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Note on the spectral inclusion theorem for Toeplitz operators

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Abstract. The author proves the spectral inclusion theorem for general Toeplitz operators defined for a certain class of function algebras. The same theorem also holds for Toeplitz operators with a continuous symbol in a symmetric domain $D \subset C^n$.

In what follows we will consider a generalization of the spectral inclusion theorem for Toeplitz operators. We will prove that it holds for Toeplitz operators defined in the context of function algebras approximating in modulus and hypodirichlet function algebras approximating in modulus. Moreover, it also holds for Toeplitz operators with a continuous symbol, defined for a bounded symmetric domain $D \subset C^N$.

Let $A \subset C(X)$ be a function algebra on a compact Hausdorff space X and let $\mu > 0$ be a regular, finite Borel measure on X. Denote by $L^2(\mu)$ the standard Hilbert space of complex μ-square integrable functions on X. We define a Hardy space $H^2(\mu)$ as the closure of A in $L^2(\mu)$. For $\varphi \in L^\infty(\mu)$ (μ — essentially bounded) we define a Toeplitz operator by

$$T_\varphi f = P(\varphi \cdot f) \quad \text{for } f \in H^2(\mu),$$

where $P\colon L^2(\mu) \to H^2(\mu)$ is an orthogonal projection.

Denote by L_φ the operator of multiplication by φ in $L^2(\mu)$. Then we can write $T_\varphi f = PL_\varphi f$.

For a bounded linear operator T in a complex Hilbert space we denote by $\sigma_\pi(T)$ the approximate point spectrum of T. In the case of $X = \Gamma$ — the unit circle, A — disc algebra, μ — Lebesgue measure it is well known that $\sigma_\pi(T_\varphi) \supset \sigma(L_\varphi)$, see [2]. Moreover, the same inclusion is also true for so-called Wiener–Hopf operators defined on a Hardy space related to a locally compact abelian group, see [3]. Now we will prove that the above inclusion holds for a Toeplitz operator defined on $H^2(\mu)$, when A is approximating in modulus. Recall that A is approximating in modulus if every continuous $v \geq 0$ on X can be uniformly approximated by moduli of functions from A.

THEOREM 1. *Let A be a function algebra on X approximating in modulus* (1). *If $\varphi \in L^\infty(\mu)$, then*

$$\sigma_\pi(T_\varphi) \supset \sigma(L_\varphi).$$

Proof. With no loss of generality we can assume that $0 \notin \sigma_\pi(T_\varphi)$, i.e., there exists an $\varepsilon > 0$ for which

$$\|T_\varphi f\|^2 \geqslant \varepsilon^2 \|f\|^2 \quad \text{for all } f \in H^2(\mu).$$

It follows that

$$\int |\varphi|^2 |f|^2 d\mu \geqslant \varepsilon^2 \int |f|^2 d\mu \quad \text{for all } f \in A,$$

and so by our assumption we get

$$\int |\varphi|^2 |f|^2 d\mu \geqslant \varepsilon^2 \int |f|^2 d\mu \quad \text{for every continuous } v \geqslant 0.$$

Therefore $0 \notin \sigma(L_\varphi)$. The proof is complete.

Denote by $\|\varphi\|_\infty$ the essential supremum of $\varphi \in L^\infty(\mu)$. The above inclusion proves the equality $\|T_\varphi\| = \|\varphi\|_\infty$. We also derive the following corollaries.

COROLLARY 1. *If $\varphi \in L^\infty(\mu)$, then the operator T_φ is quasi-nilpotent if and only if $\varphi = 0$.*

For any operator T write $W(T) = \{(Tf, f),\ \|f\| = 1\}$. Then we get

COROLLARY 2. *If $\varphi \in L^\infty(\mu)$, then $\overline{W(T_\varphi)} = \overline{W(L_\varphi)}$ (the closure).*

COROLLARY 3. *If $\varphi \in C(X)$, then T_φ is compact if and only if $\varphi = 0$.*

COROLLARY 4 (see [2]). *If $\varphi \in C(X)$ and F is a non-zero function holomorphic in an open set which includes $\sigma(T_\varphi)$ and such that $F(T_\varphi) = 0$, then T_φ is a scalar.*

Note also that Theorem 1 includes the spectral inclusion theorem for a Wiener–Hopf operator (mentioned above) in the case of a compact abelian group. For further properties of Toeplitz operators in the context of a function algebra approximating in modulus consult [6].

The next theorem shows that the spectral inclusion theorem holds also for Toeplitz operators related to hypodirichlet algebras. Recall that A is hypodirichlet on X if the uniform closure of $\operatorname{Re} A$ has a finite codimension in $C_R(X)$ and the linear span of $\log|A^{-1}|$ is dense in $C_R(X)$. The standard example of such an algebra is $R(X)$, where X is a compact subset of the complex plane C whose complement has a finite number

(1) It is enough to assume that the linear span of $\{|v|^2,\ v \in A\}$ is dense in $C_R^+(X)$.

of components [4]. If A is hypodirichlet on X, then every $\xi \in \mathrm{Sp}\, A$ (spectrum of A) has a finite-dimensional set of representing measures M_ξ and has a unique logmodular measure $m \in M_\xi$.

We define $H^2(m)$ and a Toeplitz operator T_φ, for $\varphi \in L^\infty(m)$, as above. Now we have the following theorem.

THEOREM 2. *Let A be a hypodirichlet algebra on X. If $\varphi \in L^\infty(m)$, then $\sigma_\pi(T_\varphi) \supset \sigma(L_\varphi)$.*

Proof. If $0 \notin \sigma_\pi(T_\varphi)$, then there exists an $\varepsilon > 0$ such that $\|T_\varphi f\|^2 \geqslant \varepsilon^2 \|f\|^2$ for every $f \in H^2(m)$. Let $E \subset X$ be an arbitrary Borel set. Then for every $K = 1, 2, \ldots$ the function $f_K = \psi_K + 1/K$ satisfies the inequality $\int \log f_K dm > -\infty$. By Theorem 10.3 of [1] there exists a $g_K \in H^2(m)$ such that $f_K = |g_K|^2$. Since

$$\int |\varphi|^2 \, |g_K|^2 dm \geqslant \varepsilon^2 \int |g_K|^2 dm,$$

we have

$$\int |\varphi|^2 (\psi_E + 1/K)\, dm \geqslant \varepsilon^2 \int (\psi_E + 1/K)\, dm.$$

Hence we get

$$\int_E |\varphi|^2 dm \geqslant \varepsilon^2 m(E)$$

and so $0 \notin \sigma(L_\varphi)$. This proves the theorem. One can also derive corollaries of Theorem 2, analogous to the previous corollaries.

Now we will consider a Hardy space of functions on a bounded symmetric domain $D \subset C^N$ $(N > 1)$. Let D be a bounded symmetric domain in C^N, $0 \in D$, with Bergan–Shilov boundary bD, M the group of holomorphic automorphism of D and K its isotropy group. K acts by unitary linear transformations on C^N and is transitive on bD, and bD has a unique normalized K – invariant measure μ. See [7]. The Hardy space $H^2(D)$ is the set of holomorphic functions in D with

$$\|f\| = \sup_{0 \leqslant r < 1} \Big(\int_{bD} |f(rt)|^2 d\mu_t \Big)^{1/2} < +\infty.$$

By Theorem 2 of [5], $H^2(D)$ is isometrically isomorphic with a certain closed subspace S of $L^2(bD, \mu)$. Consult [5] for various characterization of S. Let $P\colon L^2(bD, \mu) \to S$ be an orthogonal projection. We define for $\mu \in L^\infty(\mu)$ a Toeplitz operator T_φ on S by $T_\varphi f = P(\varphi \cdot f)$.

We prove the following

THEOREM 3. *If $\varphi \in C(bD)$, then $\sigma_\pi(T_\varphi) \supset \sigma(L_\varphi)$.*

Proof. Let $\varphi(\mathring{x}) = 0$, where $\mathring{x} \in bD$. Koranyi defined in [7] a function $P(u, z) = |S(u, z)|^2 S(z, z)^{-1}$ for $u \in U$ and $z \in D$, where $\overline{U} = bD$ (the closure of U) and $S(z, w)\colon D\times D\to C$ — the Szegö kernel of D. Moreover, for every fixed $z\in D$ the function $S(u, z) = S_z(u)$ extends uniquely to the function $S_z(\cdot)\in S$. By Theorem 4.7 of [7], the function $P(u, z)$ has the following properties:

(a) $P(u, z)\geqslant 0$ for all $u\in bD$, $z\in D$,

(b) $\int_{\Gamma} P(u, z)\,d\mu = 1$ for all $z\in D$,

(c) for every $\eta > 0$

$$\lim_{z\to u_0}\int_{|u-u_0|>\eta} P(u, z)\,d\mu(u) = 0 \quad \text{for all } u_0\in bD.$$

Now, using these properties, it is easy to construct a sequence $f_n\in S$ ($\|f_n\| = 1$) such that

$$\|T_\varphi f_n\|\to 0, \quad n\to\infty. \tag{1}$$

Indeed, for a sequence $D\ni\lambda_n\to\mathring{x}$ we put

$$f_n(u) = S(u, \lambda_n)\big(S(\lambda_n, \lambda_n)\big)^{-1/2}.$$

Then by (b) $\|f_n\| = 1$ and

$$(*)\qquad \|T_\varphi f_n\|^2\leqslant\|L_\varphi f_n\|^2 = \int_{bD}|\varphi|^2|f_n|^2 d\mu$$

$$\leqslant\int_{|u-\mathring{x}|\leqslant\eta}|\varphi|^2|f_n|^2 d\mu + \int_{|u-\mathring{x}|>\eta}|\varphi|^2|f_n|^2 d\mu.$$

For a given $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$\int_{|u-\mathring{x}|\leqslant\eta}|\varphi|^2|f_n|^2 d\mu\leqslant\tfrac{1}{2}\varepsilon \quad \text{for } n = 1, 2, \ldots$$

And by (c) there exists an n_0 such that

$$\int_{|u-\mathring{x}|>\eta}|\varphi|^2|f_n|^2 d\mu\leqslant\tfrac{1}{2}\varepsilon \quad \text{for all } n\geqslant n_0.$$

Thus inequality $(*)$ proves (1).

Remark([2]). We do not know whether Theorem 3 holds for every $\varphi\in L^\infty(\mu)$.

([2]) Now we know that Theorem 3 also holds for any D (bounded, symmetric) which is biholomorphic with a ball or a polydisc in C^n.

References

[1] P. Abern and D. Sarason, *The H^p-spaces of a class of function algebras*, Acta Math. 117 (1967).
[2] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. 123 (1963).
[3] L. A. Coburn and R. G. Douglas, *On C^*-algebras of operators on a half-space I*, Inst. Hautes Etudes Sci. Publ. Math. 40 (1971).
[4] T. W. Gamelin, *Uniform algebras*, Englewood Cliffs, N. Y., 1969.
[5] K. T. Hahn and J. Mitchell, *H^p-spaces on bounded symmetric domains*, Ann. Polon. Math. 28 1 (1973), p. 89–95.
[6] J. Janas, *Toeplitz operators for a certain class of function algebras*, Studia Math. 55 (1976), p. 157–161.
[7] A. Koranyi, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Ann. Math. 82. No. 2 (1965).

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