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On some applications of Bogoliubov method for hyperbolic equations

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Abstract. In this paper there is given a method of construction of approximate solution of a partial differential equation of the form

$$z''_{xy}(x, y) = \varepsilon_1\varepsilon_2 f(x, y, z(x, y)), \tag{1}$$

where f is a given continuous function and ε_1, ε_2 are small parameters.

The idea of the method of construction of these approximations is connected with a theorem of Bogoliubov type concerning (1). This theorem was presented by the author of this paper in [2].

1. Introduction. The idea of the method, which is due to Bogoliubov, was expressed by him in 1945 in monograph [1]: *On some statical methods in mathematical physics*. In this monograph a particular problem concerned with the properties of the solutions of the equations in standard form on an infinite time interval is considered. However, the idea and the methods of proofs of a number of theorems are vere effective as well as versatile, and can be applied to investigate a sufficiently wide range of problems.

The purpose of this paper is to present some application of Bogoliubov method for hyperbolic equations. A Bogoliubov type theorem for hyperbolic equation was presented by the author of this paper in [2]. This theorem will be used in this paper.

We consider the equation of the form

$$z''_{xy}(x, y) = \varepsilon_1\varepsilon_2 f\big(x, y, z(x, y)\big), \tag{1}$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are small parameters. Let J denote a bounded open interval of R. Assuming that for every $z \in J$ we have

$$\lim_{\substack{T_1\to\infty \\ T_2\to\infty}} \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} f(x, y, z)\,dx\,dy = f_0(z), \tag{2}$$

simultaneusly with equation (1) the avereged equation

$$w''_{xy}(x, y) = \varepsilon_1\varepsilon_2 f_0\big(w(x, y)\big) \tag{3}$$

is considered. The solutions of equations (1) and (3), corresponding to $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, will be denoted by $z^{\varepsilon_1\varepsilon_2}$ and $w^{\varepsilon_1\varepsilon_2}$, respectively.

In paper [2] the following theorem was proved:

THEOREM 1. *Let the function $f(x, y, z)$ occurring on the right-hand side of equation* (1) *satisfy the following conditions*:

(a) *$f(x, y, z)$ is uniformly bounded and continuous in z uniformly with respect to (x, y) such that $0 \leqslant x < \infty$, $0 \leqslant y < \infty$, $z \in J$,*

(b) *the limit* (2) *exists for every $z \in J$,*

(c) *there exist $a > 0$ and $b > 0$ such that equation* (3) *has, for $\varepsilon_1 = \varepsilon_2 = 1$ a unique solution w defined on $D = \{0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\}$ and such that*

$$w(0, y) = \tau(y) \quad \text{for } y \in \langle 0, b\rangle,$$
$$w(x, 0) = \sigma(x) \quad \text{for } x \in \langle 0, a\rangle, \tag{4}$$

where $\tau, \sigma \in C^1$.

Then for every $\mu > 0$ there exists $\delta > 0$ such that for $0 < \varepsilon_1 < \delta$, $0 < \varepsilon_2 < \delta$ the solution $z^{\varepsilon_1\varepsilon_2}$ of (1) *with boundary conditions*

$$z^{\varepsilon_1\varepsilon_2}(0, y) = \tau(y) \quad \text{for } y \in \langle 0, b\rangle,$$
$$z^{\varepsilon_1\varepsilon_2}(x, 0) = \sigma(x) \quad \text{for } x \in \langle 0, a\rangle \tag{5}$$

satisfies for $0 \leqslant x \leqslant a/\varepsilon_1$ and $0 \leqslant y \leqslant b/\varepsilon_2$ the inequality

$$|z^{\varepsilon_1\varepsilon_2}(x, y) - w^{\varepsilon_1\varepsilon_2}(x, y)| < \mu.$$

2. The construction of approximations of the solution of (1). Assume that the conditions of Theorem 1 are fulfilled. Let Ξ be the solution of (3) with boundary conditions (4), i. e., $\Xi = w^{\varepsilon_1\varepsilon_2}$. The function Ξ will be called the *first approximation of the solution* $z^{\varepsilon_1\varepsilon_2}$ of (1) with boundary conditions (5). Let us denote this function by z_1. Setting in (1)

$$z(x, y) = \Xi(x, y) + v_1(x, y) + u_1(x, y),$$

where

$$v_1(x, y) = \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \big[f\big(\xi, \eta\, \Xi(\xi, \eta)\big) - f_0\big(\Xi(\xi, \eta)\big)\big]\, d\xi\, d\eta,$$

we have

$$\frac{\partial^2 u_1(x, y)}{\partial x\, \partial y} = \varepsilon_1\varepsilon_2 \big[f\big(x, y, \Xi(x, y) + v_1(x, y) + u_1(x, y)\big) - f\big(x, y, \Xi(x, y)\big)\big]. \tag{6}$$

Let

$$\frac{\partial^2 u_1^*(x, y)}{\partial x\, \partial y} = \varepsilon_1\varepsilon_2 f_0'\big(\Xi(x, y)\big) \cdot u_1^*(x, y) + \varepsilon_1\varepsilon_2 \big[f\big(x, y, \Xi(x, y) + v_1(x, y)\big) - f\big(x, y, \Xi(x, y)\big)\big] \tag{7}$$

be the averaged equation corresponding to (6). Suppose that u_1^* is a solution of (7) with boundary conditions $u_1^*(x, 0) = u_1^*(0, y) = 0$ for $x, y \geqslant 0$. The function $z_2 = \Xi + v_1 + u_1^*$ will be called the *second approximation of the solution* $z^{\varepsilon_1\varepsilon_2}$ of (1) with boundary conditions (5). Setting now in (1)

$$z(x, y) = z_2(x, y) + v_2(x, y) + u_2(x, y),$$

where

$$v_2(x, y) = \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \Big[f\big(\xi, \eta, z_2(\xi, \eta)\big) - f\big(\xi, \eta, z_1(\xi, \eta) + v_1(\xi, \eta)\big) - \\ - f_0'\big(\Xi(\xi, \eta)\big) \cdot u_1^*(\xi, \eta)\Big] d\xi d\eta,$$

we have

$$\frac{\partial^2 u_2(x, y)}{\partial x \partial y} = \varepsilon_1\varepsilon_2 \Big[f\big(x, y, z_2(x, y) + v_2(x, y) + u_2(x, y)\big) - \\ - f\big(x, y, u_2(x, y)\big)\Big]. \tag{8}$$

Let

$$\frac{\partial^2 u_2^*(x, y)}{\partial x \partial y} = \varepsilon_1\varepsilon_2 f_0'\big(z_1(x, y)\big) \cdot u_2^*(x, y) + \varepsilon_1\varepsilon_2 \Big[f\big(x, y, z_2(x, y) + \\ + v_2(x, y)\big) - f\big(x, y, z_2(x, y)\big)\Big] \tag{9}$$

be the averaged equation corresponding to (8) and let u_2^* be a solution of (9) with boundary conditions $u_2^*(x, 0) = u_2^*(0, y) = 0$ for $x, y \geqslant 0$. The function $z_3 = z_2 + v_2 + u_2^*$ will be called the *third approximation of the solution* $z^{\varepsilon_1\varepsilon_2}$ of (1) with boundary conditions (5). In a similar way we obtain the k-th approximation of the form

$$z_k = z_{k-1} + v_{k-1} + u_{k-1}^*,$$

where

$$v_{k-1}(x, y) = \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \Big[f\big(\xi, \eta, z_{k-1}(\xi, \eta)\big) - f\big(\xi, \eta, z_{k-2}(\xi, \eta) + v_{k-2}(\xi, \eta)\big) - \\ - f_0'\big(\Xi(\xi, \eta) \cdot u_{k-2}(\xi, \eta)\big) d\xi d\eta,$$

and u_{k-1}^* is a solution of

$$\frac{\partial^2 u_{k-1}^*}{\partial x \partial y} = \varepsilon_1\varepsilon_2 f_0'\big(z_{k-1}(x, y)\big) \cdot u_{k-1}^*(x, y) + \\ + \varepsilon_1\varepsilon_2 \Big[f\big(x, y, z_{k-1}(x, y) + v_{k-1}(x, y) - f\big(x, y, z(x, y)\big)\Big] \tag{10}$$

with boundary conditions $u_{k-1}^*(x, 0) = u_{k-1}^*(0, y) = 0$.

3. Convergence of the approximation sequence. The proof of the desired approximation theorem is preceded by two lemmas.

LEMMA 2. *Let the functions f, f_0, f_z' and f_0' fulfil, for $x, y \geqslant 0$, $z \in J$, the following conditions:*

(a) *f and f'_z are continuous with respect to $x, y \geqslant 0$ and Lipschitzian with respect to z with a constant N;*

(b) *For every $z \in J$ there exists the limit* (2) *such that*

$$f'_0(z) = \lim_{\substack{T_1\to\infty\\ T_2\to\infty}} \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} f'_z(x, y, z)\,dx\,dy;$$

(c) *f_0 and f'_0 are Lipschitzian with the constant N and such that*

$$|f_0(\Xi)| \leqslant N, \qquad |f'_0(\Xi)| \leqslant N.$$

Then for every $\delta > 0$ there exist ν_1 and ν_2 such that

$$\Big|\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f\big(\xi, \eta, \Xi(\xi, \eta)\big) - f_0\big(\Xi(\xi, \eta)\big)\big]\,d\xi\,d\eta\Big| < \delta$$

and

$$\Big|\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f'_z\big(\xi, \eta, \Xi(\xi, \eta)\big) - f'_0\big(\Xi(\xi, \eta)\big)\big]\,d\xi\,d\eta\Big| < \delta$$

for $\varepsilon_1, \varepsilon_2 < \nu_1$ and $\varepsilon_1, \varepsilon_2 < \nu_2$ respectively.

Proof. Let $z_{ij} \in J$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ and let $\hat{z}$ be the function defined by $z(x, y) = \hat{z}_{ij}$ for $(x, y) \in D_{ij}$, where

$$D_{ij} = \{x_{i-1} < x \leqslant x_i;\ y_{j-1} < y \leqslant y_j\};$$

$$0 = x_0 < x_1 < \ldots < x_n = A, \qquad 0 = y_0 < y_1 < \ldots < y_m = B;$$

where A, B are sufficiently large numbers. Suppose $\hat{z}_\delta$ is such that

$$|\Xi(x, y) - \hat{z}_\delta(x, y)| < \frac{\delta}{18abN} \quad \text{for } 0 \leqslant x \leqslant \frac{a}{\varepsilon_1},\ 0 \leqslant y \leqslant \frac{b}{\varepsilon_2}$$

and such that $\hat{z}_\delta(x, y) = \hat{z}(x, y)$. For $0 \leqslant x \leqslant A$, $0 \leqslant y \leqslant B$ and every $\delta > 0$ we have

$$\Big|\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f\big(\xi, \eta, \Xi(\xi, \eta)\big) - f_0\big(\Xi(\xi, \eta)\big)\big]\,d\xi\,d\eta\Big|$$

$$\leqslant \varepsilon_1\varepsilon_2\Big|\int_0^x\int_0^y \big[f\big(\xi, \eta, \Xi(\xi, \eta)\big) - f\big(\xi, \eta, \hat{z}_\delta(\xi, \eta)\big)\big]\,\xi s\,d\eta\Big| +$$

$$+\varepsilon_1\varepsilon_2\Big|\int_0^x\int_0^y \big[f\big(\xi, \eta, \hat{z}_\delta(\xi, \eta)\big) - f_0\big(\hat{z}_\delta(\xi, \eta)\big)\big]\,d\xi\,d\eta\Big| +$$

$$+\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big|f_0\big(\hat{z}_\delta(\xi, \eta)\big) - f_0\big(\Xi(\xi, \eta)\big)\big|\,d\xi\,d\eta$$

$$\leqslant 2\varepsilon_1\varepsilon_2 N\int_0^x\int_0^y |\Xi(\xi, \eta) - \hat{z}_\delta(\xi, \eta)|\,d\xi\,d\eta + \varepsilon_1\varepsilon_2 A(x, y),$$

where

$$\Lambda(x, y) = \Big| \int_0^x \int_0^y \big[f\big(\xi, \eta, \hat{z}_\delta(\xi, \eta)\big) - f_0\big(\hat{z}_\delta(\xi, \eta)\big)\big]\, d\xi\, d\eta \Big|.$$

For $(x, y) \in D_{k+1,l+1}$, where $k < n-1$, $l < m-1$, we have

$$\Lambda(x, y) \leqslant \sum_{i=1}^{k} \sum_{j=1}^{l} \Big| \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} [f(\xi, \eta, \hat{z}_{i,j}) - f_0(\hat{z}_{ij})]\, d\xi\, d\eta \Big| +$$
$$+ \sum_{j=1}^{l} \Big| \int_{x_k}^{x} \int_{y_{j-1}}^{y_j} [f(\xi, \eta, \hat{z}_{k+1,j}) - f_0(\hat{z}_{k+1,j})]\, d\xi\, d\eta \Big| +$$
$$+ \Big| \int_{x_k}^{x} \int_{y_l}^{y} [f(\xi, \eta, \hat{z}_{k+1,l+1}) - f_0(\hat{z}_{k+1,l+1})]\, d\xi\, d\eta \Big| +$$
$$+ \sum_{i=1}^{k} \Big| \int_{x_{i-1}}^{x_i} \int_{y_l}^{y} [f(\xi, \eta, \hat{z}_{i,l+1}) - f_0(\hat{z}_{i,l+1})]\, d\xi\, d\eta \Big| \leqslant \sum_{i=1}^{8} J_i,$$

where

$$J_1 = \sum_{i=1}^{k} \sum_{j=1}^{l} \Big| \int_0^{x_i} \int_0^{y_j} [f(\xi, \eta, \hat{z}_{ij}) - f_0(\hat{z}_{ij})]\, d\xi\, d\eta -$$
$$- \int_0^{x_i} \int_0^{y_{j-1}} [f(\xi, \eta, \hat{z}_{ij}) - f_0(\hat{z}_{ij})]\, d\xi\, d\eta \Big|,$$

$$J_2 = \sum_{i=1}^{k} \sum_{j=1}^{l} \Big| \int_0^{x_{i-1}} \int_0^{y_{j-1}} [f(\xi, \eta, \hat{z}_{ij}) - f_0(\hat{z}_{ij})]\, d\xi\, d\eta -$$
$$- \int_0^{x_{i-1}} \int_0^{y_j} [f(\xi, \eta, \hat{z}_{i,j}) - f_0(\hat{z}_{i,j})]\, d\xi\, d\eta \Big|,$$

$$J_3 = \sum_{j=1}^{l} \Big| \int_0^{x} \int_0^{y_j} [f(\xi, \eta, \hat{z}_{k+1,j}) - f_0(\hat{z}_{k+1,j})]\, d\xi\, d\eta -$$
$$- \int_0^{x} \int_0^{y_{j-1}} [f(\xi, \eta, \hat{z}_{k+1,j}) - f_0(\hat{z}_{k+1,j})]\, d\xi\, d\eta \Big|,$$

$$J_4 = \sum_{j=1}^{l} \Big| \int_0^{x_k} \int_0^{y_{j-1}} [f(\xi, \eta, \hat{z}_{k+1,j}) - f_0(\hat{z}_{k+1,j})]\, d\xi\, d\eta -$$
$$- \int_0^{x_k} \int_0^{y_j} [f(\xi, \eta, \hat{z}_{k+1,j}) - f_0(\hat{z}_{k+1,j})]\, d\xi\, d\eta \Big|,$$

$$J_5 = \sum_{i=1}^{k} \Big| \int_0^{x_i} \int_0^{y} [f(\xi, \eta, \hat{z}_{i,l+1}) - f_0(\hat{z}_{i,l+1})]\, d\xi\, d\eta -$$
$$- \int_0^{x_i} \int_0^{y_l} [f(\xi, \eta, \hat{z}_{i,l+1}) - f_0(\hat{z}_{i,l+1})]\, d\xi\, d\eta \Big|,$$

$$J_6 = \sum_{i=1}^{k}\Big|\int_0^{x_{i-1}}\int_0^{y_l}[f(\xi,\eta,\hat{z}_{i,l+1})-f_0(\hat{z}_{i,l+1})]\,d\xi\,d\eta - \int_0^{x_{i-1}}\int_0^{y}[f(\xi,\eta,\hat{z}_{i,l+1})-f_0(\hat{z}_{i,l+1})]\,d\xi\,d\eta\Big|,$$

$$J_7 = \Big|\int_0^{x}\int_0^{y}[f(\xi,\eta,\hat{z}_{k+1,l+1})-f_0(\hat{z}_{k+1,l+1})]\,d\xi\,d\eta - \int_0^{x}\int_0^{y_l}[f(\xi,\eta,\hat{z}_{k+1,l+1})-f_0(\hat{z}_{k+1,l+1})]\,d\xi\,d\eta\Big|,$$

$$J_8 = \Big|\int_0^{x_k}\int_0^{y_l}[f(\xi,\eta,\hat{z}_{k+1,l+1})-f_0(\hat{z}_{k+1,l+1})]\,d\xi\,d\eta - \int_0^{x_k}\int_0^{y}[f(\xi,\eta,\hat{z}_{k+1,l+1})-f_0(\hat{z}_{k+1,l+1})]\,d\xi\,d\eta\Big|.$$

In virtue of (2), for every $\delta > 0$ there exists K such that

$$\left|\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} f(x,y,z)\,dx\,dy - f_0(z)\right| < \frac{\delta}{9Kab(2l-1)}$$

for $T_1, T_2 > K$ and every $z \in J$.

There exists $\nu_1 > 0$ such that for $0 < \varepsilon_1 < \nu_1$ and $0 < \varepsilon_2 < \nu_2$ we have $a/\varepsilon_1 > K$ and $b/\varepsilon_2 > K$. Let us now take $x_1, x_2, \ldots, x_{k+1} \in (K, a/\varepsilon_1)$ and $y_1, y_2, \ldots, y_{l+1} \in (K, b/\varepsilon_2)$. Since

$$J_1 \leqslant \sum_{i=1}^{k}\sum_{j=1}^{l} x_iy_j\left|\frac{1}{x_iy_j}\int_0^{x_i}\int_0^{y_j} f(\xi,\eta,\hat{z}_{ij})\,d\xi\,d\eta - f_0(\hat{z}_{ij})\right| + \sum_{i=1}^{k}\sum_{j=2}^{l} x_iy_{j-1}\left|\frac{1}{x_iy_{j-1}}\int_0^{x_i}\int_0^{y_{j-1}} f(\xi,\eta,\hat{z}_{ij})\,d\xi\,d\eta - f_0(\hat{z}_{ij})\right|,$$

for $\varepsilon_1 < \nu_1$ and $\varepsilon_2 < \nu_2$ we have

$$J_1 < k\frac{ab}{\varepsilon_1\varepsilon_2}(2l-1)\frac{\delta}{9Kab(2l-1)} = \frac{1}{\varepsilon_1\varepsilon_2}\cdot\frac{\delta}{9}.$$

In similar way we obtain

$$J_r < \frac{1}{\varepsilon_1\varepsilon_2}\cdot\frac{\delta}{9} \quad \text{for } r = 2, 3, \ldots, 8.$$

Then for $0 < \varepsilon_1 < \nu_1$, $0 < \varepsilon_2 < \nu_1$ and $0 \leqslant x \leqslant a/\varepsilon_1$, $0 \leqslant y \leqslant b/\varepsilon_2$ we have

$$\left|\varepsilon_1\varepsilon_2\int_0^{x}\int_0^{y}\big[f(\xi,\eta,\Xi(\xi,\eta)) - f_0(\Xi(\xi,\eta))\big]\,d\xi\,d\eta\right| < \delta.$$

In a similar way we obtain the existence of $\nu_2 > 0$ such that for $\varepsilon_1, \varepsilon_2 < \nu_2$ and $0 \leqslant x \leqslant a/\varepsilon_1$, $0 \leqslant y \leqslant b/\varepsilon_2$

$$\left| \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \left[f_z'(\xi, \eta, \Xi(\xi, \eta)) - f_0'(\Xi(\xi, \eta))\right] d\xi\, d\eta \right| < \delta.$$

LEMMA 3. *Let Q and W fulfil, for $x, y \geqslant 0$, the following conditions:*
(a) $Q(x, y) \geqslant 0$, $W(x, y) \geqslant 0$ *for* $x, y \geqslant 0$,
(b) *Q is non-decreasing with respect to x and y,*
(c) *for every $c \geqslant 0$ and $x, y \geqslant 0$*

$$W(x, y) \leqslant Q(x, y) + c \int_0^x \int_0^y W(s, t)\, ds dt.$$

Then for $x, y \geqslant 0$

$$W(x, y) \leqslant Q(x, y) \exp(cxy).$$

THEOREM 4. *Assume that the conditions of Lemma 2 are fulfilled and let*

$$|f_0(\Xi(x, y))| \leqslant N, \qquad |f_0'(\Xi(x, y))| \leqslant N.$$

There exists $\varepsilon_0 > 0$ such that for $\varepsilon_1 < \varepsilon_0$ and $\varepsilon_2 < \varepsilon_0$

$$\lim_{k\to\infty} |z^{\varepsilon_1\varepsilon_2}(x, y) - z_k(x, y)| = 0$$

uniformly for $0 \leqslant x \leqslant a/\varepsilon_1$, $0 \leqslant y \leqslant b/\varepsilon_2$, where z_k is the k-th apporoximation of the solution $z^{\varepsilon_1\varepsilon_2}$ of (1) *with boundary conditions* (5).

Proof. Substituting $z(x, y) = z_k(x, y) + h(x, y)$ into (1) we have

$$h(x, y) = \varepsilon_1\varepsilon_2 \int_0^x \int_0^y f\big(\xi, \eta, z_k(\xi, \eta) + h(\xi, \eta)\big) d\xi\, d\eta - z_k(x, y).$$

Since

$$\begin{aligned} z_k(x, y) &= z_{k-1}(x, y) + \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \big[f\big(\xi, \eta, z_{k-1}(\xi, \eta)\big) - f\big(\xi, \eta, z_{k-2}(\xi, \eta) + \\ &\qquad + v_{k-2}(\xi, \eta)\big) - f_0'\big(\Xi(\xi, \eta)\big) \cdot u_{k-2}^*(\xi, \eta)\big] d\xi\, d\eta + \\ &\quad + \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \big[f_0'\big(\Xi(\xi, \eta)\big) \cdot u_{k-1}^*(\xi, \eta) + f\big(\xi, \eta, z_{k-1}(\xi, \eta) + \\ &\qquad + v_{k-1}(\xi, \eta)\big) - f\big(\xi, \eta, z_{k-1}(\xi, \eta)\big)\big] d\xi\, d\eta \\ &= z_{k-2}(x, y) + \varepsilon_1\varepsilon_2 \int_0^x \int_0^y \big[f\big(\xi, \eta, z_{k-2}(\xi, \eta)\big) - f\big(\xi, \eta, z_{k-3}(\xi, \eta) + \\ &\qquad + v_{k-2}(\xi, \eta)\big) - f_0'\big(\Xi(\xi, \eta)\big) \cdot u_{k-3}^*(\xi, \eta)\big] d\xi\, d\eta + \end{aligned}$$

$$+\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f_0'\big(\Xi(\xi,\eta)\big)\cdot u_{k-2}^*(\xi,\eta)+f\big(\xi,\eta,z_{k-2}(\xi,\eta)+ +v_{k-2}(\xi,\eta)\big)-f\big(\xi,\eta,z_{k-2}(\xi,\eta)\big)\big]d\xi\,d\eta+$$

$$+\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f\big(\xi,\eta,z_{k-1}(\xi,\eta)\big)-f\big(\xi,\eta,z_{k-2}(\xi,\eta)+v_{k-2}(\xi,\eta)\big)- -f_0'\big(\Xi(\xi,\eta)\big)\cdot u_{k-2}^*(\xi,\eta)\big]d\xi\,d\eta+$$

$$+\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f_0'\big(\Xi(\xi,\eta)\big)\cdot u_{k-1}^*(\xi,\eta)+f\big(\xi,\eta,z_{k-1}(\xi,\eta)+ +v_{k-1}(\xi,\eta)\big)-f\big(\xi,\eta,z_{k-1}(\xi,\eta)\big)\big]d\xi\,d\eta$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$=\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f_0'\big(\Xi(\xi,\eta)\big)\cdot u_{k-1}^*(\xi,\eta)+f\big(\xi,\eta,z_{k-1}(\xi,\eta)+v_{k-1}(\xi,\eta)\big)\big]d\xi\,d\eta,$$

therefore

$$h(x,y)=\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f\big(\xi,\eta,z_k(\xi,\eta)+h(\xi,\eta)\big)- -f\big(\xi,\eta,z_k(\xi,\eta)\big)\big]d\xi\,d\eta+v_k(x,y), \tag{11}$$

where

$$v_k(x,y)=\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f\big(\xi,\eta,z_k(\xi,\eta)\big)-f\big(\xi,\eta,z_{k-1}(\xi,\eta)+ +v_{k-1}(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\cdot u_{k-1}^*(\xi,\eta)\big]d\xi\,d\eta.$$

By the mean-value theorem there exists a number $\lambda\in(0,1)$ such that

$$v_k(x,y)=\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f_z'\big(\xi,\eta,z_{k-1}(\xi,\eta)+v_{k-1}(\xi,\eta)+\lambda u_{k-1}^*(\xi,\eta)\big)- -f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)\big]\cdot u_{k-1}^*(\xi,\eta)\,d\xi\,d\eta+ +\varepsilon_1\varepsilon_2\int_0^x\int_0^y \big[f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\big]u_{k-1}^*(\xi,\eta)\,d\xi\,d\eta. \tag{12}$$

Since

$$\big|f_z'\big(\xi,\eta,z_{k-1}(\xi,\eta)+v_{k-1}(\xi,\eta)+\lambda u_{k-1}^*(\xi,\eta)\big)-f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)\big|$$
$$\leqslant\big|f_z'\big(\xi,\eta,z_2(\xi,\eta)\big)-f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)\big|+\big|f_z'\big(\xi,\eta,z_2(\xi,\eta)+v_2(\xi,\eta)\big)-$$
$$-f_z'\big(\xi,\eta,z_2(\xi,\eta)\big)\big|+\big|f_z'\big(\xi,\eta,z_3(\xi,\eta)\big)-f_z'\big(\xi,\eta,z_2(\xi,\eta)+v_2(\xi,\eta)\big)\big|+$$
$$+\big|f_z'\big(\xi,\eta,z_3(\xi,\eta)+v_3(\xi,\eta)\big)-f_z'\big(\xi,\eta,z_3(\xi,\eta)\big)\big|+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\big|f_z'\big(\xi,\eta,z_{k-1}(\xi,\eta)\big)-f_z'\big(\xi,\eta,z_{k-2}(\xi,\eta)+v_k(\xi,\eta)\big)\big|+$$
$$+\big|f_z'\big(\xi,\eta,z_{k-1}(\xi,\eta)+v_{k-1}(\xi,\eta)+\lambda u_{k-1}^*(\xi,\eta)\big)-f_z'\big(\xi,\eta,z_k(\xi,\eta)\big)\big|,$$

then we have

$$(13)\qquad v_k(x,y) \leqslant \varepsilon_1\varepsilon_2 N \int_0^x\int_0^y \Big[\big(|v_1(\xi,\eta)|+|u_1^*(\xi,\eta)|+\ldots+|v_{k-1}(\xi,\eta)|+ \\ +\lambda|u_{k-1}^*(\xi,\eta)|\big)u_{k-1}^*(\xi,\eta)\Big]d\xi d\eta + \\ +\varepsilon_1\varepsilon_2\left|\int_0^x\int_0^y u_{k-1}^*(\xi,\eta)\big[f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\big]d\xi d\eta\right|$$

for $0\leqslant x\leqslant a/\varepsilon_1$ and $0\leqslant y\leqslant b/\varepsilon_2$.

For any functions F, $G\in C^1(D_{xy})$, where $D_{xy}=\{(\xi,\eta)\colon\ 0\leqslant\xi\leqslant x,\ 0\leqslant\eta\leqslant y\}$, we have

$$\int_0^x\int_0^y F_x'(\xi,\eta)\cdot G(\xi,\eta)\,d\xi d\eta = -\int_0^x\int_0^y F(\xi,\eta)\cdot G_x'(\xi,\eta)\,d\xi d\eta + \\ +\int_{\beta(D_{xy})} F(x,\eta)\cdot G(x,\eta)\,d\eta,$$

$$\int_0^x\int_0^y F_y'(\xi,\eta)\cdot G(\xi,\eta)\,d\xi d\eta = -\int_0^x\int_0^y F(\xi,\eta)\cdot G_y'(\xi,\eta)\,d\xi d\eta + \\ +\int_{-\beta(D_{xy})} F(\xi,y)\cdot G(\xi,y)\,d\xi;$$

$\beta(D_{xy})$ denotes the boundary of D_{xy}.

Taking now

$$G(\xi,\eta)=u_{k-1}^*(\xi,\eta),\qquad F(\xi,\eta)=\int_0^x\big[f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\big]d\xi$$

we obtain

$$\int_0^x\int_0^y u_{k-1}^*(\xi,\eta)\big[f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\big]d\xi d\eta \\ = -\int_0^x\int_0^y\left\{\frac{\partial u_{k-1}^*(\xi,\eta)}{\partial x}\int_0^\xi\big[f_z'\big(\tau,\eta,\Xi(\tau,\eta)\big)-f_0'\big(\Xi(\tau,\eta)\big)\big]d\tau\right\}d\xi d\eta + \\ +\int_0^y\left\{u_{k-1}^*(x,\eta)\int_0^x\big[f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\big]d\xi\right\}d\eta.$$

Taking

$$G(\xi,\eta)=\frac{\partial u_{k-1}^*(\xi,\eta)}{\partial x},$$

$$F(\xi,\eta)=\int_0^\xi\left\{\int_0^\eta\big[f_z'\big(\xi,\eta,\Xi(\xi,\eta)\big)-f_0'\big(\Xi(\xi,\eta)\big)\big]d\xi\right\}d\eta$$

we have

$$-\int_0^x\int_0^y\left\{\frac{\partial^2 u_{k-1}^*(\xi,\eta)}{\partial x\,\partial y}\int_0^\xi\int_0^\eta\left[f_z'(\tau,\sigma,\Xi(\tau,\sigma))-f_0'(\Xi(\tau,\sigma))\right]d\tau\,d\sigma\right\}d\xi\,d\eta+$$

$$+\int_0^x\left\{\frac{\partial u_{k-1}^*(\xi,\eta)}{\partial x}\int_0^\xi\int_0^y\left[f_z'(\tau,\eta,\Xi(\tau,\eta))-f_0'(\Xi(\tau,\eta))\right]d\tau\,d\eta\right\}d\xi$$

$$=\int_0^x\int_0^y\left\{\frac{\partial u_{k-1}^*(\xi,\eta)}{\partial x}\int_0^\xi\left[f_z'(\tau,\eta,\Xi(\tau,\eta))-f_0'(\Xi(\tau,\eta))\right]d\tau\right\}d\xi\,d\eta.$$

Hence and by (13) we obtain

$$(14)\quad |v_k(x,y)|\leqslant \varepsilon_1\varepsilon_2 N\int_0^x\int_0^y\left[\left(|v_1(\xi,\eta)|+|u_1^*(\xi,\eta)|+\ldots+\right.\right.$$

$$\left.\left.+|v_{k-1}(\xi,\eta)|+\lambda|u_{k-1}^*(\xi,\eta)|\right)u_{k-1}^*(\xi,\eta)\right]d\xi\,d\eta+$$

$$+\varepsilon_1\varepsilon_2\int_0^x\int_0^y\left\{\left|\frac{\partial^2 u_{k-1}^*(\xi,\eta)}{\partial x\partial y}\right|\left|\int_0^\xi\int_0^\eta\left[f_z'(\tau,\sigma,\Xi(\tau,\sigma))-\right.\right.\right.$$

$$\left.\left.\left.-f_0'(\Xi(\tau,\sigma))\right]d\tau\,d\sigma\right|\right\}d\xi\,d\eta+$$

$$+\varepsilon_1\varepsilon_2\int_0^x\left\{\left|\frac{\partial u_{k-1}^*(\xi,\eta)}{\partial x}\right|\left|\int_0^\xi\int_0^y\left[f_z'(\tau,\eta,\Xi(\tau,\eta))-\right.\right.\right.$$

$$\left.\left.\left.-f_0'(\Xi(\tau,\eta))\right]d\tau\,d\eta\right|\right\}d\xi+$$

$$+\varepsilon_1\varepsilon_2|u_{k-1}^*(x,y)|\left|\int_0^x\int_0^y\left[f_z'(\xi,\eta,\Xi(\xi,\eta))-\right.\right.$$

$$\left.\left.-f_0'(\Xi(\xi,\eta))\right]d\xi\,d\eta\right|+$$

$$+\varepsilon_1\varepsilon_2\int_0^y\left\{\left|\frac{\partial u_{k-1}^*(x,\eta)}{\partial y}\right|\left|\int_0^x\int_0^\eta\left[f_z'(\xi,\sigma,\Xi(\xi,\sigma))-\right.\right.\right.$$

$$\left.\left.\left.-f_0'(\Xi(\xi,\sigma))\right]d\xi\,d\sigma\right|\right\}d\eta$$

for $0\leqslant x\leqslant a/\varepsilon_1$ and $0\leqslant y\leqslant b/\varepsilon_2$. The solution u_{k-1}^* of (10) with boundary conditions $u_{k-1}^*(x,0)=u_{k-1}^*(0,y)=0$ has the form

$$u_{k-1}^*(x,y)=\varepsilon_1\varepsilon_2\int_0^x\int_0^y V(x,y,\xi,\eta)\left[f\left(\xi,\eta,z_{k-1}(\xi,\eta)+v_{k-1}(\xi,\eta)\right)-\right.$$

$$\left.-f\left(\xi,\eta,z_{k-1}(\xi,\eta)\right)\right]d\xi\,d\eta,$$

where $V(x, y, \xi, \eta)$ is the Riemann function of (10). In this case the function $V(x, y, \xi, \eta)$ is continuous with respect to (x, y) and (ξ, η), thus the function r defined by $r(x, y) = \sup\{|V(x, y, \xi, \eta)|;\ (\xi, \eta) \in D_{xy}\}$ is continuous. Denoting

$$M = \sup\{\sup_{D_{xy}} |V(x, y, \xi, \eta)|;\ 0 \leqslant x \leqslant a/\varepsilon_1,\ 0 \leqslant y \leqslant b/\varepsilon_2\}$$

we obtain for $0 \leqslant x \leqslant a/\varepsilon_1$ and $0 \leqslant y \leqslant b/\varepsilon_2$

$$|u^*_{k-1}(x, y)| \leqslant MNab\,\|v_{k-1}\|,$$

where

$$\|v_{k-1}\| = \sup\{|v_{k-1}(x, y)|;\ 0 \leqslant x \leqslant a/\varepsilon_1,\ 0 \leqslant y \leqslant b/\varepsilon_2\}.$$

Hence and by (10)

$$\left|\frac{\partial u^*_{k-1}(x, y)}{\partial x}\right| \leqslant \varepsilon_1 N^2 b^2 Ma\,\|v_{k-1}\| + \varepsilon_1 Nb\,\|v_{k-1}\|,$$

$$\left|\frac{\partial u^*_{k-1}(x, y)}{\partial y}\right| \leqslant \varepsilon_2 N^2 a^2 Mb\,\|v_{k-1}\| + \varepsilon_2 Na\,\|v_{k-1}\|,$$

$$\left|\frac{\partial^2 u^*_{k-1}(x, y)}{\partial x \partial y}\right| \leqslant \varepsilon_1\varepsilon_2 N^2 Mab\,\|v_{k-1}\| + \varepsilon_1\varepsilon_2 N\,\|v_{k-1}\|.$$

In virtue of Lemma 2, there exist numbers $\nu_1 > 0$, $\nu_2 > 0$ such that for $\varepsilon_1 < \min(\nu_1, \nu_2)$, $\varepsilon_2 < \min(\nu_1, \nu_2)$ we have

$$\begin{aligned} |v_k(x, y)| \leqslant (Nab)^2 M(\,\|v_1\| + MNab\,\|v_1\| + \ldots + \|v_{k-1}\| + \\ + MNab\,\|v_{k-1}\|)\,\|v_{k-1}\| + N(MNab+1)ab\,\delta\,\|v_{k-1}\| + \\ + Nb(MNab+1)a\,\delta\,\|v_{k-1}\| + Na(MNab+1)b\,\delta\,\|v_{k-1}\| + MNab\,\|v_{k-1}\| \end{aligned}$$

for $0 \leqslant x \leqslant a/\varepsilon_1$ and $0 \leqslant y \leqslant b/\varepsilon_2$. Then for $\varepsilon_1 < \varepsilon_0$, $\varepsilon_2 < \varepsilon_0$ and $0 \leqslant x \leqslant a/\varepsilon_1$, $0 \leqslant y \leqslant b/\varepsilon_2$, where $\varepsilon_0 = \min(\nu_1, \nu_2)$, we have

$$(14)\qquad |v_k(x, y)| \leqslant Nab\left\{(1 + MNab)\left[abM \sum_{r=1}^{k-1} \|v_r\| + 3\delta\right] + M\delta\right\}\|v_{k-1}\|.$$

Denoting

$$P = \max\{Nab,\ (1 + MNab),\ abM,\ 3,\ M\}$$

we obtain for $k = 2, 3, \ldots$

$$|v_k(x, y)| < P^2\Big(\sum_{r=1}^{k-1} A_r\Big)\|v_{k-1}\|,$$

where $A_1 = (2P+1)\delta$ and $A_r = P\,\|v_{k-1}\|$ for $r = 2, 3, \ldots, k-1$.

Hence

$$|v_2(x,y)| < P^2\delta^2(2P+1),$$
$$|v_3(x,y)| < P^4\delta^3(2P+1)^2(1+P^3\delta),$$
$$|v_4(x,y)| < P^6\delta^4(2P+1)^3(1+P^3\delta)^2[1+P^5\delta^2(2P+1)],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Since

$$(2P+1) > 1, \quad (1+P^3\delta) > 1, \ldots,$$

then

$$|v_2(x,y)| < P^2\delta^2(1+2P),$$
$$|v_3(x,y)| < P^4\delta^3(1+2P)^2(1+P^3\delta),$$
$$|v_4(x,y)| < P^6\delta^4(1+2P)^4(1+P^3\delta)^2(1+P^5\delta^2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|v_k(x,y)| < P^{2(k-1)}\delta^k(2P+1)^{2^{k-2}}\prod_{r=1}^{k-2}(1+P^{2r+1}\delta^r)^{2^{k-r-2}}.$$

Since $(1+P^{2r+1}\delta^r) > 1$, then for $k > 2$ and $\delta < 1/P$ we have

$$|v_k(x,y)| < P^{2k-1}\delta^k(2P+1)^{2^{k-2}}\Big[\prod_{r=1}^{k-2}(1+P^{2r+1}\delta^r)\Big]^{2^{k-3}}$$
$$\leqslant P^{2k-1}\delta^k(3P)^{2^{k-2}}\left\{\frac{1}{(k-2)^{k-2}}\Big[\sum_{r=1}^{k-2}\big(1+(P\delta)^rP\big)\Big]^{k-2}\right\}^{2^{k-3}}$$
$$\leqslant P^{2k-1+(k/2)\cdot 2^{k-2}}\delta^k\cdot 3^{2^{k-2}}\left\{\frac{1}{(k-2)^{k-2}}\Big[\sum_{r=1}^{k-2}\big(1+(P^2\delta)\big)^r\Big]^{k-2}\right\}^{2^{k-3}}$$
$$\leqslant P^{2k-1+(k/2)\cdot 2^{k-2}}\delta^k\cdot 3^{2^{k-2}}\left\{\sum_{r=1}^{k-2}\left(1+\frac{1}{k-2}\,\frac{P^2\delta}{1-P^2\delta}\right)^{k-2}\right\}^{2^{k-3}}.$$

But $3 \leqslant P$; then

$$|v_k(x,y)| \leqslant 2^{(2k-1)+\frac{k+2}{2}2^{k-2}}\delta^k\left\{1+\frac{1}{k-2}\binom{k-1}{1}\frac{P^2\delta}{1-P^2\delta}+\right.$$
$$\left.+\frac{1}{(k-2)^2}\binom{k-2}{2}\left(\frac{P^2\delta}{1-P^2\delta}\right)^2+\ldots+\frac{1}{(k-2)^{k-2}}\left(\frac{P^2\delta}{1-P^2\delta}\right)^{k-2}\right\}^{2^{k-}}$$
$$\leqslant P^{\varphi(k)}\delta^k\left\{1+\frac{P^2\delta}{1-P^2\delta}+\frac{1}{2!}\left(\frac{P^2\delta}{1-P^2\delta}\right)^2+\ldots+\frac{1}{(k-2)!}\left(\frac{\delta P^2}{1-P^2\delta}\right)^{k-2}\right\}^{2^{k-3}}$$
$$\leqslant P^{\varphi(k)}\delta^k\exp\left(\frac{2^{k-3}P^2\delta}{1-P^2\delta}\right),$$

where

$$\varphi(k) = (2k-1) + \frac{k-2}{2} 2^{k-2}.$$

From (11) we obtain for $0 \leqslant x \leqslant a/\varepsilon_1$ and $0 \leqslant y \leqslant b/\varepsilon_2$

$$|h(x, y)| \leqslant \varepsilon_1 \varepsilon_2 N \int_0^x \int_0^y |h(\xi, \eta)| d\xi d\eta + |v_k(x, y)|.$$

Hence

$$|h(x, y)| \leqslant P^{\varphi(k)} \delta^k \exp\left(\frac{2^{k-3} P^2 \delta}{1 - P^2 \delta}\right) \cdot \exp(Nab).$$

Taking now $\delta < P^{\frac{-\varphi(k)}{k}}$ we have

$$\lim_{k \to \infty} |z_k(x, y) - z^{\varepsilon_1 \varepsilon_2}(x, y)| = 0$$

uniformly for $0 \leqslant x \leqslant a/\varepsilon_1$ and $0 \leqslant y \leqslant b/\varepsilon_2$, where $\varepsilon_1 < \varepsilon_0$, $\varepsilon_2 < \varepsilon_0$.

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