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Central motions

by RONALD A. KNIGHT (Kirksville, Mo., U.S.A.)

Abstract. We obtain the following extension of a well-known result on metric phase spaces which are locally compact or complete. The set of Poisson stable points of a nonwandering flow on a locally compact Hausdorff phase space is dense in the phase space. Moreover, this extends Birkhoff's classical conclusions for certain Euclidean n-subspaces to locally compact Hausdorff phase spaces. We conclude our observations by verifying that any closed trajectory satisfying the Baire property is homeomorphic to a point, a circle, or the real line.

The classical concept of central motions introduced by Birkhoff (see [3]) are considered anew in this paper. Birkhoff directed considerable attention toward developing a theory for qualitatively determining all types of solutions or motions and their interrelationships for dynamical systems. (Autonomus systems of differential equations of the general form $dx_i/dt = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$, whose right members are continuous in some region of R^n were referred to as dynamical systems.) Birkhoff demonstrated that in a closed n-dimensional manifold M there is a set M_1 of central motions (nonwandering motions) towards which all other motions of the system tend asymptotically. Using transfinite induction the largest closed subset M_0 whose points are all nonwandering with respect to M_0 was obtained. This set was called the set of central motions of M and was shown to coincide with the closure of the set of Poisson stable points.

Nemytskii and Stepanov [4] carry out this construction for a generalized dynamical system on a compact metric phase space with identical consequences. The extension to a locally compact metric space is an easy next step. In view of these results, the set of central motions or center of a continuous flow on a Hausdorff phase space has been defined to be the closure of the set of Poisson stable points (see [1]).

Bhatia and Hajek give the following generalization of Birkhoff's classifi-

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cation of the center of a nonwandering flow. If the phase space X of a nonwandering flow is metric and either locally compact or complete, then the set of Poisson stable points is dense in X, i.e., X is the center. Our principal task here is to show that this result extends to locally compact Hausdorff phase spaces. We also show that the set of wandering points tend uniformly to the extended nonwandering set so that Birkhoff's concept of a center coincides with the generalized definition whenever the phase space is locally compact Hausdorff. We give an example to show that a flow on a nonlocally compact Hausdorff phase space need not enjoy this property. Finally, closed orbits satisfying the Baire property are classified.

Throughout the paper we assume that there is a given flow (X, π) on a Hausdorff phase space X. We shall denote the orbit, limit set, and prolongational limit set of x in X by $C(x)$, $L(x)$, and $J(x)$ respectively. The unilateral sets carry the appropriate $+$ or $-$ superscript. A point x is *positively* (*negatively*) *Poisson stable* provided $x \in L^+(x)$ $(x \in L^-(x))$ and x is *Poisson stable* if it is both positively and negatively Poisson stable. A point x is *nonwandering* if $x \in J(x)$. The reader may consult [1] and [2] for basic dynamical system concepts used herein.

We now obtain our major result.

THEOREM 1. *The set of Poisson stable points of a nonwandering flow on a locally compact phase space is dense.*

Proof. Let (X, π) be nonwandering and X be locally compact. Select a non-Poisson stable point x and a relatively compact open neighborhood V of x. Since $x \in J(x)$, $V \cap Vt_1 \neq \emptyset$ for some $t_1 > 1$. Denote $V \cap Vt_1$ by V_1. Each point of V_1 is nonwandering so that there exists a $t_2 < -2$ such that $V_2 = V_1 \cap V_1 t_2$ is nonempty. Similarly, there is a $t_3 > 3$ such that $V_3 = V_2 \cap V_2 t_3 \neq \emptyset$. Evidently, we can proceed inductively to construct a sequence (V_n) of relatively compact subneighborhoods of V with $V_{n+1} \subset V_n$ and $V_{n+1}(-t_{n+1}) \subset V_n$ for each $n \geqslant 1$. Let $M = \bigcap \bar{V}_n$. Then for any y in M, there exist subsequences (t_{k_i}) and (t_{m_i}) of (t_n) such that (yt_{k_i}) and (yt_{m_i}) converge in the compact set M as $t_{k_i} \to +\infty$ and $t_{m_i} \to -\infty$. Thus, $L^+(y) \cap M \neq \emptyset$ and $L^-(y) \cap M \neq \emptyset$ for each y in M. Next, let B denote the collection of all limit sets $L^+(z)$ for $z \in M$ and let A be a chain in B. Then $M \cap (\bigcap_A L^+(q)) = \bigcap_A (M \cap L^+(q))$ contains at least one point p. Thus, $L^+(p) \subset L^+(q)$ for each $L^+(q) \in A$. By the dual to Zorn's Lemma, B contains a minimal element $L^+(z)$. Let z_0 be an element of $L^+(z) \cap M$. Then $L^+(z_0) \subset L^+(z) \subset L^+(z_0)$ so that we have $z_0 \in L^+(z) = L^+(z_0)$. Also z_0 in M implies that $L^-(z_0) \cap M \neq \emptyset$. For $p \in L^-(z_0) \cap M$ we have $L^+(p) \subset L^-(z_0)$. Furthermore, $L^+(p) \cap M \neq \emptyset$ implies $L^+(z_0) \subset L^+(p)$, and hence, $z_0 \in L^-(z_0)$. Thus, z_0 is Poisson stable. Since each relatively compact neighborhood V of x contains a Poisson stable point z_0, x is contained in the closure of the set of Poisson stable points. The proof is complete.

COROLLARY. *Let X be locally compact. The set of Poisson stable points is dense in the set of central motions* (in the sense of Birkhoff).

By noting that the center M of a flow on a locally compact space X contains the interior G^0 of the set G of nonwandering points, we obtain the following corollary. This statement is a consequence of $x \in J^+(x) \cap G^0 = J^+_{G^0}(x)$ (the prolongation of x relative to G^0) for each $x \in G^0$ (see 3.24.9 of [1]), i.e., $X = M \cup \overline{(X \setminus G)}$.

COROLLARY. *If X is locally compact, then the set consisting of the Poisson stable points and the wandering points is dense in X.*

The set of central motions of a wandering flow on a locally compact Hausdorff phase space need not coincide with the set of nonwandering points. Example 3.10 of [4] is such a flow. On the other hand, if X is not locally compact, the set of central motions need not be dense in X even though the flow is nonwandering. We give such an example on a nonlocally compact noncomplete metric space. The flow given in Example 4.06 of [4] is defined on a torus T. There is precisely one critical point p, $L^+(x) = \{p\} = L^-(y)$ for exactly one orbit $C(x)$ and exactly one orbit $C(y)$, where $L^-(x) = L^+(y) = T$, and $T = L(z)$ for each noncritical point z. Let $X = C(y)$. Then $X = C(y) = L^+_X(y)$ (the positive limit set of y relative to X) whereas $L^-_X(y) = \emptyset$. Thus, $(X, \pi|X)$ is a nonwandering flow containing no Poisson stable points.

Next, we show that the set of nonwandering points of the extended flow (X^*, π^*), where X is locally compact, uniformly attracts each wandering point.

THEOREM 2. *Let X be locally compact. Then for any neighborhood V^* of the nonwandering set of (X^*, π^*) there exists a $T_V > 0$ such that $V^*[0, T_V] = X^*$.*

Proof. Let V^* be an open neighborhood of the nonwandering set M^* of X^*, where (X^*, π^*) is the extended flow on the one point compactification of X. For any wandering point x we have $\emptyset \neq L^{*\pm}(x) \subset M^*$ since all limit sets consist of nonwandering points. No orbit can be frequently out of V^* in either direction. We define $t_x = \inf\{t \in \boldsymbol{R}^+ \colon xt_0 \notin \overline{V^*},\ xt_1 \in V^*, \text{ and } 0 < t_0 < t_1 < t\}$ for each $x \in X^*$. That $0 \leqslant t_x < +\infty$ for each $x \in X^*$ is obvious. Define $T_V = \sup\{t_x \colon x \in \partial V^*\}$. For each $x \in \partial V^*$ there exists an open neighborhood V_x of x such that $V_x t_0 \subset X^* \setminus \overline{V^*}$ and $V_x(t_x + \varepsilon_x) \subset V^*$, where $\varepsilon_x > 0$ and $0 < t_0 < t_x$. Thus, for each $z \in V_x$, $t_z < t_x + \varepsilon_x$. Since ∂V^* is compact there is a finite cover $\{V_{x_1}, \ldots, V_{x_n}\}$ of ∂V^*, and hence, $T_V \leqslant \max\{t_{x_i} + \varepsilon_{x_i} \colon 1 \leqslant i \leqslant n\}$. For any $x \notin V^*$ let $\tau = \inf\{t \in \boldsymbol{R}^- \colon x[t, 0] \subset X^* \setminus V^*\}$. Evidently $L^*(x) \subset M^*$ implies $-\infty < \tau \leqslant 0$. Now $x \in (x\tau)[0, T_V] \subset V^*[0, T_V]$ since $-\tau < t_{x\tau} \leqslant T_V$. Hence, $X^* \subset V^*[0, T_V]$.

Since each invariant strong attractor is stable (2.15, [1] or 8.15, [2]) the following corollary is evident.

COROLLARY. *If X is locally compact, then the flow is nonwandering if and only if the extended nonwandering set is a strong attractor.*

We now turn our attention briefly to classification of orbits satisfying the Baire property.

PROPOSITION 1. *An orbit $C(x)$ is homeomorphic to R if and only if x is not unilaterally Poisson stable.*

Proof. Let h: $R \to C(x)$ be a homeomorphism of R onto $C(x)$. Then $h(t) = x\pi_x^{-1} h(t)$ for each $t \in R$. The mapping $\pi_x^{-1} h$: $R \to R$ is a bijection, and hence, $h^{-1}\pi_x$ is a continuous bijection of R. Thus, $h^{-1}\pi_x$ is a homeomorphism. The mapping $\pi_x = h(h^{-1}\pi_x)$ is a homeomorphism. Let (xt_i) be a net such that $xt_i \to x$. Then $\pi_x^{-1}(xt_i) \to \pi_x^{-1}(x) = 0$. Hence, $x \notin L(x)$.

Conversely, let $x \notin L(x)$ and suppose that V is an open interval with $\pi_x(V)$ not open. Then there is a net (xt_i) in $C(x) \setminus \pi_x(V)$ converging to a point xt_0 in $\pi_x(V)$. Some subnet (t_{n_i}) of (t_i) converges to a point t_1 because no subnet of (t_i) can diverge to $+\infty$ or $-\infty$. The fact that x is a regular point implies $t_1 = t_0$. But this means that (t_{n_i}) is ultimately in V, and hence, that (xt_{n_i}) is ultimately in $\pi_x(V)$ which is absurd. Hence, π_x is an open map.

COROLLARY. *An orbit $C(x)$ is homeomorphic to R if and only if π_x is a homeomorphism.*

PROPOSITION 2. *A closed orbit $C(x)$ satisfying the Baire property is homeomorphic to R, S_1, or a single point.*

Proof. The map f: $C(x) \to S_1$ defined by $f(xt) = c$ is $(2\pi t/T)$, where T is the fundamental period of a periodic point x is a homeomorphism. Whenever $C(x)$ is a critical orbit it is a single point. Finally, let x be a regular trajectory. The sets $L(x) \setminus x[-n, n]$, $n = 1, 2, 3, \dots$ are each open dense subsets of $L(x)$. Hence $\bigcap (L(x) \setminus x[-n, n]) = L(x) \setminus \bigcup x[-n, n] = L(x) \setminus C(x) = \emptyset$ is dense in $L(x)$ by the Baire property so that $L(x) = \emptyset$. By Proposition 1, π_x is a homeomorphism. The proof is complete.

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NORTHEAST MISSOURI STATE UNIVERSITY

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