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Geometric objects with finitely determined germs

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Abstract. We prove that if a natural bundle admits a section-germ which is finitely determined, then its fibre dimension is not greater than the dimension of the base manifold. This in turn implies that the order of the bundle is not greater than 2 (or 3 if the base is one-dimensional). Applying a classification of such natural bundles, we indicate geometric objects which are 0 or 1-determined. Their canonical forms are given.

The concept of finite determinacy of map-germs with respect to action of some contact groups was introduced by Mather in [2]. It is due to Gollek [1] that this concept was transfered to differential geometry for germs of geometric objects. The latter are subject to group action of germs of local diffeomorphisms of the base manifold. Except for functions, the action is different from that considered by Mather. The main difficulty in applying Mather theory is that the Lie derivative is not $C^{\infty}(M)$-linear with respect to the vector argument. Nevertheless, the Mather necessary condition of finite determinacy is applicable to geometric objects, as it was shown by Gollek.

We are going to prove that only bundles whose fibre dimension is not greater than the dimension of the base manifold may admit finitely determined section-germs. We indicate such germs in bundles satisfying this condition and having homogeneous fibres. We also prove some new canonical form theorems for geometric objects. In particular, we give a version of the Darboux theorem for density 1-forms.

Basic notations:

$D_x(M)$ group of germs of local diffeomorphisms of M preserving x,
$G(n, k) = D_0^k(\boldsymbol{R}^n)$ group of k-jets at 0 of members of $D_0(\boldsymbol{R}^n)$,
$E = H^r(M) \times_{G(n,r)} F$ associated bundle to the r-frame bundle on M,
$\varepsilon_x(E)$ set of section-germs of E at x,
$J_x^k(E)$ fibre over x of k-jet bundle $J^k E$,
$\varepsilon_n(F)$ set of germs at 0 of C^{∞}-maps from $\boldsymbol{R}^n$ into F,
$T_n^k F$ manifold of k-jets at 0 of members of $\varepsilon_n(F)$.

1. Preliminaries. The associated fibre bundle E defined above is called a *bundle of geometric objects of order* r and of type which is determined by the action of the differential group $G(n, r)$ on the manifold F. Sections $M \to F$ are fields of geometric objects of this type. Recall (see [3]) that E is a natural bundle over M, that is, each local diffeomorphism φ of M has its prolongation to a local isomorphism $E\varphi$ of E which covers φ and commutes with restrictions to open subsets. The fact that E is of order r means that the restriction of $E\varphi$ to a fibre over $x \in \mathrm{Dom}\,\varphi$ depends only on the k-jet at x of φ.

Let $\sigma\colon U \to E$ be a local section. For each $\varphi \in \mathrm{Diff}(M)$ defined on U,

$$\varphi_* \sigma = E\varphi \circ \sigma \circ \varphi^{-1} \tag{1.1}$$

is a section of E defined on $\varphi(U)$. On passing to germs we get an action of the group $D_x(M)$ on $\varepsilon_x(E)$. Since $j_x^k(\varphi_* \sigma)$ depends only on $j_x^{r+k}\varphi$ and $j_x^k \sigma$, this action can be factorized to an action of $D_x^{r+k}(M)$ on $J_x^k E$. For $\sigma \in \varepsilon_x(E)$ denote

$$\mathrm{Orb}_x \sigma = D_x(M) \cdot \sigma \quad \text{and} \quad \mathrm{Orb}\, j_x^k \sigma = D_x^k(M) \cdot j_x^k \sigma.$$

DEFINITION 1.1. A germ $\sigma \in \varepsilon_x(E)$ is called *k-determined* (with respect to $D_x(M)$) if for any germ $\sigma' \in \varepsilon_x(E)$, $j_x^k \sigma' = j_x^k \sigma$ implies that $\sigma' \in \mathrm{Orb}_x \sigma$.

We may consider the case $r = 0$, in which we define $E = M \times F$. Then sections of E are C^∞-maps from M into F and $\varphi_* \sigma = \sigma \circ \varphi^{-1}$. In this case the k-determinacy with respect to $D_x(M)$ coincides with that of Mather by "right equivalence" (cf. [2]). If $\dim F > 1$ then only the germs of submersions are finitely determined. This is why Gollek conjectured that $\dim F \leqslant \dim M$ should be a condition of the existence of finitely determined germs. We are going to prove it.

2. Proof of Gollek conjecture. Since the problem of finite determinacy is local, we can work in a local trivialization of E by a chart into $\boldsymbol{R}^n \times F$, $n = \dim M$. Assume that $x = 0$; then $\varepsilon_0(E) = \varepsilon_n(F)$ and $J_0^k E = T_n^k F$. The germ-group $D = D_0(\boldsymbol{R}^n)$ acts on $\varepsilon_n(F)$ by

$$D \times \varepsilon_n(F) \ni (\varphi, \sigma) \to \mathrm{germ}_0 \{x \to E(t_{-x} \circ \varphi \circ t_{\varphi^{-1}(x)}) \sigma \circ \varphi^{-1}(x)\},$$

where t_x is the translation of $\boldsymbol{R}^n$ by x. On replacing the $germ_0$ by *k-jet at* 0 we get the action of $G(n, r+k)$ on $T_n^k F$. If $0 \in \boldsymbol{R}^n$ corresponds to $x \in M$ in this trivialization, then D-equivalence of germs from $\varepsilon_n(F)$ corresponds to $D_x(M)$-equivalence of germs from $\varepsilon_x(E)$. In what follows we will consider finitely determined germs from the set $\varepsilon_n(F)$ with respect to the group D.

THEOREM 2.1. *If $\varepsilon_n(F)$ contains a finitely determined germ, then $\dim F \leqslant n$.*

Proof. Actually, we are going to prove a little more: if there exists $\sigma \in \varepsilon_n(F)$ and an integer $k \geqslant 0$, such that for any $\sigma' \in \varepsilon_n(F)$ and all $l \geqslant k$ we have

(2.1) $$j_0^k \sigma' = j_0^k \sigma \quad \text{implies} \quad j_0^l \sigma' \in \operatorname{Orb} j_0^l \sigma,$$

then $\dim F \leqslant n$.

Let $m = \dim F$ and let $\pi_k^l \colon T_n^l F \to T_n^k F$ be the canonical projection. For convennience, write $\sigma^s = j_0^s \sigma$. We define

$$G^l(\sigma^k) = \{\alpha \in G(n, r+l);\ \alpha\sigma^k = \sigma^k\}, \qquad F^l(\sigma^k) = (\pi_k^l)^{-1}(\sigma^k).$$

(2.1) says that the subgroup $G^l(\sigma^k)$ acts transitively on $F^l(\sigma^k)$. Since $G^l(\sigma^l)$ is the isotropy group of this action, we are given

(2.2) $$\dim F^l(\sigma^k) = \dim G^l(\sigma^k) - \dim G^l(\sigma^l).$$

There is

(2.3) $$\dim F^l(\sigma^k) = \dim T_n^l F - \dim T_n^k F = m\left[\binom{n+l}{l} - \binom{n+k}{k}\right]$$

and

(2.4) $$\dim G(n, r+l) = n\left[\binom{n+r+l}{r+l} - 1\right]$$

(since the dimension of the space of all Taylor expansions of functions $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^m$ up to order l is $m\binom{n+l}{l}$). Moreover,

(2.5) $$\dim G^l(\sigma^k) = \dim G(n, r+l) - \dim G(n, r+k) + \dim G^k(\sigma^k).$$

We have

(2.6) $$\binom{n+r+l}{r+l} = \varepsilon_l \binom{n+l}{l}, \quad \text{where} \quad \varepsilon_l = \frac{(n+r+l)\ldots(n+l+1)}{(l+r)\ldots(l+1)}$$

and, r being fixed, $\varepsilon_l \to 1$ when $l \to \infty$.

From formulae (2.2)–(2.6) we get by direct computation

$$(m - n\varepsilon_l)\binom{n+l}{l} + \dim G^l(\sigma^l) = (m - n\varepsilon_k)\binom{n+k}{k} + \dim G^k(\sigma^k).$$

Since it holds for arbitrarily great l, it follows that $m \leqslant n$, which is what was to be proved.

Now we can apply the author's result from [6]:

Let E be a natural bundle with base M and standard fibre F. Suppose that $n = \dim M$, $m = \dim F$ and $k =$ order of E. Then

$$k \leqslant \max\left\{\frac{m}{n-1};\ \frac{m}{n}+1\right\} \qquad \text{if } n \geqslant 2$$

and

$$k \leqslant 2m+1 \qquad \text{if } n = 1$$

(the latter bound is due to Epstein and Thurston, cf. [3]).

Combining this and Theorem 2.1 we get

COROLLARY 2.2. *Under assumption of Theorem 2.1 we obtain additionally that the order of E is not greater than 2 if $n \geqslant 2$ and not greater than 3 if $n = 1$.*

3. Bundles which may admit finitely determined section-germs. Each $G(n, r)$-space F can be decomposed into a union of homogeneous spaces (orbits) on which the group acts transitively. We will consider the existence of finitely determined germs in $\varepsilon_n(F)$ when F is a homogeneous $G(n, r)$-space with admissible data: $m = \dim F \leqslant n$, $n \geqslant 2$ and $r = 1$ or 2. The natural bundles with such standard fibres were classified by the autor in [5]. Here is the list of them, giving a full classification up to local isomorphisms of natural bundles (which is sufficient for our purpose):

$r = 1$

1. $\Lambda^n T^* M$ ($\cong \Lambda^n TM$) bundle of n-forms on M,
2. TM tangent bundle,
3. $T^* M$ cotangent bundle,
4. PTM projective tangent bundle,
5. $PT^* M$ projective cotangent bundle,
6. $D^c TM$ bundle of vector densities of weight $c \neq 0, 1$,
7. $D^c T^* M$ bundle of covector densities of weight $c \neq 0, 1$,
8. $\Lambda^{n-1} TM$ bundle of $(n-1)$-vectors,
9. $\Lambda^{n-1} T^* M$ bundle of $(n-1)$-forms,
10. $PTM \times_M \Lambda^n T^* M$,
11. $PT^* M \times_M \Lambda^n T^* M$,
12. $G_2 M$ ($\dim M = 4$) Grassmann bundle of 2-planes,
13. $E_1 \subset E_2 \subset TM$ ($\dim M = 3$) bundle of flag distributions,
14. $CO(M)$ ($\dim M = 2$) bundle of conformal metrics,
15. $CH(M)$ ($\dim M = 3$) bundle of conformal hyperbolic metrics.

$r = 2$

16. $\Gamma_c M$ bundle of contracted linear connections,
17. $P(\operatorname{div} TM)$ projective divergence bundle.

(For homogeneity, vector bundles are to be taken without zero section.)

Neither two of the listed bundles are locally isomorphic, including those with different weights c.

We are going to indicate sections in trivializations $\boldsymbol{R}^n \times F$ of these bundles whose germs at 0 are 0 or 1-determined. We shall do it by showing that they are D-equivalent to a canonical form, whenever the 0 or 1-jet at 0 satisfies a (necessarily invariant) condition.

Some of canonical form theorems of this type are well known; we recall them now for a later use. The numerals will mean that we consider a section of the bundle placed under such a number in the above list.

Ad 1. If w is an n-form on $\mathbf{R}^n$ such that $w(0) \neq 0$, then there exists $\varphi \in D$ for which $\varphi_* w = dx^1 \wedge \ldots \wedge dx^n$ near 0.

Ad 2. For a vector field X with $X(0) \neq 0$, its local canonical form is $\partial/\partial x^1$.

Ad 3. Let ω be an 1-form on $\mathbf{R}^n$ such that $\omega(0) \neq 0$ and $d\omega(0)$ has maximum rank. By the Darboux theorem there is $\varphi \in D$ such that $\varphi_* \omega = \mathring{\omega}$ near the origin, where $\mathring{\omega}$ is defined as follows:

$$(3.1) \qquad \begin{array}{ll} \sum_{i=1}^{m} x^i dx^{i+m} + dx^{1+m} & \text{if } n = 2m, \\ \sum_{i=1}^{m} x^i dx^{i+m} + dx^0 & \text{if } n = 2m+1. \end{array}$$

Ad 4. Let $\langle X \rangle$ be a 1-dimensional distribution on $\mathbf{R}^n$ generated by a nowhere zero vector field X. Then $\langle \partial/\partial x^1 \rangle$ is its canonical form defined, as in all other cases to follow, in a neighbourhood of 0.

Ad 9. If u is an $(n-1)$-form on $\mathbf{R}^n$ and $u(0) \neq 0$, $du(0) \neq 0$, then its canonical form is

$$u = (x^1 + 1)\, dx^2 \wedge \ldots \wedge dx^n.$$

(see Terng [4]).

4. A generalization of the Darboux theorem. Let π be a field of covector densites (e.g. a density 1-form) of weight $c \neq 0$ on $\mathbf{R}^n$. For $\varphi \in D$ we have

$$(4.1) \qquad \varphi_* \pi = |J|^c (\pi \circ \varphi^{-1}) \circ D\varphi^{-1}, \qquad J = \det(D\varphi^{-1}),$$

which is the "transformation rule" of this type of geometric objects. (The standard fibre of the bundle $D^c T^* M$ is $\mathbf{R}^{n^*}$ on which $G(n, 1) = GL(n)$ acts by $A \cdot \pi = |\det A^{-1}|^c \pi \cdot A^{-1}$.) Denote $\varphi\pi = (\pi \circ \varphi^{-1}) \circ D\varphi^{-1}$, the right-hand side being the transformation rule of 1-forms. Then (4.1) can be written

$$(4.2) \qquad \varphi_* \pi = |J|^c \varphi\pi.$$

Let $\pi = \pi_i dx^i$ (where π_i are not functions but densities). We may define a formal exterior derivative $d\pi = \partial_i \pi_j dx^i \wedge dx^j$ (no tensor !). Using (4.2) we get

$$(4.3) \qquad d(\varphi_* \pi) = d(|J|^c) \wedge (\varphi\pi) + |J|^c d(\varphi\pi).$$

We see that the rank of $d\pi$ depends on the local chart.

DEFINITION 4.1. We will say that π has maximum pfaffian class at a point $x \in \mathbf{R}^n$ if

$$(4.4) \qquad \begin{array}{ll} \pi(x) \wedge (d\pi(x))^{m-1} \neq 0 & \text{when } n = 2m, \\ \pi(x) \wedge (d\pi(x))^{m} \neq 0 & \text{when } n = 2m+1. \end{array}$$

An easy computation shows that this is an invariant notion. We can see also that, π having maximum pfaffian class, $d\pi$ has maximum rank m if $n = 2m+1$ and it has rank m or $m-1$ if $n = 2m$.

LEMMA 4.2. *Let $n = 2m$ and let π have maximum class at the origin. Then there exists a local chart near 0 in which $d\pi(0)$ has maximum rank m.*

Proof. Suppose that the rank of $d\pi(0)$ is $m-1$. Then the image W of the map $d\pi(0)$: $\boldsymbol{R}^n \to \boldsymbol{R}^{n^*}$, defined by $v \to i(v)\,d\pi(0)$ (interior product), has codimension 2. Take an 1-form $\theta \in \boldsymbol{R}^{n^*}$ such that $\theta \notin W$ and $\theta \wedge \pi(0) \neq 0$. Since J can be an arbitrary non-zero function, there exists $\varphi \in D$ with

$$D\varphi(0) = \mathrm{id} \quad \text{and} \quad d(|J|^c)(0) = \theta.$$

Then $\varphi\pi(0) = \pi(0)$ and $d(\varphi\pi)(0) = d\pi(0)$. Consequently, (4.3) gives

$$d(\varphi_* \pi)(0) = \theta \wedge \pi(0) + d\pi(0).$$

From the choice of θ and from (4.4) it follows that $d(\varphi_* \pi)(0)$ has maximum rank m (that is, its m-th exterior power is not 0).

LEMMA 4.3. *Let π be of maximum pfaffian class at 0. There exists $\varphi \in D$ such that $\varphi_* \pi = f\pi_0$, where f is a positive function and π_0 has the Darboux form* (3.1).

Proof. In view of Lemma 4.2 one can assume that $d\pi(0)$ has maximum rank. Since also $\pi(0) \neq 0$, by the Darboux theorem, there exists $\varphi \in D$ such that $\varphi\pi = \pi_0$. Then, writing $f = |J|^c$ in (4.2) we obtain the required form.

In given coordinates $x^1, \ldots, x^n$ we define g_π as follows

$$(d\pi)^m = g_\pi\, dx^1 \wedge \ldots \wedge dx^n \quad \text{if } n = 2m,$$
$$\pi \wedge (d\pi)^m = g_\pi\, dx^1 \wedge \ldots \wedge dx^n \quad \text{if } n = 2m+1.$$

(One can check that in the case $n = 2m+1$, g_π is a density of weight $(m+1)c$, so it is a concomitant of first order of the field π.) In particular, we have $g_{\pi_0} = m!$ for both the forms (3.1). There is also

$$g_{\varphi\pi} = J(g_\pi \circ \varphi^{-1}), \quad \text{hence } g_{\pi_0} = Jm! \tag{4.5}$$

and

$$g_{f\pi_0} = a(f), \quad \text{where } a(f) = f^{m+1}m! \text{ if } n = 2m+1$$

and

$$a(f) = (m-1)!\,f^{m-1}\left(\sum_{i=1}^{m} x^i \frac{f}{x^i} + \frac{f}{x^1}\right) + m!\,f^m \tag{4.6}$$

if $n = 2m$.

From the above it follows that

$$\text{if} \quad \varphi\pi_0 = f\pi_0, \quad \text{then} \quad J = \frac{1}{m!}a(f). \tag{4.7}$$

THEOREM 4.4. *Let π be a density 1-form of weight $c \neq 0$ on $\mathbf{R}^n$. Suppose that π has maximum pfaffian class at the origin and that $c \neq -1/(m+1)$ if $n = 2m+1$. Then there exists $\varphi \in D$ such that $\varphi_* \pi = \pi_0$ near the origin, where π_0 is a density 1-form on $\mathbf{R}^n$ defined by* (3.1).

Proof. By Lemma (4.3) we may supoose that $\pi = f\pi_0$, $f > 0$, near the origin. We are going to show that there exists a function h defined in a neighbourhood of 0, such that $h > 0$, $a(h) > 0$ and

$$\frac{1}{m!} a(h)^c h = f. \tag{4.8}$$

Assume that $n = 2m+1$. Then $a(h) = h^{m+1} m!$ and equation (4.8) gives $h^{(m+1)c+1} = f$. From this we get h as it was required.

Now, let $n = 2m$. From (4.6) and (4.8) we obtain a quasi-linear differential equation of first order

$$(m-1)!\, h^{m-1} \left(\sum_{i=1}^{m} x^i \frac{\partial h}{\partial x^i} + \frac{\partial h}{\partial x^1} \right) = \left(\frac{\partial f}{|h|} m! \right)^{1/c} - h^m m!. \tag{4.9}$$

Since the characteristic vector of this equation is non-zero near the origin, there exists a solution h defined in a neighbourhood of 0. We impose the initial condition $h(0) > 0$. From (4.9) it follows that $a(h) > 0$.

Since we have $g_{h\pi_0} = a(h)$, different from 0 at the origin, it follows that $d(h\pi_0)(0)$ has maximum rank. As also $h\pi_0(0) \neq 0$, by Darboux theorem, there exists φ such that $h\pi_0 = \varphi\pi_0$. According to (4.7) we have then $J = \det(D\varphi^{-1}) = (1/m!)\, a(h) > 0$. For this φ there is

$$\varphi_* \pi_0 = J^c \varphi\pi_0 = (1/m!)\, a(h)^c h\pi_0 = f\pi_0 = \pi,$$

near the origin. This completes the proof.

Theorem 4.4 says that the germ at 0 of π is 1-determined (the condition involves its first jet at 0).

The bundle $D^c T^* M$ for $c = 1$ is isomorphic with the bundle $\Lambda^{n-1} TM$; their correspondence in local coordinates is by

$$\pi \to i(\pi) \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n}$$

and has invariant character. Therefore there exist also 1-determined section-germs of $(n-1)$-vectors (in the proof of Theorem 4.4 the case $c = 1$ was admited).

Clearly, there are no 0-determined section-germs in $D^c T^* M$. Note also that the case $n = 2m+1$ and $c = -1/(m+1)$ remains unsettled.

5. Further cases for $r = 1$. We continue to look for 0 and 1-determined germs in the bundles listed in Section 3. As so far, we shall do it by finding their canonical forms.

Ad 6. Let X be a vector density of weight $c \neq 0, 1$ on $\boldsymbol{R}^n$. Its transformation rule is

$$\varphi_* X = |J|^c (D\varphi \cdot X)\varphi^{-1}, \tag{5.1}$$

where $\varphi \in D$ and $J = \det(D\varphi^{-1})$.

Assume that $X(0) \neq 0$. There exists φ such that $(D\varphi \cdot X)\varphi^{-1}\, \partial/\partial x^1$ near the origin. So, $\varphi_* X = f\, \partial/\partial x^1$, where we put $f = |J|^c$.

Consider another change of local coordinates:

$$\varphi = \{y^1 = \int_0^{x^1} h(x)\,dx^1,\ y^2 = x^2, \ldots, y^n = x^n\}, \tag{5.2}$$

where h is a positive function. Then $J = (h \circ \varphi^{-1})^{-1}$ and

$$\varphi_* \frac{\partial}{\partial x^1} = \left(\frac{\partial y^j}{\partial x^1}\frac{\partial}{\partial y^j}\right)\circ \varphi^{-1} = (h \circ \varphi^{-1})\frac{\partial}{\partial y^1}. \tag{5.3}$$

On applying this to (5.1) we get

$$\varphi_*(f\,\partial/\partial x^1) = |J|^c f \circ \varphi^{-1}\, \varphi_*\, \partial/\partial x^1 = (h \circ \varphi^{-1})^{-c+1} f \circ \varphi^{-1}\, \partial/\partial y^1.$$

Since $c \neq 1$, we can choose h such that $\varphi_* X = \partial/\partial y^1$, so it is the canonical form of X.

Thus we proved that if $X(0) \neq 0$ then the germ of X at the origin is 0-determined.

For $c = 1$, the bundle $D^c TM$ is isomorphic with $\Lambda^{n-1} T^* M$ by invariant correspondence

$$D^1 TM \ni X \to u = i(X)(dx^1 \wedge \ldots \wedge dx^n) \in \Lambda^{n-1} T^* M. \tag{5.4}$$

Given a field X, let u be defined as above. Then $X(0) \neq 0$ implies $u(0) \neq 0$. From

$$du = n(\operatorname{div} X)\,dx^1 \wedge \ldots \wedge dx^n$$

we see that the divergence of X is a density of weight 1. Moreover, condition $du(0) \neq 0$ is equivalent to $(\operatorname{div} X)(0) \neq 0$. Making use of the Terng canonical form for u (see Section 3), that is $u = (x^1 + 1)\,dx^2 \wedge \ldots \wedge dx^n$, we may calculate X from (5.4) to be

$$X = (x^1 + 1)\,\partial/\partial x^1,$$

near the origin.

So we proved that the germ of X at 0 is 1-determined if X and $\operatorname{div} X$ are non-zero at the origin.

Ad 10. Let $\langle X \rangle$ be a section of $PT\boldsymbol{R}^n$ (X is a vector field on $\boldsymbol{R}^n$) and w an n-form on $\boldsymbol{R}^n$, $w(0) \neq 0$. We can suppose that $X = \partial/\partial x^1$ near 0. Write $w(x) = h(x)\,dx^1 \wedge \ldots \wedge dx^n$. On applying transformation (5.2) we have

$$\varphi_* w = h \circ \varphi^{-1} J dy^1 \wedge \ldots \wedge dy^n = dy^1 \wedge \ldots \wedge dy^n$$

and, by (5.3), $\langle \varphi_* X \rangle = \langle \partial/\partial y^1 \rangle$.

Thus we proved that the canonical form of the pair $(\langle X \rangle, w)$ is

$$\left(\left\langle \frac{\partial}{\partial x^1} \right\rangle, dx^1 \wedge \ldots \wedge dx^n\right)$$

near 0, and consequently, that the germ of it at 0 is 0-determined if $w(0) \neq 0$.

Ad 5. Let $\langle \pi \rangle = \{f\pi : f \text{ non-zero function, } \pi \text{ 1-form}\}$ be section of $PT^* M$. Then $d(f\pi) = df \wedge \pi + f d\pi$ and its rank may depend on f. We define an invariant property of $\langle \pi \rangle$ similarly as we did for density 1-forms: we say that it has maximum class if π satisfies condition (4.4).

LEMMA 5.1. *If $\langle \pi \rangle$ has maximum class at x, then there exists f such that $d(f\pi)$ has maximum rank at x.*

The proof is analogous as for density 1-forms.

Assume that π is chosen so that $d\pi(x)$ has maximum rank. Since also $\pi(x) \neq 0$, by Darboux theorem, there exist local coordinates near x in which π has canonical form (3.1). Its equivalency class is the canonical form of $\langle \pi \rangle$. Thus a germ of $\langle \pi \rangle$ of maximum class at x is 1-determined.

Ad 11. Let $\langle \pi \rangle$ be a section of $PT^* \mathbf{R}^n$, of maximum class at the origin, and let w be an n-form on $\mathbf{R}^n$, $w(0) \neq 0$. According to what we proved above, one can assume that π has canonical form π_0 defined in (3.1). Let $w = f dx^1 \wedge \ldots \wedge dx^n$ in the same system of coordinates. There exists a function h, $h(0) > 0$, which satisfies equation

$$\frac{1}{m!} a(h) = \frac{1}{|f|} \tag{5.5}$$

in a neighbourhood of 0, where $a(h) = h^{m+1} m!$ if $n = 2m+1$ and it is defined by (4.6) if $n = 2m$. In the latter case, (5.5) is a quasi-linear differential equation which has solutions. Let g_π be defined as in Section 4 (see formulae (4.5) and further). Since $g_{h\pi_0} = a(h) > 0$, it follows that $d(h\pi_0)^m(0) \neq 0$, so $d(h\pi_0)$ has maximum rank at 0. Therefore there exists φ such that $\varphi_* \pi_0 = h \circ \varphi^{-1} \pi_0$. According to (4.7) we have $J = \det(D\varphi^{-1}) = (1/m!) a(h \circ \varphi^{-1})$.

On the other hand, using (5.5), we get

$$\varphi_* w = (f \circ \varphi^{-1}) J dy^1 \wedge \ldots \wedge dy^n = (\operatorname{sgn} f) dy^1 \wedge \ldots \wedge dy^n,$$

if $\varphi : (x^i) \ni (y^i)$.

Thus the canonical form of the pair $(\langle \pi \rangle, w)$ near the origin is

$$(\langle \pi_0 \rangle, \pm dy^1 \wedge \ldots \wedge dy^n). \tag{5.6}$$

Remark. π being an 1-form, g_π is a density of weight 1 (see (4.5)). Since $g_{f\pi} = f^{m+1} g_\pi$ for $n = 2m+1$, $\operatorname{sgn} g_\pi = \operatorname{sgn} g_{f\pi}$ if $m+1$ is even. Then the coincidence, or not, of the signs of g_π and of the density determined by the n-form w is an invariant of the pair in question. This invariant decides on the sign $+$ or $-$ in (5.6).

Ad 13. Let $E_1 = \langle X \rangle$ and $E_2 = \langle X, Y \rangle$ be distributions on M^3. We may assume that $X = \partial/\partial x^1$ and $Y = \partial/\partial x^2 + f\,\partial/\partial x^3$. Suppose that $X \wedge Y \wedge [X, Y](0) \neq 0$; hence $\partial f/\partial x^1(0) \neq 0$. Apply φ defined by

$$y^1 = f, \quad y^i = x^i \quad \text{for } i = 2, \ldots, n$$

near 0. There is

$$\frac{\partial}{\partial x^1} = \frac{\partial f}{\partial x^1}\frac{\partial}{\partial y^1}, \quad \frac{\partial}{\partial x^2} = \frac{\partial f}{\partial x^2}\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}, \quad \frac{\partial}{\partial x^3} = \frac{\partial f}{\partial x^3}\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^3}.$$

Then

$$E_1 = \left\langle \frac{\partial}{\partial y^1} \right\rangle \quad \text{and} \quad E_2 = \left\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} + y^1 \frac{\partial}{\partial y^3} \right\rangle$$

and this is the canonical form of the pair (E_1, E_2). So, the germ of this pair at 0 is 1-determined.

Ad 14. Let $\langle g \rangle$ be a conformal metric on M^2. It is well known that M^2 is locally conformally isometric with Euclidean space E^2. Hence the canonical form of $\langle g \rangle$ is $\langle dx^2 + dy^2 \rangle$. It is 0-determined.

Ad 15. A similar theorem is true also for each conformal hyperbolic metric on M^2. This is equivalent to the more known fact that every almost complex structure on M^2 is integrable. The equivalency follows from the fact that the (1, 1)-tensor

$$f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has the same isotropy subgroup as $\langle dx\,dy \rangle$; hence the corresponding homogeneous spaces are isomorphic.

6. Cases of bundles of order 2. Ad 16. The standard fibre of the divergence bundle $\mathrm{Div}(TM)$ is $\boldsymbol{R}^n \times \boldsymbol{R}$ and its sections are pairs (X, d), where X is a nowhere zero vector field and d a real valued quantity. They are subject to transformation rule:

$$\varphi_*(X, d) = \big(D\varphi \cdot X,\ d + X(\log|J|)\big)\varphi^{-1} \quad (Xf = \partial_X f), \tag{6.1}$$

where this time J stands for $\mathrm{Det}(D\varphi)$. In particular, $(X, \mathrm{div}\,X)$ is such a pair on $\boldsymbol{R}^n$.

Let $\langle X, d \rangle$ be a section of $P(\mathrm{Div}\,T\boldsymbol{R}^n)$. Suppose $X = \partial/\partial x^1$ and choose φ to be:

$$y^1 = \int_0^{x^1} \Big(\exp\big(-\int_0^{x^1} d(x)\,dx^1\big)\Big)dx^1, \quad y^i = x^i \quad \text{for } i = 2, \ldots, n.$$

Then

$$X(\log J) = -d \quad \text{and} \quad D\varphi \cdot X = \frac{\partial y^1}{\partial x^1} \frac{\partial}{\partial y^1} \neq 0.$$

So

$$\varphi_* \langle X, d \rangle = \left\langle \frac{\partial}{\partial y^1}, 0 \right\rangle,$$

which is the canonical form of this section. We conclude that its germ at the origin is 0-determined.

Ad 17. If Γ^i_{jk} are Christoffel symbols of a linear connection, $\Gamma_k = \Gamma^i_{ik}$, then $\Gamma = \Gamma_k dx^k$ is the contracted linear connection object. The transformation rule of Γ is

(6.2) $$\varphi_* \Gamma = (\Gamma \circ \varphi^{-1}) \circ D\varphi^{-1} + d(\log |J|),$$

where $J = \det(D\varphi^{-1})$. The first part of the right-hand side is just the transformation of an 1-form; denote it by $\varphi\Gamma$. From (6.2) we see that the exterior derivative of Γ, $d\Gamma = \partial_i \Gamma_j dx^i \wedge dx^j$ is a 2-form and

(6.3) $$d(\varphi_* \Gamma) = (d\Gamma) \circ D\varphi^{-1} \otimes D\varphi^{-1}.$$

Suppose that $d\Gamma(0)$ has maximum rank. Then there exists φ such that $\varphi\Gamma = \Gamma_0$ has canonical form (3.1). For this φ we have

$$\varphi_* \Gamma = \Gamma_0 + df \quad \text{for } df = d(\log |J|).$$

There is $d(\Gamma_0 + df) = d\Gamma_0$, so $d(\Gamma_0 + df)(0)$ has also maximum rank. Therefore there exists another φ such that $\varphi(\Gamma_0 + df) = \Gamma_0$. From this and from the fact that $d(\varphi_* \Gamma) = d(\varphi\Gamma)$ we get, using (6.3),

$$d\Gamma_0 = (d\Gamma_0) \circ D\varphi^{-1} \otimes D\varphi^{-1}.$$

Since $\det(d\Gamma_0)$ is different from 0 near the origin, it follows from the above that $|\det D\varphi^{-1}| = 1$, so $|J| = 1$. Hence, applying (6.2) we obtain

$$\varphi_* (\Gamma_0 + df) = \varphi(\Gamma_0 + df) = \Gamma_0,$$

so Γ_0 is the canonical form of Γ.

Conclusion. If $\det d\Gamma(0) \neq 0$ then the germ of Γ at 0 is 1-determined.

In this way we showed that in each considered case there are 0 or 1-determined section-germs.

A section whose every germ is 0-determined is integrable (e.g. is constant in a local trivialization). Necessarily, such a section takes values in a subbundle which has homogeneous fibre. The results of this work allow to give a full classification of bundles whose every section is integrable. It is easy to see that in the two unsettled cases (density 1-forms when $n = 2m+1$,

$c = -1/(m+1)$, and 2-dimensional distributions on M^4) there are no 0-determined germs. Thus we have

THEOREM 6.1. *The only geometric object fields on manifolds of dimension greater than one, which are canonically integrable, are arbitrary* (*nowhere zero in case of vector bundles*) *sections of the following natural bundles*:

$$\Lambda^n T^* M,\ TM,\ PTM,\ PTM \times_M \Lambda^n T^* M,\ D^c TM$$

$$\textit{for } c \neq 1,\ CO(M^2),\ CH(M^2) \textit{ and } P(\mathrm{Div}\, TM)$$

(*classified up to local isomorphism*).

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