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The zeros of functions of finite order in C^n

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Abstract. We study classes of holomorphic functions in C^n whose zeros accumulate close to certain real hyperplanes and present several applications to holomorphic functions which arise frequently in analysis. These classes of holomorphic functions generalize the functions of class $\mathscr{A}$, holomorphic functions in the half plane in C whose zeros lie close to the boundary.

Let $z = (z_1, \ldots, z_n) \in C^n$ and let $\|z\| = (\sum_{j=1}^{n} |z_j|^2)^{1/2}$ be the Euclidian norm. By a cone Γ in C^n we will always mean an open cone with vertex at the origin; Γ is an open set such that $tz \in \Gamma$ for all $t > 0$ when $z \in \Gamma$. A holomorphic function f is of finite order in Γ if there exists $k > 0$ such that $|f(z)| \leqslant C_k \exp \|z\|^k$ for all $z \in \Gamma$ and the order ϱ of f is the infimum of all k for which this holds. A holomorphic function f of order ϱ is of finite type in Γ if, for some $B > 0$, $|f(z)| \leqslant C \exp B \|z\|^{\varrho}$ for all $z \in \Gamma$.

In the study of entire functions of one complex variable (or more generally functions holomorphic in the upper half-plane), a great deal of attention has been paid to the so-called functions of class $\mathscr{A}$, those functions of exponential type (i.e. $\varrho = 1$ and finite type) in the upper half-plane whose zeros lie close to the real axis. By close to the real axis, one means that if $r_k e^{i\theta_k}$ are the zeros of in the upper half-plane, then

$$\sum_k \frac{\sin \theta_k}{1 - r_k} < \infty. \tag{1}$$

These functions generalize the idea of functions all of whose zeros are real. Many characterizations of such functions exist (cf. Boas [2], Chapters VII and VIII, Levin [7], Chapter V).

It is our purpose here to extend these ideas by studying those holomorphic functions f of finite order in cones $\Gamma \in C^n$ bounded by a finite number of $(2n-1)$-dimensional hyperplanes and whose zeros lie close to the boundary of Γ. We then give several examples of functions which possess this property, functions which arise in quite a natural way in the theory of entire functions. This will permit a rather fine characterization of their zero sets.

Let $X_\xi = \{z \in C^n\colon \operatorname{Re}\langle \xi, z\rangle > 0\}$. Suppose $\Gamma = \bigcap_{i=1}^{M} X_{\xi_i}$. We will say that a function f holomorphic in a neighbourhood of $\bar{\Gamma}$ is in class $\mathscr{G}\mathscr{A}^{(\mu,\nu)}(\Gamma)$ (*generalized class* $\mathscr{A}$ *of order* (μ, ν)) if the distance of its zero set to the boundary of Γ is of the order $r^\mu(1+\log^+ r)^\nu$ ($\log^+ a = \sup(\log a, 0)$). In order to render this more precise, we need a measure of the zero set of f. Since $\log|f|$ is R^{2n} subharmonic in Γ, if Δ is the Laplacian in R^{2n}, $\Delta = 4\sum_{j=1}^{n} \partial^2/\partial z_j \partial \bar{z}_j$, then $\sigma = \frac{1}{2\pi}\Delta \log|f|$, taken as a distribution, defines a positive measure in Γ. For $n = 1$, this is just the Dirac measure (with multiplicity) of the zero set of f, while for $n > 1$ it is the $(2n-2)$-dimensional area (with multiplicities) of the zero set of f (cf. Lelong [5], [6]). Then the zeros of f lie close to $\partial\Gamma$ of order (μ, ν) if

$$\int_\Gamma \inf_i \frac{\operatorname{Re}\langle \xi_i, a\rangle\, d\sigma(a)}{(1+\|a\|^{2n-1+\mu})(1+\log^+\|a\|)^\nu} < \infty. \tag{2}$$

For $n = 1$ and Γ the upper half-plane, this means that the zeros $a_k = r_k e^{i\theta_k}$ satisfy $\sum_k \frac{\sin\theta_k}{(1+r_k^\mu)(1+\log^+ r_k)^\nu} < \infty$ so the functions of class $\mathscr{A}$ are the functions in $\mathscr{G}\mathscr{A}^{(1,0)}(\Gamma)$ by (1).

We begin by developing an integral formula which gives necessary and sufficient conditions for a function f holomorphic in a neighbourhood of Γ and of order ϱ in Γ to be in $\mathscr{G}\mathscr{A}^{(\varrho,\nu)}(\Gamma)$. From this, we will develop several sufficient conditions which will be relatively easy to verify in practice. We note that the criteria for the class $\mathscr{G}\mathscr{A}^{(\mu,\nu)}(\Gamma)$ can be applied with some interesting results to entire functions of global order ϱ strictly greater than μ.

Our first application is to Fourier transforms. Let $\Xi = \{\xi^{(j)}\}_{j=1}^{M}$ be a set of vectors in R^n such that any subset of less than n is linearly independent and let $K(\Xi, \xi_0) = \{\xi \in R^n\colon \langle \xi^{(j)}, \xi - \xi_0\rangle < 0, j = 1, \ldots, M\}$. Let $\hat{\Xi} = \{\xi \in \bar{K}\colon \langle \xi - \xi_0, \xi^{(\lambda_j)}\rangle = 0,\ j = 1, \ldots, n-1$, for some choice $\lambda_1 < \lambda_2 < \ldots < \lambda_{n-1}\}$ and $\Gamma(\Xi) = \{z \in C^n\colon \operatorname{Re} i\langle \xi, z\rangle > 0, \xi \in \Xi\}$. Let $\mu = \mu_1 + i\mu_2$ be a bounded complex measure on K such that for either $j = 1$ or $j = 2$, μ_j is a positive measure in a neighbourhood of ξ_0 and $\int_{B(\xi_0,t)\cap K} d\mu_j \geq Ct^\alpha$ for some α, $0 \leq \alpha < +\infty$. Then if $f(z) = \int_K \exp - i\langle \xi, z\rangle\, d\mu(\xi)$, $f \in \mathscr{G}\mathscr{A}^{(0,\nu)}(\Gamma(\Xi))$ for every $\nu > 2$ and if $\alpha = 0$, $f \in \mathscr{G}\mathscr{A}^{(0,\nu)}(\Gamma(\Xi))$ for every $\nu > 1$. Slightly better results hold for $n = 1$.

Our second example applies to exponential polynomials. Let $\Xi = \{\xi^{(j)}\}_{j=1}^{m}$ be a finite set of points in C^n no two of which are co-linear over R: Let $K(\Xi) = \{\xi \in C^n\colon \operatorname{Re}\langle \xi, z\rangle \leq \sup_j \operatorname{Re}\langle \xi^{(j)}, z\rangle\ \forall z \in C^n\}$ and $\Gamma_j = \{z \in C^n\colon$

$\operatorname{Re}\langle \xi^{(j)}-\xi^{(k)}, z\rangle > 0, k \neq j\}$. Let $f(z) = \sum_{j=1}^{M} P_j(z)\exp\langle \xi^{(j)}, z\rangle + s(z)$, where the p_j are polynomials and $s(z) = \int_K \exp\langle \xi, z\rangle\, d\mu(\xi)$ for some bounded measure μ on K such that $\operatorname{supp}\mu \cap \{\xi^{(j)}\} = \emptyset$. Then $f \in \mathscr{G}\mathscr{A}^{(0,\nu)}(\Gamma_j)$ for every $\nu > 2$. If the P_j are all constants, then $f \in \mathscr{G}\mathscr{A}^{(0,\nu)}(\Gamma_j)$ for all $\nu > 1$.

Finally, we show that if $f(z) = f_1(z) + f_2(z)\exp iz_1^p$ for p an integer and f_1 and f_2 entire functions of order at most ϱ, then $f(z) \in \mathscr{G}\mathscr{A}^{(\mu,0)}(\Gamma_j)$ for $\mu > \varrho \geqslant 1$ or for $\mu = 1 > \varrho$, where $\Gamma_j = \left\{z\colon \frac{2\pi(j-1)}{p} < \arg z_1 < \frac{2\pi j}{p}\right\}$. Thus, we can apply these ideas to entire functions whose global order can be arbitrarily large.

We note that these results are capable of being generalized in the direction of proximate orders (cf. [7]) with a greater generality of application and a slight improvement in the estimates but at a cost of greater complication, and hence greater confusion, in the calculations. We did not judge the slight improvement in results worth the confusion in terms of ideas, and so we leave this line of pursuit to the interested reader.

1. The integral formula. Let $\Gamma = \{z \in C^n\colon \operatorname{Re}\langle \xi_i, z\rangle > 0,\ i = 1, \ldots, M\}$. The ξ_i are of course only determined up to positive multiples, so we can always assume without loss of generality that $\|\xi_i\| = 1$; in what follows, this assumption will always be made implicitly. Let $\Gamma_r = \Gamma \cap \{z\colon \|z\| < r\}$.

We let

$$F_{(\mu,\nu)}(\|z\|) = \frac{1}{\|z\|^{\mu+2n-1}(\log\|z\|)^{\nu}} \quad \text{for } \|z\| > 1. \tag{3}$$

Then $\Delta(x_1 F_{(\mu,\nu)}(\|z\|)) = x_1 \tilde{F}_{(\mu,\nu)}(\|z\|)$, where

$$\tilde{F}_{(\mu,\nu)}(\|z\|) = \frac{1}{\|z\|^{\mu+2n-1}(\log\|z\|)^{\nu}}\left[(\mu+2n-1)(\mu-1) + \frac{\nu(2\mu+2n-2)}{\log\|z\|} + \frac{\nu(\nu+1)}{(\log\|z\|)^2}\right]. \tag{4}$$

By the rotational invariance of the Laplacian, it follows that

$$\Delta(\operatorname{Re}\langle \xi_i, z\rangle F_{(\mu,\nu)}(\|z\|)) = \operatorname{Re}\langle \xi_i, z\rangle \tilde{F}_{(\mu,\nu)}(\|z\|).$$

Let $\lambda > 1$, $\Gamma(\lambda, R) = \Gamma_R - \bar{\Gamma}_\lambda$, $h_i^{(\mu,\nu)}(z) = \operatorname{Re}\langle \xi_i, z\rangle[F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)]$, $S_i = \{z \in \Gamma\colon \operatorname{Re}\langle \xi_i, z\rangle < \operatorname{Re}\langle \xi_j, z\rangle,\ j \neq i\}$ and $S_{ij} = \bar{S}_i \cap \bar{S}_j \cap \Gamma$. We will denote by $d\tau_p$ the Lebesgue measure on p-dimensional Euclidian space. We begin by collecting some facts essential for what follows.

Lemma 1 (cf. Levin [7], p. 21). *For $n = 1$, let $f(z)$ be holomorphic in the circle $|z| < 2eR$, with $f(0) = 1$, and let η satisfy $0 < \eta < 3e/2$. Then* *inside the circle* $|z| \leqslant R$ *but outside a family of excluded circles the sum of whose radii is not greater that* $4\eta R$

$$\log|f(z)| > -\left(2+\log\frac{3e}{2\eta}\right)\log M(2eR),$$

where $M(t) = \sup\limits_{\|z\|=t} |f(z)|$.

LEMMA 2. *Let* f *be holomorphic in a neighbourhood* $\mathcal{O}$ *of* Γ_R. *Then* $\log|f|$ *is integrable on* $\partial\Gamma \cap B(0, R)$ *and on* S_{ij} *for every* i, j.

Proof. Let $x \in \partial\Gamma$. Then x is contained in a $(2n-1)$ real dimensional face Y. Since the zero set of a holomorphic function is locally of finite $(2n-2)$ dimensional measure (cf. Lelong [5]), there exists a ball $B(x, r) \subset \mathcal{O}$ and $y \in Y \cap \partial\Gamma$ such that $d(y, x) < r/3$ and $f(y) \neq 0$. Lemma 2 implies that $\log|f|$ is integrable in $B(y, r/2) \cap Y \cap \partial\Gamma$, which contains $B(x, r/6) \cap Y \cap \partial\Gamma$. Thus, since $\partial\Gamma \cap \overline{B(0, R)}$ is compact in $\mathcal{O}$, $\log|f|$ is integrable on this set. The argument for S_{ij} is identical. Q.E.D.

Let $\alpha \in \mathscr{C}_0^\infty(B(0, 1))$ such that α depends only on $\|z\|$, $\alpha \equiv 1$ in a neighbourhood of the origin, and $\int \alpha(z)\, d\lambda(z) = 1$. We set $\alpha_\varepsilon(z) = \frac{1}{\varepsilon^{2n}}\alpha\left(\frac{z}{\varepsilon}\right)$. If f is function which is locally integrable, we set $f_\varepsilon(z) = f * \alpha_\varepsilon(z) = \int f(z-z')\alpha_\varepsilon(z')\, d\lambda(z')$. If f is subharmonic in a domain Ω, then f_ε is $\mathscr{C}^\infty$ and subharmonic in $\Omega_\varepsilon = \{z\colon d(z, \complement\Omega) > \varepsilon\}$, $f_\varepsilon \geqslant f$ in Ω_ε and $f_\varepsilon \downarrow f$.

THEOREM 1. *Let* $\Gamma = \{z\colon \operatorname{Re}\langle\xi_i, z\rangle > 0, i = 1, \ldots, M\}$ *and* $h_{(\mu,\nu)}(z) = \inf h_i^{(\mu,\nu)}(z)$ *in* $\Gamma(\lambda, R)$, $\lambda > 3$. *If* f *is holomorphic in a neighbourhood of* Γ *and* $Q = \Gamma \cap S_{2n-1}$, *then if* $\sigma_f = \Delta \frac{1}{2\pi}\log|f|$,

$$\begin{aligned}
&\int\limits_{\Gamma(\lambda,R)} h_{(\mu,\nu)}(z)\, d\sigma_f(z)\\
&\quad = \frac{1}{2\pi}\int\limits_{\Gamma(\lambda,R)} \inf_i \operatorname{Re}\langle\xi_i, z\rangle\, \tilde{F}_{(\mu,\nu)}(\|z\|)\log|f(z)|\, d\tau_{2n}(z) +\\
&\quad + \frac{1}{2\pi}\int\limits_{\partial\Gamma\cap[B(0,R)-B(0,\lambda)]} \big(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\big)\log|f(z)|\, d\tau_{2n-1}(z) -\\
&\quad - \frac{1}{2\pi}\sum_{i<j}\int\limits_{S_{ij}\cap\Gamma(\lambda,R)} \big(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\big)\|\xi_i - \xi_j\|\log|f(z)|\, d\tau_{2n-1}(z) +\\
&\quad + \frac{1}{2\pi R^\mu}\left[\frac{(\mu+2n-1)}{(\log R)^\nu} + \frac{\nu}{(\log R)^{\nu+1}}\right]\int\limits_Q \log|f(R\omega)|\, S_1(\omega)\, d\tau_{2n-1}(\omega) +\\
&\quad + A(f, \lambda, R),
\end{aligned}$$

where $1 \geqslant S_1(\omega) \geqslant 0$ *and* $|A(f; R, \lambda)| \leqslant C_{f,\lambda}$ *independent of* R. *(*$F_{(\mu,\nu)}(\|z\|)$ *and* $\tilde{F}_{(\mu,\nu)}(\|z\|)$ *are defined by* (3) *and* (4), *respectively.)*

Proof. We begin by supposing $g \in \mathscr{C}^{\infty}$ in $\Gamma(0, R)$. Let $S_i(\lambda', R) = S_i \cap \Gamma(\lambda', R)$. Clearly $\int_{\Gamma(\lambda', R)} h_{(\mu,\nu)} \Delta g d\tau_{2n} = \sum_i \int_{S_i(\lambda', R)} h_i^{(\mu,\nu)} \Delta g d\tau_{2n}$. We apply Green's theorem to each of these integrals. Here $\vec{\eta}_i$ is the external unit normal to $\partial S_i(\lambda', R)$,

$$\int_{S_i(\lambda',R)} h_i^{(\mu,\nu)} \Delta g d\tau_{2n} = \int_{S_i(\lambda',R)} g \operatorname{Re}\langle \xi_i, z\rangle \tilde{F}_{(\mu,\nu)}(\|z\|) d\tau_{2n}(z) + \\ + \int_{\partial S_i(\lambda',R)} \left[\frac{dg}{d\vec{\eta}_i} h_i^{(\mu,\nu)}(z) - \frac{d}{d\vec{\eta}_i} h_i^{(\mu,\nu)}(z) g(z)\right] d\tau_{2n-1}(z).$$

We note that on $\partial B(0, R)$ and on $\partial\Gamma \cap S_i(\lambda', R)$; $h_i^{(\mu,\nu)} \equiv 0$, so

$$\int_{S_i(\lambda',R)} h_i^{(\mu,\nu)} \Delta g d\tau_{2n} = \int_{S_i(\lambda',R)} g \operatorname{Re}\langle \xi_i, z\rangle \tilde{F}_{(\mu,\nu)}(\|z\|) d\tau_{2n}(z) +$$

$$+\frac{1}{R^{\mu}}\left[\frac{(\mu+2n-1)}{(\log R)^{\nu}} + \frac{\nu}{(\log R)^{\nu+1}}\right] \int_{\partial S_i(\lambda',R) \cap \partial B(0,R)} \operatorname{Re}\langle \xi_i, \omega\rangle g(R\omega) d\tau_{2n-1}(\omega) +$$

$$+ \int_{\partial\Gamma \cap \partial S_i(\lambda,R)} g\left(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\right) d\tau_{2n-1}(z) +$$

$$+ \sum_{j \neq i} \left\{ \int_{S_{ij}} \left[\frac{dg}{d\vec{\eta}_i} h_i^{(\mu,\nu)} - \frac{dh_i^{(\mu,\nu)}}{d\vec{\eta}_i} g\right] d\tau_{2n-1}(z) + \right. \\ \left. + \int_{\partial S_i(\lambda',R) \cap \partial B(0,1)} \left[\frac{dg}{d\vec{\eta}_i} h_i^{(\mu,\nu)} - \frac{d}{d\vec{\eta}_i} h_i^{(\mu,\nu)} g\right] (\lambda'\omega) \lambda'^{2n-1} d\tau_{2n-1}(\omega) \right\}.$$

We note that on S_{ij}, $\vec{\eta}_i = \dfrac{\bar{\xi}_i - \bar{\xi}_j}{\|\xi_i - \xi_j\|}$, $\vec{\eta}_j = -\vec{\eta}_i$ and $\operatorname{Re}\langle \xi_i, z\rangle = \operatorname{Re}\langle \xi_j, z\rangle$; hence

$$\frac{d}{d\vec{\eta}_i}\left[\operatorname{Re}\langle \xi_i, z\rangle \left(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\right)\right] + \frac{d}{d\vec{\eta}_j}\left[\operatorname{Re}\langle \xi_j, z\rangle \left(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\right)\right] \\ = \|\xi_i - \xi_j\| \left(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\right) \quad \text{on } S_{ij}.$$

Thus, summing over i, we get

$$\int_{\Gamma(\lambda',R)} h_{(\mu,\nu)} \Delta g d\tau_{2n} = \int_{\Gamma(\lambda',R)} g \inf_i \operatorname{Re}\langle \xi_i, z\rangle \tilde{F}_{(\mu,\nu)}(\|z\|) d\tau_{2n}(z) +$$

$$+\frac{1}{R^{\mu}}\left[\frac{(\mu+2n-1)}{(\log R)^{\nu}} + \frac{\nu}{(\log R)^{\nu+1}}\right] \int_{Q} \inf_i \operatorname{Re}\langle \xi_i, \omega\rangle g(R\omega) d\tau_{2n-1}(\omega) +$$

$$+ \int_{\partial\Gamma \cap [B(0,R) - B(0,\lambda')]} g\left(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\right) d\tau_{2n-1}(z) -$$

$$-\sum_{i<j}\int_{S_{ij}\cap[B(0,R)-B(0,\lambda')]}\|\xi_i-\xi_j\|\,[F_{(\mu,\nu)}(\|z\|)-F_{(\mu,\nu)}(R)]\,g\,d\tau_{2n-1}(z)+$$

$$+\int_Q\left[\frac{d}{d\vec{\zeta}}h_{(\mu,\nu)}(\lambda'\omega)g(\lambda'\omega)\lambda'^{2n-1}-\frac{d}{d\vec{\zeta}}g(\lambda'\omega)h_{(\mu,\nu)}(\lambda'\omega)\lambda'^{2n-1}\right]d\tau_{2n-1}(\omega),$$

where $\vec{\zeta}$ is the exterior unit normal to $B(0,\lambda')$. But

$$\text{(5)}\qquad \frac{d}{d\vec{\zeta}}[g(\lambda'\omega)]\lambda'^{2n-1}h_{(\mu,\nu)}(\lambda'\omega)=\frac{d}{d\vec{\zeta}}[g(\lambda'\omega)\lambda'^{2n-1}h_{(\mu,\nu)}(\lambda'\omega)]-$$

$$-g(\lambda'\omega)(2n-1)\lambda'^{2n-2}h_{(\mu,\nu)}(\lambda'\omega)-g(\lambda'\omega)\lambda'^{2n-1}\frac{d}{d\vec{\zeta}}h_{(\mu,\nu)}(\lambda'\omega).$$

We now integrate from $\lambda/2$ to λ and use (5) to get

$$\text{(6)}\qquad \int_{\Gamma(\lambda,R)}h_{(\mu,\nu)}\Delta g\,d\tau_{2n}=\int_{\Gamma(\lambda,R)}g\inf_i\operatorname{Re}\langle\xi_j,z\rangle\tilde{F}_{(\mu,\nu)}(\|z\|)\,d\tau_{2n}(z)+$$

$$+\frac{1}{R^\mu}\frac{(\mu+2n-1)}{(\log R)^\nu}+\frac{\nu}{(\log R)^{\nu+1}}\int_Q\inf_i\operatorname{Re}\langle\xi_j,\omega\rangle g(R\omega)\,d\tau_{2n-1}(\omega)+$$

$$+\int_{\partial\Gamma\cap[B(0,R)-B(0,\lambda)]}g\big(F_{(\mu,\nu)}(\|z\|)-F_{(\mu,\nu)}(R)\big)\,d\tau_{2n-1}(z)-$$

$$-\sum_{i<j}\int_{S_{ij}\cap[B(0,R)-B(0,\lambda)]}\|\xi_i-\xi_j\|\big(F_{(\mu,\nu)}(\|z\|)-F_{(\mu,\nu)}(R)\big)\,dg\tau_{2n-1}(z)+A(g,\lambda,R),$$

where

$$\tfrac{1}{2}\lambda A(g,\lambda,R)=-\int_{\lambda/2}^{\lambda}\Big(\int_{\Gamma(0,\lambda)-\Gamma(0,\lambda')}h_{(\mu,\nu)}\Delta g\,d\tau_{2n}\Big)d\lambda'+$$

$$+\int_{\lambda/2}^{\lambda}\int_{\Gamma(0,\lambda)-\Gamma(0,\lambda')}g\inf_i\operatorname{Re}\langle\xi_i,z\rangle\tilde{F}_{(\mu,\nu)}(\|z\|)\,d\tau_{2n}\,d\lambda'+$$

$$+\int_{\lambda/2}^{\lambda}\int_{\partial\Gamma[B(0,\lambda)-B(0,\lambda')]}g\big(F_{(\mu,\nu)}(\|z\|)-F_{(\mu,\nu)}(R)\big)\,d\tau_{2n-1}\,d\lambda'-$$

$$-\sum_{i>j}\int_{\lambda/2}^{\lambda}\int_{S_{ij}\cap[B(0,\lambda)-B(0,\lambda')]}\|\xi_i-\xi_j\|\big(F_{(\mu,\nu)}(\|z\|)-F_{(\mu,\nu)}(R)\big)g\,d\tau_{2n-1}(z)\,d\lambda'+$$

$$+2\int_{\lambda/2}^{\lambda}\int_{\Gamma\cap\partial B(0,1)}\frac{d}{d\vec{\zeta}}h_{(\mu,\nu)}(\lambda'\omega)g(\lambda'\omega)\lambda'^{2n-1}\,d\tau_{2n-1}(\omega)\,d\lambda'-$$

$$-\int_Q[g(\lambda\omega)\lambda^{2n-1}h_{(\mu,\nu)}(\lambda\omega)-g(\tfrac{1}{2}\lambda\omega)(\tfrac{1}{2}\lambda)^{2n-1}h_{(\mu,\nu)}(\tfrac{1}{2}\lambda\omega)]\,d\tau_{2n-1}(\omega)+$$

$$+\int_{\lambda/2}^{\lambda}\int_Q g(\lambda\omega)(2n-1)\lambda'^{2n-2}h_{(\mu,\nu)}(\lambda'\omega)\,d\tau_{2n-1}(\omega)\,d\lambda'.$$

Since $\log|f|$ is a subharmonic function, it is locally integrable. Furthermore, it is integrable on every set $A_\lambda=\Gamma\cap\partial B(0,\lambda)$. To see this, we note that, if $f(y_0) \neq 0$, $\int_{\partial D} \log|f(x)| P(x, y) dS(x) \geqslant \log|f(y_0)|$, and since f is bounded above, $\log|f(x)|$ is integrable. But there exists a domain D with $\mathscr{C}^2$ boundary in $\mathcal{O}$ such that $A_\lambda \subset \partial D$, which proves the assertion.

We now apply (6) to $g_\varepsilon = \frac{1}{2\pi} \log|f| * \alpha_\varepsilon$ and let $\varepsilon \to 0$. We note that there is a constant A such that $A \geqslant g_\varepsilon \geqslant \log|f|$, so all the integrals converge by the Lebesgue dominated convergence theorem. Q.E.D.

COROLLARY 1. *Let f be holomorphic in a neighbourhood of Γ and of order μ in Γ. Then a sufficient condition for $f \in \mathscr{G}\mathscr{A}^{(\mu,\nu)}(\Gamma)$ is that for some $\lambda > 3$, there exists $M_{f,\lambda}$ such that*

$$\int_{\Gamma(\lambda,R)} \log|f(z)| \inf_i \operatorname{Re}\langle \xi_i, z\rangle \tilde{F}_{(\mu,\nu)}(\|z\|) d\tau_{2n}(z) +$$
$$+ \int_{\partial\Gamma \cap [B(0,R) - B(0,\lambda)]} \log|f(z)| \big(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\big) d\tau_{2n-1} -$$
$$- \sum_{i<j} \int_{S_{ij} \cap \Gamma(\lambda,R)} \log|f(z)| \, \|\xi_i - \xi_j\| \big(F_{(\mu,\nu)}(\|z\|) - F_{(\mu,\nu)}(R)\big) d\tau_{2n-1}(z) < M_{f,\lambda}$$

for all R.

We now state a more restrictive condition, which in practise is actually what we shall verify. We set $\log^- a = \sup(-\log a, 0)$.

COROLLARY 2. *Let f be a holomorphic function in a neighbourhood of Γ of order μ in Γ. Then a sufficient condition for f to be in $\mathscr{G}\mathscr{A}^{(\mu,\nu)}(\Gamma)$ is that*

$$\int_{\Gamma(\lambda,R)} \log^{\pm} |f(z)| \inf_i \operatorname{Re}\langle \xi_i, z\rangle \tilde{F}_{(\mu,\nu)}(\|z\|) d\tau_{2n}(z) +$$
$$+ \int_{\partial\Gamma \cap [B(0,R) - B(0,\lambda)]} \log^+ |f(z)| F_{(\mu,\nu)}(\|z\|) d\tau_{2n-1}(z) +$$
$$+ \sum_{i<j} \int_{S_{ij} \cap \Gamma(\lambda,R)} \log^- |f(z)| F_{(\mu,\nu)}(\|z\|) d\tau_{2n-1}(z) < +\infty,$$

where we take the + sign in the first term if $\mu \geqslant 1$ and the minus sign if $\mu < 1$.

2. Fourier transforms. It is well known that a function bounded on the real axis and holomorphic and of exponential type in the upper half-plane is in class $\mathscr{A}$ (this follows, for instance, from Corollary 2) so that the Fourier transform of a bounded measure with compact support is in class $\mathscr{A}$ for $n = 1$. But with some rather mild supplementary hypotheses, one can conclude much more.

THEOREM 2. *Let $n = 1$ and let $\mu = \mu_1 + i\mu_2$ be a bounded complex measure on the interval $A = \{-\infty < y \leqslant a < +\infty\}$ with real part μ_1 and* *imaginary part* μ_2 *such that for some* $\alpha, 0 \leqslant \alpha < +\infty$ *and some* $\delta > 0$ *one of the following holds*: $+\int_{\{|y-a| \leqslant t\} \cap A} d\mu_1(t) \geqslant Ct^\alpha$, $-\int_{\{|y-a| \geqslant t\} \cap A} d\mu_1(t) \geqslant Ct^\alpha$, $+\int_{\{|y-a| \geqslant t\} \cap A} d\mu_2(t) \geqslant Ct^\alpha$ *or* $-\int_{\{|y-a| \leqslant t\} \cap A} d\mu_2(t) \geqslant Ct^\alpha$ *for* $t < \delta$.

Let $f(z) = \int_{-\infty}^{a} \exp - iztd\mu(t)$. *Then* $f \in \mathcal{G}.\mathcal{A}^{(0,\nu)}(y \geqslant 0)$ *for every* $\nu > 2$. *If* $\alpha = 0$, *then* $f \in \mathcal{G}.\mathcal{A}^{(0,\nu)}(y \geqslant 0)$ *for* $\nu > 1$.

Proof. We consider the function

$$g(z) = f(z) \exp iza = \int_{-\infty}^{a} \exp - iz(t-a) \, d\mu(t)$$

which has the same zeros as $f(z)$. We note that

$$\Big| \int_{-\infty}^{a} \exp y(t-a) \, d\mu(t) \Big| \leqslant C \exp - \delta y.$$

Let $\mu(t) = \int_{t}^{a} d\mu(t)$, $t < a$. We assume that $a-\delta$ is a point of continuity for $\mu(t)$. Since there are at most a countable number of discontinuities for $\mu(t)$, we lose no generality. Let $\mu_0 = \lim_{t \to a} \mu(t)$. If $\mu_0 \neq 0$, then

$$\Big| \int_{a-\delta}^{a} \exp y(t-a) \, d\mu(t) \Big| \geqslant C > 0.$$

Assume that $\mu_0 = 0$. Integrating by parts, we find that

$$\int_{a-\delta}^{a} \exp y(t-a) \, d\mu(t) = \left[-\exp y(t-a) \mu(t) \right]_{a-\delta}^{a} + y \int_{a-\delta}^{a} \exp y(t-a) \mu(t) \, dt$$

$$= \mu(a-\delta) \exp - y\delta + y \int_{a-\delta}^{a} \exp y(t-a) \, d\mu(t) \, dt$$

and it follows from the hypotheses that

$$\Big| \int_{a-\delta}^{a} \exp y(t-a) \mu(t) \, dt \Big| \geqslant \frac{C}{y^{1+\alpha}} \int_{-y\delta}^{0} \exp W \cdot W^\alpha \, dW \geqslant \frac{C}{y^{1+\alpha}}$$

so for y sufficiently large, $\log |g(y)| \geqslant -\frac{1}{2} \alpha \log |y| - C''$.

Since $|g(z)|$ is bounded on the x-axis, it is enough, by Corollary 2 to Theorem 1, to show that

$$\int_{y \geqslant 0} \frac{y \log |g(x+iy)| \, d\tau_2(z)}{(1+(x^2+y^2))^3 (1+\log^+(x^2+y^2))^\nu} < +\infty \quad \text{for } \nu > 2.$$

But since we may assume that $|g(z)| \leqslant 1$ for $y \geqslant 0$,

$$\int_{y \geqslant 0} \frac{y \log |g(x+iy)|\, d\tau_2(z)}{(1+(x^2+y^2))^3 (1+\log^+ (x^2+y^2))^\nu}$$
$$\geqslant \int_{y \geqslant 0} \frac{y}{(1+y)(1+\log^+ y)^\nu} \frac{\log |g(x+iy)|\, d\tau_2(z)}{(x^2+y^2)}.$$

By the Poisson Integral Formula for the upper half-plane, since $\log |g|$ is a subharmonic function, we have

$$\int \frac{y \log |g(x+iy)|\, dx}{(x^2+y^2)} \geqslant \log |g(2y)| \geqslant -c \log |y| - c'$$

and since

$$\int_{y \geqslant 2} \frac{\log |y|\, dy}{(1+y)(\log y)^\nu} < +\infty \quad \text{for } \nu > 2,$$

the conclusion now follows. For $\alpha = 0$, $\log |g(y)| \geqslant C > -\infty$ and the proof is similar. Q.E.D.

We now prove a higher dimensional version of the same theorem (with minor modifications).

Let $\Xi = \{\xi^{(j)}\}_{j=1}^M$ be a system of vectors in $\boldsymbol{R}^n$ with $M \geqslant n$ such that any subset of at most n vectors forms a linearly independent system. Let $K(\Xi, \xi_0) = \{\xi \in \boldsymbol{R}^n \colon \langle \xi^{(j)}, \xi - \xi_0 \rangle < 0,\ j = 1, \ldots, M\}$.

Let $\hat{\Xi} = \{\xi \in \bar{K} \colon \|\xi\| = 1, \langle \xi - \xi_0, \xi^{(\lambda_j)} \rangle = 0,\ j = 1, \ldots, n-1$ for some choice $\lambda_1 < \lambda_2 < \ldots < \lambda_{n-1}\}$.

Note that if $M = n$ there are exactly n vectors in $\hat{\Xi}$.

Set $\Gamma(\Xi) = \{z \in \boldsymbol{C}^n \colon \operatorname{Re} i\langle \xi, z \rangle > 0, \xi \in \hat{\Xi}\}$. If $\Xi' \subset \Xi$, then $K(\Xi, \xi_0) \subset K(\Xi', \xi_0)$ and so for $\xi \in \bar{K}(\Xi, \xi_0)$ we can write $\xi = \sum_{j=1}^{Q} c_j(\xi_j - \xi_0)$ with $c_j \geqslant 0$ and $\xi_j \in \hat{\Xi}'$ so $\Gamma(\Xi') \subset \Gamma(\Xi)$.

THEOREM 3. *Let $\mu = \mu_1 + i\mu_2$ be a bounded complex measure on $K(\Xi, \xi_0)$ for $\Xi = \{\xi^{(j)}\}_{j=1}^M$ ($M \geqslant n$ as above) with real part μ_1 and imaginary part μ_2 such that for some α, $0 \leqslant \alpha < +\infty$ and some $\delta > 0$ one of the following holds: $\mu_1|_{B(\xi_0,\delta) \cap K}$ is a positive measure and $\int_{B(\xi_0,t) \cap K} d\mu_1(\xi) \geqslant Ct^\alpha$ for $t < \delta$ or $\mu_2|_{B(\xi_0,\delta) \cap K}$ is a positive measure and $\int_{B(\xi_0,t) \cap K} d\mu_2(\xi) \geqslant Ct^\alpha$ for $t < \delta$.*

If $f(z) = \pm \int_K \exp - i\langle \xi, z \rangle\, d\mu(\xi)$, then $f \in \mathcal{G}\mathcal{A}^{(0,\nu)}(\Gamma(\Xi))$ for every $\nu > 2$ if $\alpha \neq 0$ and $\nu > 1$ if $\alpha = 0$.

Proof. We consider the function

$$g(z) = f(z) \exp i\langle \xi_0, z \rangle = \int_K \exp - i\langle \xi - \xi_0, z \rangle\, d\mu(\xi)$$

which has the same zeros as $f(z)$. We remark that $g(z)$ is bounded in $\Gamma(\Xi)$ for if $z \in \Gamma(\Xi)$, then $\operatorname{Re} i\langle \xi, z\rangle > 0$ for some $\xi \in \hat{\Xi}$ and so $\sup_{\bar{B}(0,1) \cap K} \operatorname{Re} i\langle \xi, z\rangle > 0$ since the supremum is attained at an extremal point. Thus $\operatorname{Re} - i\langle \xi, z\rangle < 0$. We assume without loss of generality that $|g(z)| \leqslant 1$. Furthermore, if Ξ' is any subset of Ξ with only n elements, then $\Gamma(\Xi') \subset \Gamma(\Xi)$ and since $\Gamma(\Xi) = \bigcup_\sigma \Gamma(\Xi'(\sigma))$, where the union is taken over all possible subsets σ of Ξ with n elements, if we can prove the theorem for Ξ with only n elements, the general result will follow. Thus we assume without loss of generality that $M = n$.

Let ξ_j be the elements of $\hat{\Xi}$, $j = 1, \ldots, n$. Since $\langle \xi - \xi_0, z\rangle \geqslant -\|\xi - \xi_0\| \, \|z\|$, if $iy = (iy_1, \ldots, iy_n) \in \Gamma(\Xi)$, then one shows as in Theorem 2 that

$$|g(iy)| \geqslant c \exp - \delta \min \langle \xi_j, y\rangle + c' \|y\|^{-d}. \tag{7}$$

We choose as local coordinates, in $\Gamma(\Xi)$, $w = (w_1, \ldots, w_n)$, $w_j = \langle \xi_{(j)}, z\rangle$ and we write $w_j = u_j + iv_j$. Then $\Gamma(\Xi) = \{w\colon \operatorname{Im} w_j \geqslant 0, j = 1, \ldots, n\}$.

If we can show that

$$-\int_{\Gamma(\Xi)} \frac{\log |g(w)| \, d\tau_{2n}(w)}{(1+\|w\|)^{2n}(1+\log^+ \|w\|^\nu)} < +\infty \tag{8}$$

and

$$-\int_{\tilde{S}_{ij}} \frac{\log |g(w)| \, d\tau_{2n-1}(w)}{(1+\|w\|^{2n-1})(1+\log^+ \|w\|)^\nu} < +\infty \quad \text{for every pair } i, j, \tag{9}$$

where $\tilde{S}_{ij} = \{w \in \Gamma(\Xi)\colon \operatorname{Im} w_i = \operatorname{Im} w_j\}$, then the result will follow from Corollary to Theorem 1.

We begin by proving (8). Then for some positive constants

$$\int_{\Gamma(\Xi)} \frac{\log |g(w)| \, d\tau_{2n}(w)}{(1+\|w\|)^{2n}(1+\log^+ \|w\|)^\nu} \geqslant C_1 \int_{\substack{v_j \geqslant 0 \\ \|v\| \geqslant 2}} \frac{dv_1 \ldots dv_n}{(1+\|v\|)^n(1+\log^+ \|v\|)^\nu} \|v\|^n \times$$

$$\times \int \frac{\log |g(u_1, v_1, \ldots, u_n, v_n)| \, du_1 \ldots du_n}{\prod_{j=1}^{n} (u_j^2 + \|v\|^2)} - C_2$$

$$\geqslant C_1 \int_{\substack{v_j \geqslant 0 \\ \|v\| \geqslant 2}} \frac{dv_1 \ldots dv_n}{(1+\|v\|)^n(1+\log^+ \|v\|)^\nu} \log \left|g(0, v_1 + \|v\|, \ldots, 0, v_n + \|v\|)\right| - C_2$$

by repeated application of the Poisson Integral Formula for the half-plane to g, and it follows from (7) that the integral on the right converges for $\alpha \neq 0$ and $\nu > 1$.

We shall prove (9) only for S_{12}, the other cases being proved in a similar manner. Let $w' = (w'_1, \ldots, w'_n)$ with $w'_1 = w_1 - w_2$ and $w'_j = w_j$, $j \geqslant 2$. Then $S_{12} = \{w' \colon \operatorname{Im} w'_1 = 0,\ \operatorname{Im} w'_j \geqslant 0,\ j \geqslant 2\}$. If we let

$$A(r) = \int_{B(0,r)\cap S_{12}} \frac{-\log |g(w')|\, d\tau_{2n-1}(w')}{(1+\|w'\|)^{2n-1}(1+\log^+\|w'\|)^{\nu}},$$

then it suffices to show that $\lim\limits_{r\to\infty} A(r) < +\infty$. Let

$$\sigma(r) = \int_{B(0,r)\cap S_{12}} -\log |g(w')|\, d\tau_{2n-1}(w').$$

Then after an integration by parts, we obtain for $r > 1$

$$(10)\qquad A(r) = \frac{\sigma(r)}{(1+r)^{2n-1}(1+\log r)^{\nu}} + (2n-1)\int_1^r \frac{\sigma(t)\,dt}{(1+t)^{2n}(1+\log t)^{\nu}} + \nu\int_1^t \frac{\sigma(t)\,dt}{(1+t)^{2n}(1+\log t)^{\nu+1}} + C.$$

Let

$$Q(r) = \int_0^r \times \ldots \times \int_0^r dv'_2 \ldots dv'_n \int_{-r}^{+r} \times \ldots \times \int_{-r}^{+r} \big(-\log |g(u'_1, 0, u'_2, v'_2, \ldots, u'_n, v'_n)|\big)\, du'_1 \ldots du'_n$$

so that $Q(r) \geqslant \sigma(r)$. By repeated applications of the Poisson Integral Formula for the half-plane, we obtain

$$\frac{Q(r)}{r^n} \leqslant C_1 \int_0^r \times \ldots \times \int_0^r dv'_2 \ldots dv'_n \int_{-r}^{+r} \times \ldots \times \int_{-r}^{+r} \prod_{j=1}^{n} \frac{r}{[(u'_j)^2 + r^2]} \times \big(-\log |g(u'_1, 0, u'_2, v'_2, \ldots, u'_n, v'_1)|\big)\, du'_1 \ldots du'_n$$

$$\leqslant C_1 \int_0^r \times \ldots \times \int_0^r dv'_2 \ldots dv'_n \int_{-\infty}^{+\infty} \times \ldots \times \int_{-\infty}^{+\infty} \prod_{j=1}^{n} \frac{r}{[(u'_j)^2 + r^2]} \times \big(-\log |g(u'_1, 0, u'_2, v'_2, \ldots, u'_n, v'_n)|\big)\, du'_1 \ldots du'_n$$

$$\leqslant C_1 \int_0^r \times \ldots \times \int_0^r dv'_2 \ldots dv'_n \int_{-\infty}^{+\infty} \frac{r}{[(u'_1)^2 + r^2]} \times \big(-\log |g(u'_1, 0, 0, v'_2 + r, \ldots, 0, v'_n + r)|\big)\, du'_1$$

$$\leqslant C_1 \int_0^r \times \ldots \times \int_0^r dv_2 \ldots dv_n \int_{-\infty}^{+\infty} \frac{r}{[u_1^2 + r^2]} \times \big(-\log |g(u_1, v_2 + r, 0, v_2 + r, 0, \ldots, 0, v_n + r)|\big) du_1$$

$$\leqslant C_1 \int_0^r \times \ldots \times \int_0^r dv_2 \ldots dv_n\big(-\log|g(0, v_2+2r, 0, v_2+r, 0, \ldots, 0, v_n+r)|\big)$$

$$\leqslant C_2 \alpha \log r \cdot r^{n-1} \quad \text{for } \alpha \neq 0 \quad \text{by (7) for } r \text{ large,}$$

$$\leqslant C_2 r^{n-1} \quad \text{for } \alpha = 0,$$

so

$$\sigma(r) \leqslant C_2 r^{2n-1} \alpha \log r \quad \text{for } \alpha \neq 0,$$

$$\sigma(r) \leqslant C_2 r^{2n-1} \quad \text{for } \alpha = 0,$$

which shows by (10) that $\lim\limits_{r\to\infty} A(r) < +\infty$. Q.E.D.

We compare the results of Theorem 3 to conclusions that we could draw from other methods.

If Γ is a connected cone in C^n and $f(z)$ is holomorphic of order ϱ in Γ and finite type, we define the indicator function of f (with respect to order ϱ) by

$$h_f^*(z) = \overline{\lim_{z'\to z}}\ \overline{\lim_{t\to\infty}} \frac{\log|f(tz')|}{t^\varrho};$$

it is pluri-subharmonic and positively homogeneous of order ϱ in Γ.

Let ϱ be any positive number. Then if $g(z)$ is the function considered in Theorem 3, $h_g^*(z) \leqslant 0$ since $g(z)$ is bounded in $\Gamma(\Xi)$, and, since by (7), $h_g^*(z_0) = 0$ for at least one point in $\Gamma(\Xi), h_g^*(z) \equiv 0$ in $\Gamma(\Xi)$ by the maximum principle. It then follows from [4] that $g(z)$ is of completely regular growth in $\Gamma(\Xi)$ since by (7) it is of regular growth along one ray and $h_g^*(z)$ is pluri-harmonic. Thus by [3], if $\Gamma' \subset \Gamma(\Xi)$ such that $\bar{\Gamma}| \cap \bar{\Gamma}(\Xi) = \{0\}$, then, for $\varepsilon > 0$, $\int\limits_{B(0,r)\cap\Gamma} \Delta\left(\frac{1}{2\pi}\log|g|\right) \leqslant \varepsilon r^{2n-2+\varrho}$ for $r > R_{\varepsilon,\varrho}$. We note that this technique says nothing about how the zero set approaches the boundary of $\Gamma(\Xi)$ and gives even weaker results in cones contained in the interior of $\Gamma(\Xi)$.

3. Exponential polynomials. Let $\Xi = \{\xi_0^{(j)}\}_{j=1}^M$ be a finite set of points in C^n no two of which are colinear and set $K(\Xi) = \{\xi \in C^n\colon \operatorname{Re}\langle\xi, z\rangle \leqslant \sup\limits_j \operatorname{Re}\langle\xi^{(j)}, z\rangle\ \forall\, z\, C^n\}, \Gamma_j(\Xi) = \{z \in C^n\colon \operatorname{Re}\langle\xi^{(j)}-\xi^{(k)}, z\rangle > 0, k \neq j\}$.

THEOREM 4. *Let* $f(z) = \sum\limits_{j=1}^M P_j(z)\exp\langle\xi^{(j)}, z\rangle + s(z)$ *for polynomials* $P_j(z)$ *and* $s(z) = \int\limits_K \exp\langle\xi, z\rangle\, d\mu(\xi)$ *for some bounded complex measure* μ *on* K *such that* $\operatorname{supp}\mu \cap \{\xi^{(j)}\} = \{\emptyset\}$. *Then* $f \in \mathcal{G}^{(0,\nu)}(\Gamma_j)$ *for every* $\nu > 2$. *If the* P *are all constant, then* $f \in \mathcal{G}\mathcal{A}^{(0,\nu)}(\Gamma_j)$ *for all* $\nu > 1$.

Proof. Let $g(z) = f(z)\exp-\langle\xi^{(j)}, z\rangle = P_j(z) + \sum\limits_{k\neq j} P_k(z)\exp\langle\xi^{(k)}-\xi^{(j)}, z\rangle + s(z)\exp-\langle\xi^{(j)}, z\rangle$. Then $|s(z)\exp-\langle\xi^{(j)}, z\rangle| \leqslant C\exp-\delta\|z\|$ in Γ_j. We shall apply Corollary 2 to Theorem 1 to $g(z)$. First we note that $\log|g(z)| \leqslant C_1 + C_2 \log^+ \|z\|$ on $\partial\Gamma_j$ so

$$\int_{\partial\Gamma_j} \frac{\log^+ |g(z)|\, d\tau_{2n-1}(z)}{(1+\|z\|)^{2n-1}(\log^+ \|z\|+1)^{\nu}} \leqslant \int_0^{\infty} \frac{C_3}{(1+t)(\log^+ t+1)^{\nu-1}}\, dt < +\infty$$

if $\nu > 2$.

Let $S_{kl} = \{z \in \Gamma_j\colon \operatorname{Re}\langle \xi^{(k)} - \xi^{(j)}, z\rangle = \operatorname{Re}\langle \xi^{(l)} - \xi^{(j)}, z\rangle \geqslant \operatorname{Re}\langle \xi^{(m)} - \xi^{(j)}, z\rangle,$ $m \neq k, l\}$ for $k \neq l \neq j$, and let $A_{kl} = S_{kl} \cap \partial B(0,1)$. Then

$$\int_{S_{kl}} \frac{\log^- |g(z)|\, d\tau_{2n-1}(z)}{(1+\|z\|)^{2n-1}(\log^+ \|z\|+1)^{\nu}} \leqslant \int_{S_{kl}} \frac{\big|\log|g(z)|\big|\, d\tau_{2n-1}(z)}{(1+\|z\|)^{2n-1}(\log^+ \|z\|+1)^{\nu}}$$
$$\leqslant \int_{A_{kl}} \int_0^{\infty} \frac{\big|\log|g(t\omega)|\big|\, dt}{(1+t)(\log^+ t+1)^{\nu}}\, d\tau_{2n-2}(\omega)$$

and we shall show that this last integral is bounded for all k, l.

Let $\eta_1 = \dfrac{\xi^{(k)} - \xi^{(j)}}{\|\xi^{(k)} - \xi^{(j)}\|}$ and $\eta_2, \ldots, \eta_n$ be such that the vectors $\{\eta_j\}$ form an orthonormal system and let $\omega' = (\omega'_1, \ldots, \omega'_n)$ with $\omega'_j = \langle \eta_j, \omega\rangle$. There exists a constant $\alpha > 0$ such that, for $\omega \in A_{kl}$, $\operatorname{Re}\langle \xi^{(m)} - \xi^{(j)}, \omega\rangle \leqslant \alpha \operatorname{Re}\langle \xi^{(k)} - \xi^{(j)}, \omega\rangle < 0$ for $m \neq j$. Let $\lambda' = \operatorname{Re}\omega'_1$. Then for $\lambda' > 0$ fixed

$$|g(t\omega') - P_j(t\omega')| \leqslant C \exp(-\alpha\lambda' t + c\log^+ t). \tag{11}$$

Set $\nu' = \dfrac{\nu - 1}{2}$ and $t_{\lambda'} = \dfrac{(\log^-(\alpha\lambda'))^{\nu'}}{\alpha\lambda'} + C_0$, where C_0 is a constant to be fixed later. After an integration by parts, we obtain

$$\int_0^{t_{\lambda'}} \frac{\big|\log|g(t\omega')|\big|\, dt}{(1+t)(\log^+ t+1)^{\nu}} = \int_0^{t_{\lambda'}} \frac{\big|\log|g(t\omega')|\big|\, dt}{(1+t_{\lambda'})(\log^+ t_{\lambda'}+1)^{\nu}} +$$
$$+ \int_0^{t_{\lambda'}} \int_0^{t} \big|\log|g(s\omega')|\big|\, ds \left[\frac{1}{(1+t)^2(\log^+ t+1)^{\nu}} + \frac{\nu}{(1+t)^2(\log^+ t+1)^{\nu+1}}\right] dt.$$

Since $g(z)$ is of order 1 and finite type, it follows from Lemma 1 that $\int_0^t \big|\log|g(s\omega')|\big|\, dt \leqslant C_1 t^2$ for $t \geqslant 1$ so that

$$\int_0^{t_{\lambda'}} \frac{\big|\log|g(t\omega')|\big|\, dt}{(1+t)(\log^+ t+1)^{\nu}} \leqslant \frac{C_2 t_{\lambda'}}{(\log^+ t_{\lambda'}+1)^{\nu}} \leqslant \frac{C_2}{\lambda'(\log^- \lambda'+1)^{\nu-\nu'}} = A(\lambda')$$

and since for $\lambda' < \frac{1}{2}$ we can choose λ' as a local coordinate on $\partial B(0,1)$, $\int_{\partial B(0,1)} A(\lambda')\, d\tau_{2n-1}(\omega') < \infty$ for $\nu > 2$.

We assume without loss of generality that $P_j(0) \neq 0$ (for if $P_j(0) = 0$, we choose a point $z_0 \notin \Gamma_j$ such that $P_j(z_0) \neq 0$ and $\Gamma_j \subset P'_j = \Gamma_j + z_0$ — the translation of Γ_j by z_0 — and we prove the theorem for Γ'_j which then implies the theorem for Γ_j). Let γ be such that $P_j(z) \neq 0$ for $\|z\| < \gamma < 1$ and let d' be the degree of $P_j(z)$. We claim that in every complex line τz, $\tau \in C$ outside a set of d' circles the sum of whose radii is d' that $|P_j(\tau z)| \geqslant |P_j(0)|/\gamma^{d'}$. To see this we let $P_z(\tau) = P_j(z\tau)$. Then $|P_z(\tau)| = \left|P_z(0) \prod_j \left(1 - \frac{\tau}{c_j(z)}\right)\right|$ and if $|\tau - c_j(z)| > 1$, then

$$|P_z(\tau)| \geqslant \left|\frac{P_z(0)}{\pi\, c_j(z)}\right| \geqslant \frac{|P_z(0)|}{\gamma^{d'}}.$$

We further assume that $|P_j(0)|/\gamma^{d'} \geqslant 2$ (if necessary, we multiply $g(z)$ by a constant). Suppose that $t \geqslant t_{\lambda'}$. Then it follows from (11) that $|g(t\omega') - P_j(t\omega')| \leqslant 1$ if we choose C_0 sufficiently large (depending on the $P_k(z)$, $k \neq j$) and so when $|P_j(t\omega')| \geqslant 2$, $|g(t\omega')| \geqslant 1$. If $t \geqslant t'_\lambda$ belongs to one of the exceptional circles c_j, then there is a point $\hat{t}_j \geqslant t_{\lambda'}$ such that $|\hat{t}_j - t| \leqslant q$ and $|g(\hat{t}_j\omega')| \geqslant 1$. Furthermore $c_j \subset \{\tau\omega' : |\tau\omega' - \hat{t}_j| \leqslant 2q\}$ and $\{\tau\omega' : |\tau\omega' - \hat{t}_j\omega'| \leqslant 2eq\} \subset \Gamma_j$ (if C_0 is sufficiently large depending on d'). Then by Lemma 1 $\int_{t\omega' \in c_j} \big|\log|g(t\omega')|\big|\,dt \leqslant C_1 q \log \hat{t}_j$ so

$$\sum_j \int_{t\omega' \in c_j} \frac{\big|\log|g(t\omega')|\big|\,dt}{(1+t)(\log^+ t + 1)^\nu} \leqslant \frac{c_2 q^2 \log \hat{t}_j}{(1+t_q)(\log^+ \hat{t}_q + 1)} \leqslant c_2 q^2.$$

In a similar straightforward calculation, one shows that

$$\int_{\Gamma_j} \frac{\big|\log|g(z)|\big|\,d\tau_{2n}(z)}{(1+\|z\|)^{2n}(\log^+ \|z\| + 1)^\nu} < +\infty \qquad \text{for } \nu > 2$$

which completes the proof. The case of an exponential sum (i.e. $P_j(z) = \text{const}$ for all j) is similar and technically even easier so we do not present it. Q.E.D.

We note that Berenstein [1] has shown that for an exponential polynomial and any cone $\Gamma \subset \Gamma_j$ (as above) such that $\bar{\Gamma} \cap \bar{\Gamma}_j = \{\emptyset\}$ then

$$\varlimsup_{r\to\infty} \frac{\int_{B(0,r)\cup\bar{\Gamma}} \Delta(\log|f|)}{r^{2n-2}\log r} \leqslant k.$$

His method could not, however, measure the rate at which the zero set tended towards the boundary.

THEOREM 5. *Let $f(z) = f_1(z) + f_2(z)\exp iz_1^p$ for p a positive integer and $f_1(z)$ and $f_2(z)$ two entire functions of order at most ϱ. Let $\Gamma_j = \left\{z : \frac{2\pi(j-1)}{p} < \arg z_1 < \frac{2\pi j}{p}\right\}$. Then for all j, $f \in \mathcal{G}\mathcal{A}^{(\mu,0)}(\Gamma_j)$ for $\mu > \varrho \geqslant 1$ or $\mu = 1 > \varrho$.*

Proof. If $g(z)$ is any entire function of order ϱ in C^n, then (cf. [6])

$$\int_{B(0,r)} \Delta(\log|g|) \leqslant C_{\varrho'} r^{\varrho'+2n-2} \quad \text{for } \varrho' > \varrho .$$

Hence we may assume without loss of generality that $\mu \leqslant \varrho$ and $f_1(z) \not\equiv 0$ for otherwise, we are through. Furthermore, we assume that $\operatorname{Re} iz_1^p \leqslant 0$ in Γ_j for otherwise we consider $f(z)\exp - iz_1^p$.

Let ϱ' be such that $\mu > \varrho' > \varrho$. Then $\log^+|f(z)| \leqslant c_1 + c_2\|z\|^{\varrho'}$ and so

$$\int_{\partial\Gamma_j} \frac{\log^+|f(z)|}{(1+\|z\|)^{2n-1+\mu}}\, d\tau_{2n-1}(z) < +\infty$$

and

$$\int_{\Gamma_j} \frac{\log^+|f(z)|}{(1+\|z\|)^{2n+\mu}}\, d\tau_{2n}(z) < +\infty .$$

Thus, in order to apply Corollary 2 to Theorem 1, it suffices to show that

$$\int_{S_j} \frac{\log^-|f(z)|}{(1+\|z\|)^{2n-1+\mu}}\, d\tau_{2n}(z) < +\infty, \quad \text{where } S_j = \left\{ z\colon \arg z_1 = \theta_j = \frac{(j-\frac{1}{2})2\pi}{p} \right\}.$$

Let $z = (z_1, z')$, $z \in C^{n-1}$. We shall show that

$$\int_{C^{n-1}} \int_0^\infty \frac{\left|\log|f(te^{i\theta_j}z)|\right|}{(1+t+\|z'\|)^{1+\mu}(1+\|z'\|)^{2n-2}}\, dt\, d\tau_{2n-2}(z') < +\infty .$$

Suppose that $f_1(z)+f_2(z) \equiv 0$. Then $f(z) = f_1(z)[1-\exp iz_1^p]$ and $f_1(z)$ is in $\mathscr{G}\mathscr{A}^{(\mu,0)}(\Gamma_j)$ since $\varrho < \mu$ and so is $g(z) = 1-\exp iz_1^p$ since it is bounded in Γ_j and satisfies (12). Thus, we suppose $f_1(z)+f_2(z) \not\equiv 0$ and hence there exists $\tilde{z}_1$ such that $|\tilde{z}_1| \leqslant \frac{1}{2}$ and $h_1(z') = f_1(\tilde{z}_1, z') + f_2(\tilde{z}_1, z') \not\equiv 0$ in C^{n-1}. Furthermore, since $f_1(z) \not\equiv 0$ we can write $f_1(z) = z_1^q \hat{f}_1(z)$ for q some non-negative integer and $\hat{f}_1(z)$ an entire function of order at most ϱ such that $\hat{f}_1(0, z') \not\equiv 0$. Let $h_2(z') = \hat{f}_1(0, z)$.

There exists a constant $c_p > 0$ such that $(w, z') \subset \Gamma_j$ f... $|w-e^{i\theta_j}| < c_p$. We assume $c_p < 2$. By Lemma 1, if $h_2(z') \neq 0$, then

$$|f_1(se^{i\theta_j}, z')| \geqslant \exp - c_p'\left[\left|\log|h_2(z')|\right| + c_3(t+\|z'\|)^{\varrho'} + c_4\right]$$

for $s \leqslant 2t$, $t \geqslant t_p$, except perhaps on a set of measure at most $c_p\hat{t}/2e$. Thus, there exists a constant c_p'' such that, setting $t(z') = \left[c_p''\left(\left|\log|h_2(z')|\right| + \right.\right.$ $\left.\left. + c\|z'\|^{\varrho} + 1\right)\right]^p$ if $t \geqslant t(z')$, $|f_2(te^{i\theta_j}, z')\exp - t^p| \leqslant \frac{1}{2}\exp - c_p\left[\left|\log|h_2(z)|\right| + \right.$ $\left. + c_3(t+\|z'\|)^{\varrho} + c_4\right]$ and so for $t(z') \leqslant t \leqslant 2\hat{t}$ except perhaps on a set measure $c_p\hat{t}/2e$

(12) $$|f(te^{i\theta_j}, z')| \geqslant \tfrac{1}{2}\exp - c_p'\left[\left|\log|h_2(z')|\right| + c_3(t+\|z'\|^{\varrho'} + c_4\right].$$

Now

$$\int_0^{t(z')} \frac{\left|\log|f(te^{i\theta_j}, z')|\right| dt}{(1+t+\|z'\|)^{1+\mu}}$$

$$= \frac{\int_0^{t(z')} \left|\log|f(te^{i\theta_j}, z')|\right| dt}{(1+t(z')+\|z'\|)^{1+\mu}} + (\mu+1) \frac{\int_0^{t(z')} \left(\int_0^t \left|\log|f(se^{i\theta_j}, z')|\right| ds\right) dt}{(1+t+\|z'\|)^{2+\mu}}$$

and since by Lemma 1,

$$\int_0^t \left|\log|f(se^{i\theta_j}, z')|\right| ds \leqslant t\left[c_5 \left|\log|h_1, (z')|\right| + c_6 (t+\|z'\|)^p + c_7\right] \quad \text{for } t \geqslant 2$$

it follows that

$$\int_0^{t(z')} \frac{\left|\log|f(te^{i\theta_j}, z')|\right|}{(1+t+\|z'\|)^{1+\mu}}$$

$$\leqslant c_8 \frac{\left|\log|h_1(z')|\right|}{(1+\|z'\|)^{\mu}} + c_9 \frac{\left|\log|h_2(z')|\right|}{(1+\|z'\|)^{\mu}} + c_{10}(t+\|z'\|)^{\varrho'-\mu} + c_{11}.$$

Let $t_n = (1+cp/2e)^n$. Then if $t_n \geqslant t(z')$, by (12) there exists t'_n with $t_n \leqslant t'_n \leqslant (1+cp/2e)\, t_n$ such that

$$\log|f(t'_n e^{i\theta_j}, z')| \geqslant \tfrac{1}{2} \exp - c'_p \left[\left|\log|h_2(z')|\right| + c_3(t+\|z'\|^{\varrho} + c_4\right]$$

and since $\{(w, z')\colon |w - t'_n e^{i\theta_j}| < t'_n c_p\} \subset \Gamma_j$ and $\log|f(te^{i\theta_j}, z')| \leqslant c_{12} + c_{13}(t+\|z'\|)^{\varrho'}$ in Γ_j, it follows by Lemma 1 that

$$\int_{t_n}^{t_{n+1}} \left|\log|f(te^{i\theta_j}, z')|\right| dt \leqslant t'_n\left[c_{14}(t+\|z'\|)^{\varrho'} + c_{15}\left|\log|h_2(z')|\right| + c_{16}\right]$$

so

$$\int_{t(z')}^{\infty} \frac{\left|\log|f(te^{i\theta_j}, z')|\right| dt}{(1+t+\|z'\|)^{\mu+1}} \leqslant \frac{c_{16} + c_{15}\left|\log|h_2(z')|\right|}{(1+\|z'\|)^{\mu+\varrho'/2}} \sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{cp}{2e}\right)^{n(\mu-\varrho'/2)}} +$$

$$+ c_{14} \sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{cp}{2e}\right)^{(\mu-\varrho')n}} \leqslant c_{17} \frac{\left|\log|h_2(z')|\right|}{(1+\|z'\|)^{\mu+\varrho'/2}} + c_{18}.$$

Since $\varrho' < \mu$,

$$\int_{C^{n-1}} \frac{(1+\|z'\|)^{\varrho'-\mu}}{(1+\|z'\|)^{2n-2}} d\tau_{2n-2}(z') < +\infty.$$

We assume without loss of generality that $h(0) \neq 0$, $j = 1, 2$ (for otherwise we choose a different origin). Then by the sub-median property for subharmonic functions

$$c_{2n-2}\log|h_j(0)| \leqslant \int_{\|z'\|=t} \frac{\log^+|h_j(z')|\,d\tau_{2n-3}(z')}{t^{2n-3}} - \int_{\|z'\|=t} \frac{\log^-|h_j(z')|\,d\tau_{2n-3}(z')}{t^{2n-3}}$$

so

$$c_{2n-2}\log|h_j(0)| + \int_{\|z'\|=t} \frac{\log^-|h_j(z')|\,d\tau_{2n-3}}{t^{2n-3}} \leqslant \int_{\|z'\|=t} \frac{\log^+|h_j(z')|\,d\tau_{2n-3}(z')}{t^{2n-3}} \leqslant t^{\varrho'-2n+3}.$$

Hence

$$\int \frac{\big|\log|h_j(z')|\big|\,d\tau_{2n-2}(z')}{(1+\|z'\|)^{2n-2+\mu+\varrho'/2}} \leqslant c_{19}+c_{20}\int_0^\infty \frac{1}{t^{1+\mu-\varrho'/2}}\,dt < +\infty.$$

Q.E.D.

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