to $g^{\wedge}$ which will be denoted by $g^{\vee}$. We have in local coordinates

$$g^{\wedge} : [a^i] \mapsto [g_{ij} a^j],$$
$$g^{\vee} : [a_i] \mapsto [g^{ij} a_j].$$

Consider the field of pseudoscalars

$$\mathbf{g} := (\det g^{\wedge})^{2n}. \tag{14}$$

PROPOSITION 9. $\mathbf{g}$ *establishes a one-to-one mapping of vector fields to pseudovector fields. If w is a vector, then* $w/\mathbf{g}$ *is a pseudovector. If* ω *is a covector, then* $\mathbf{g}g\ \omega$ *is a pseudovector.*

Thus the mapping

$$K^{\wedge} : (\text{covector fields}) \to (\text{covector fields}),$$
$$\omega \mapsto K_{(-)}(\mathbf{g}g^{\vee}\omega)$$

is well defined and linear.

DEFINITION 6. We define the *pseudoscalar curvature* of a Hlavatý connection to be the following field

$$k := \frac{1}{n-1}\operatorname{trace}(K^{\wedge}).$$

Thus we have, in local coordinates,

$$k = \frac{1}{n-1}\mathbf{g}g^{jl}H_{.jl}^{\cdot}. \tag{15}$$

We supply some considerations on the Weyl geometry of surfaces in the conformal space: We denote by $\mathrm{Co}(R^n)$ a group of conformal transformations of R^n including translations. We denote by C^n, where $n \geqslant 3$, the n-dimensional conformal space, which is viewed upon as a Klein space $(R^n, \mathrm{Co}(R^n))$. Keeping in mind the previous notations we consider the trivial bundle $W(C^n) = C^n \times \mathrm{Co}(R^n)$ and the global cross-section

$$s_0 : C^n \to W(C^n),$$
$$x \mapsto \text{unity of } \mathrm{Co}(R^n).$$

Thus C^n is provided with a standard Weyl metric, G, which has a matrix of components at s_0 equal to $[\delta^i_j]$. In order to check some Hlavatý connection in $W(C^n)$ it is sufficient to define n functions $\Gamma_1, \ldots, \Gamma_n$ on C^n. Thus Theorem 1 yields the formulas for computing the remaining components of a connection. We have at s_0

$$\Gamma^k_{ij} = \Gamma_i \delta^k_j + \Gamma_j \delta^k_i - \Gamma^k_{\cdot} \delta_{ij},$$

where $\Gamma^k_{\cdot} := \delta^{kh}\Gamma_h$. Then we have for any field a of pseudotensors

$$\nabla^G_i a^k = \partial_i a^k + \Gamma_l a^l \delta^i_k - \Gamma_k a^i.$$

Now we compute the corresponding curvature tensor H. We have

$$(16)\qquad \begin{aligned} H_{ijl}{}^k = {} & \partial_i \Gamma_l \delta^k_j - \partial_j \Gamma_l \delta^k_i + \partial_j \Gamma^k_{\cdot} \delta_{li} - \partial_i \Gamma_k \delta_{lj} + \\ & + \Gamma_l(\Gamma_j \delta^k_i - \Gamma_i \delta^k_j) + (\Gamma_i \delta_{lj} - \Gamma_i \delta_{li})\Gamma^k_{\cdot} + \\ & + (\Gamma_h \Gamma^h_{\cdot})(\delta_{il}\delta^k_j - \delta_{jl}\delta^k_i). \end{aligned}$$

Hence

$$H_{\cdot jl}{}^{\cdot} = (2-n)(\partial_j \Gamma_l - \Gamma_j \Gamma_l) + (-\partial_h \Gamma^h_{\cdot} + (2-n)\Gamma_h \Gamma^h)\delta_{jl}$$

and

$$k = -2\partial_h \Gamma^h_{\cdot} + (2-n)\Gamma_h \Gamma^h_{\cdot}.$$

We assume still that $n \geqslant 3$. Let S be a smooth ν-dimensional surface in C^n. We fix for a moment a point $p \in S$. We denote by S_p the vector space tangent to S at p. Let $[\boldsymbol{i}_1, \ldots, \boldsymbol{i}_n]$ be the standard frame in C^n. There exist conformal mappings which send $0 \in C^n$ to p in such a way that the vectors $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_\nu$ are mapped into S_p. The same can be done with points which lie dose to p. Thus there exists an open neighbourhood $\mathscr{U}$ of p in S and a smooth mapping which assigns to each $q \in \mathscr{U}$ a frame $[e_1(q), \ldots, e_\nu(q), e_{\nu+1}(q), \ldots, e_n(q)]$ so that the first ν of them span a space S_p and the last $n-\nu$ of them span the subspace which is vertical to the first one. We let the group $R \times GL(R^\nu) \times GL(R^{n-\nu})$ to act on those frames as follows:

$$\begin{aligned} & \big([\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n], (t, [A^\beta_\alpha], [B^L_H])\big) \\ & \qquad \mapsto \Big(\Big[t, \sum_1^\nu A^\beta_1 \boldsymbol{e}_\beta, \ldots, \sum_1^\nu A^\beta_\nu \boldsymbol{e}_\beta\Big], \Big[t, \sum_{\nu+1}^n B^L_{\nu+1}\boldsymbol{e}_L, \ldots, \sum_{\nu+1}^n B^L_n \boldsymbol{e}_L\Big]\Big). \end{aligned}$$

This treatment yields a subbundle of $W(C^n|S)$ which consists of those frames which are split into two parts so that one part spans S_p and the other spans its orthogonal complement. We denote the obtained subbundle by $V(C^n, S)$.

We denote by π the natural projection which maps $V(C^n, S)$ onto $W(S)$. Then any bundle which is associated with $V(C^n, S)$ may be projected in a natural way to a bundle which is associated with $W(C^n)$. We denote the associated projection also by π. Thus π sends vector fields on C to vector fields on S, pseudovector fields on C to pseudovector fields on S, etc. For computations in local coordinates we proceed as follows: If we have, for instance, a pseudovector a, then we assign it to some frame from $V(C^n, S)$ and we truncate the last $n-\nu$ components of a.

THEOREM 2. *Assume that there is fixed a standard Weyl metric G and some Hlavatý connection C on $W(C^n)$. Thus the immersion $S \to C^n$ induces a Weyl metric g on S and a Hlavatý connection in $W(S)$.*

Proof. If a, b are pseudovectors on S, then we put $g(a, b) = G(a, b)$. In order to define a connection on S we assume

$$\nabla_v^C a = \pi \circ \nabla_v^{\bar{C}} a.$$

We have to show that there holds the identity

$$\partial_v g(a, b) = g(\nabla_v^C a, b) + g(a, \nabla_v^C b)$$

(see Definition 4 (i)). In fact, we have

$$\begin{aligned} \partial_v g(a, b) &= \partial_v G(a, b) = g(\nabla_v^{\bar{C}} a, b) + g(a, \nabla_v^{\bar{C}} b) \\ &= g(\nabla_v^C a, b) + g(a, \nabla_v^C b) + 0 + 0. \end{aligned}$$

Let $[x_1, \ldots, x_\nu]$ be a local field of frames on S. Since $\bar{C}$ is the Hlavatý connection, then we have

$$\nabla_{x_\alpha}^C x_\beta = \pi(\nabla_{x_\alpha}^{\bar{C}} x_\beta) = \pi(\nabla_{x_\beta}^{\bar{C}} x_\alpha) = \nabla_{x_\beta}^C x_\alpha$$

(see Definition 4 (ii)). Then the local components Γ_α, $\Gamma_{\alpha\beta}^\varkappa$ of the connection C may be found from the formulas

$$\nabla_{x_\alpha}^C x_\beta = \Gamma_{\alpha\beta}^\varkappa x_\varkappa, \quad \nabla_{x_\alpha}^C \theta = \Gamma_\alpha \theta,$$

θ being some pseudoscalar, for instance g. This finishes the proof.

From now on we restrict ourselves to the case when $\nu = n-1$ and we assume that S is orientable. Thus each space S_p is spanned by one vector. We equip S with a field of pseudovectors

$$\begin{aligned} \boldsymbol{n}\colon S &\to S' \\ p &\mapsto \boldsymbol{n}(p), \end{aligned}$$

such that $G(\boldsymbol{n}, \boldsymbol{n}) = 1$. It follows from the construction of the connection C that there exists a mapping h such that there holds the identity

$$\Delta_v^{\bar{C}} a = \nabla_v^C a + h_v(a)\boldsymbol{n}. \tag{17}$$

This is the analogue of the classical "Gauss equation". Since $\underset{v}{\nabla^{C}} a$, $\underset{v}{\nabla^{\bar{C}}} a$ and $\boldsymbol{n}$ are pseudovectors then $h_v a$ is a scalar field. We set

$$\nabla^{\bar{C}}_v \boldsymbol{n} = b_v. \tag{18}$$

We shall prove that b_v is a field of pseudovectors on S. We have on s: $0 = \underset{v}{\partial} G(\boldsymbol{n}, \boldsymbol{n}) = 2G(\boldsymbol{n}, \underset{v}{\nabla^{\bar{C}}} \boldsymbol{n})$ and it follows that b_v is orthogonal to $\boldsymbol{n}$.

PROPOSITION 10. *The mapping $v \mapsto b_v$ may be expressed by g and by h as follows:*

$$b = -g^{\check{}} \circ h$$

(Weingarten equality).

Proof. We have $\partial_v G(n, a) = G(b_v, a) + G\big(n, \underset{v}{\nabla^{C}} a + h_v(a) n\big)$. Hence $g(b_v, a) = -h_v(a)$, that is $g^{\hat{}}(b_v) = -h_v(\cdot)$. Thus we obtain $b_v = -g^{\check{}} \circ h_v$. We have the following local expression

$$b^{\beta}_{a} = -g^{\beta\varkappa} h_{\varkappa a}. \tag{19}$$

Now we shall deduce some identities which are analogous to those of Riemannian geometry. In view of (17) and (18) we have

$$\begin{aligned} \nabla^{\bar{C}}_w \nabla^{\bar{C}}_v a &= \nabla^{\bar{C}}_w \big(\nabla^{C}_v a + h_v(a)\boldsymbol{n}\big) \\ &= \nabla^{C}_w \nabla^{C}_v a + h_v(a) b_w + \big(\nabla^{\bar{C}}_w h_v(a) + h_w(\nabla^{C}_v a)\big)\boldsymbol{n}. \end{aligned}$$

The above identity implies the following relation between the curvature tensors $\bar{H}$ and H which correspond to the connections $\bar{C}$ and C respectively:

$$\begin{aligned} \bar{H}_{w,v} a &= H_{w,v} a + h_w(a) b_v - h_v(a) b_w + \\ &\quad + \big(\nabla^{\bar{C}}_w h_v(a) - \nabla^{\bar{C}}_v h_w(a) + h_w(\nabla^{C}_v a) - h_v(\nabla^{C}_w a)\big)\boldsymbol{n}. \end{aligned}$$

Hence we obtain

PROPOSITION 11. *If $\bar{H}$ is identically 0, then we have*

$$H_{v,w} a = h_w(a) b_v - h_v(a) b_w.$$

Using (19) we have the following expression in local coordinates:

$$H_{\mu\nu\lambda}{}^{\varkappa} = h_{\mu\lambda} g^{\varkappa\varrho} h_{\varrho\nu} - h_{\nu\lambda} g^{\varkappa\varrho} h_{\varrho\mu}. \tag{20}$$

THEOREM 3. *If $n = 3$ and $\nu = 2$ and $\bar{H}$ vanishes identically on S, then the pseudoscalar curvature of the connection C is equal to $\boldsymbol{g}$* $\det(b)$.

This theorem may be obtained by an easy computation from (15) and (20). This is a certain modification of "theorema egregium" of Gauss, However, it does not seem to be of great interest as far as we do not know whether any non-trivial connections with vanishing H do exist (see formula (16)).

Final remark. There exist surfaces in C^n such that there do not exist smooth, non-vanishing tangent vector fields. Theorem 2 assures the existence of a Weyl metric of a Hlavatý connection on a surface. This fact implies that the form γ (see formula (8)) with local components $\Gamma_1, \ldots, \Gamma_n$ can not be identified with any vector or a covector field on the basic manifold. The same fact follows from (6). Thus any attempt to define a Weyl structure by equipping a manifold with a Riemannian scalar product and with some complementary vector field is a misunderstanding. Such an approach agrees neither with a global point of view, nor with fundamental notions assumed in the theory of geometric objects [4]. In view of our remark on p. 190 there exists an other way to define a Weyl structure on the manifold M. Namely a pseudotensor which defines a Weyl metric may be identified with a class of positive proportionality of Riemannian tensor fields. There is attached a connection form in such a way that the parallel transport of angle measures is assured. Cf. [2].

References

[1] R. Crittenden, *Covariant differentiation*, Quart. J. Math. Oxford Ser. 13, 25 (1962), p. 285–289.

[2] B. G. Folland, *Weyl manifolds*, J. Differential Geometry 4 (2) (1970), p. 145–154.

[3] Gh. Gheorghiev, R. Miron, Dan I. Papuc, *Geometria analitică şi diferenţială, II*, Bucureşti 1969.

[4] S. Gołąb, *Rachunek tensorowy*, sec. ed., Warszawa 1962.

[5] V. Hlavatý, *Les courbes de la variété W_n*, Mémor. Sci. Math., Paris 1934.

[6] — *Théorie d'immersion d'une W_m dans W_n*, Ann. Soc. Polon. Math. 21 (1949).

[7] O. Kowalski, *Partial curvature structures and the conformal geometry of submanifolds* (to appear).

[8] A. Lichnerowicz, *Théorie globale des connexions et des groupes d'holonomie*, Roma 1955.

[9] А. П. Норден, *Пространства аффинной связности*, Москва–Ленинград 1950.

[10] A. Szybiak, *Covariant differentiation of geometric objects*, Dissertationes Mathematicae LVI, 1967.

[11] H. Weyl, *Reine Infinitesimalgeometrie*, Math. Z. 2 (1918), p. 348–411.

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