ANNALES
POLONICI MATHEMATICI
LIII.3 (1991)

Discrete maximum principle in parabolic boundary-value problems*

by ANDRZEJ KARAFIAT (Kraków)

Abstract. The purpose of this paper is to extend the maximum principle for parabolic BVP to the discrete case and also the discrete maximum principle proved by P. G. Ciarlet to the parabolic case. Some properties of a discrete Green function are given. A stability condition for the FDM with irregular grid is obtained.

1. Introduction. We set, for simplicity, $(t, x) = (t, x^1, \ldots, x^N)$,

$$u_{,t} = \frac{\partial u}{\partial t}, \quad u_{,i} = \frac{\partial u}{\partial x^i}, \quad u_{,ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \quad u_{,n} = \frac{\partial u}{\partial n},$$

where n is the outward normal vector.

In the theory of partial differential equations of parabolic type the following theorems on differential inequalities are known (for detailed formulations of the theorems we refer to the cited monographs):

THEOREM 1.1 ([7], p. 34). *Let $D = (0, T) \times \Omega$, where $\Omega \subset \mathbf{R}^N$ is a bounded domain. Let*

$$1^\circ \qquad Lu = u_{,t} - \Big[\sum_{i,j=1}^{N} a_{ij}(x, t) u_{,ij} + \sum_{i=1}^{N} b_i(x, t) u_{,i} + c(x; t) u\Big],$$

and assume that

2° *L is parabolic,*
3° *a, b, $c \in C^0(D)$,*
4° *the derivatives of u appearing in 1° are in $C^0(D)$,*
5° *$c \leqslant 0$,*
6° *$Lu \leqslant 0$ in D.*

If u attains a positive maximum in D at $P_0 \in D$, then $u \equiv u(P_0)$ in D.

THEOREM 1.2 ([7], p. 41). *Let*

$$\partial D_0 = \{0\} \times \Omega, \quad \partial D_T = \{T\} \times \Omega, \quad S = \partial D \backslash (\partial D_0 \cup \partial D_T)$$

* The paper was partially supported by the contract CPBP 02.01 of the Polish Academy of Sciences.

be the three parts of the boundary ∂D of D. If 1°–5° hold and
6° $Lu = 0$,

then

$$\max_{(x,t)\in D} |u(x, t)| \leqslant \max_{(x,t)\in \partial D_0 \cup S} |u(x, t)|.$$

THEOREM 1.3 ([15], Th. 63.1). *Let* $x = (x^1, \ldots, x^N)$,

$$U_x = (u_{,1}, \ldots, u_{,N}), \qquad U_{xx} = (u_{,11}, u_{,12}, \ldots, u_{,NN}),$$

let functions $\alpha(t, x) \geqslant 0$, $\beta(t, x) > 0$ *be defined on the boundary* ∂D *and let* $\Sigma_\alpha = \{(t, x) \in \partial D\colon \alpha(t, x) > 0\}$. *Suppose that*

(i) *the domain D is regular enough and non degenerate (cf. [15]),*
(ii) *the function $f(t, x, u, U_{\alpha x}, U_{xx})$ defined on D is elliptic,*
(iii) *the functions u, v are regular enough,*
(iv) $u_{,t}(t, x) < f(t, x, u, U_x, U_{xx})$, $v_{,t}(t, x) \geqslant f(t, x, v, V_x, V_{xx})$ *in the set* $G = \{(t, x)\colon u(t, x) = v(t, x)\}$,
(v) $u(0, x) < v(0, x)$,
(vi) $\beta(t, x)[u(t, x) - v(t, x)] - \alpha(t, x)(u - v)_{,n}(t, x) < 0$, *on* Σ_α, $u(t, x) - v(t, x) < 0$, *on* $\partial D \backslash \Sigma_\alpha$.

Then $u(t, x) < v(t, x)$ *in* D.

THEOREM 1.4 ([15], Th. 64.1). *Under general assumptions similar to those of Theorem. 1.3 if the strong inequalities are replaced by weak ones then* $u(t, x) \leqslant v(t, x)$ *in* D.

If we approximate the solution u of $Lu = f$ by a finite difference method (FDM) it is natural to expect that the approximative solution U will have analogous properties. This idea was used in proofs of convergence and stability for the classical FDM ([4, 6]). When a FDM with irregular grid was introduced by R. H. Mac Neal [11] and this idea was followed by many other authors ([9, 10] and references in [13]), a demand for general theorems, valid for any grid, appeared. Monotonicity of matrices and a sufficient condition for it (L. Collatz [3]) were a basis for general maximum principles in linear cases (T. Styś [14]). The theorem of P. G. Ciarlet [2] is an example of a discrete maximum principle for elliptic BVP. Inequalities for solutions of difference equations at a regular mesh, analogous to those for the corresponding differential equations, were proved for both linear and nonlinear boundary-value problems of elliptic and parabolic types (A. Fitzke [5], Z. Kowalski [8], M. Malec [12], T. Styś [14]). They were used in proofs of convergence of FDM at a regular mesh.

The purpose of the present paper is an extension of Theorems 1.1–1.4 to discrete solutions of the given equations obtained by both explicit and implicit FDM. For the explicit method a stability condition will be obtained too.

Consider a bounded domain $\Omega \subset \mathbf{R}^N$ with boundary $\partial\Omega$ and the domain $D = (0, T) \times \Omega \subset \mathbf{R}^{N+1}$ with boundary $\partial D = \partial D_0 \cup \partial D_T \cup \partial D_1 \cup \partial D_2$, defined as follows:

$$\partial D_0 = \{0\} \times \Omega, \quad \partial D_T = \{T\} \times \Omega,$$
$$\partial D_1 \cup \partial D_2 = \partial D \backslash (\partial D_0 \cup \partial D_T), \quad \partial D_1 \cap \partial D_2 = \emptyset,$$

where ∂D_1, ∂D_2 will be defined by boundary conditions.

We set

$$\partial D_B = \partial D_0 \cup \partial D_1, \quad \bar{D} = D \cup \partial D.$$

Next we assume that a mesh of nodes x_i, $i \in I$, is fixed in Ω, and moreover

(1) $$x_i \in \begin{cases} \Omega & \text{if } i \in I_{\text{int}} \text{ (internal nodes)}, \\ \partial\Omega & \text{if } i \in I_{\text{bd}} \text{ (boundary nodes)}, \end{cases}$$

$I = I_{\text{int}} \cup I_{\text{bd}}$.

For a given sequence $0 = t_0 < t_1 < \ldots < t_M = T$, $k = t_{j+1} - t_j = \text{const}$, a mesh of nodes in $\bar{D}$ consists of the points (t_n, x_i), $n \in \{0, \ldots, M\}$, $i \in I$. The (t_n, x_i) for $n \in \{1, \ldots, M-1\}$, $i \in I_{\text{int}}$ are internal nodes. Others are boundary nodes. For each $i \in I$ let $G(i)$ be the set of indices of nodes belonging to the difference star with central node x_i. Let $h(i)$ be the radius of this star and $h = \max_i h(i)$.

We assume that for each $(t_n, x_i) \in \partial D_2$,

(2) $$(t_n, x_j) \in D, \quad \forall j \in G(i),\, j \neq i.$$

2. Difference inequalities for the explicit method. Consider the heat equation

(3) $$u_{,t} - c^2 u_{,xx} = 0$$

in $D \subset \mathbf{R}^2$ with the Dirichlet boundary condition

(4) $$u(t, x) = g(t, x),$$

for $(t, x) \in \partial D_B$, and $\partial D_2 = \emptyset$.

The explicit FDM provides the following equations for the approximative solution U:

(5) $$U(t_{n+1}, x_i) = \lambda U(t_n, x_i - h) + (1 - 2\lambda) U(t_n, x_i) + \lambda U(t_n, x_i + h)$$

$\forall n \in \{0, \ldots, M-1\}$, $i \in I_{\text{int}}$, where $\lambda = c^2 k h^{-2}$.

If the stability condition $\lambda \leqslant 0.5$ is satisfied, then

(6) $$\min_j U(t_n, x_j) \leqslant U(t_{n+1}, x_i) \leqslant \max_j U(t_n, x_j)$$

$\forall n \in \{0, \ldots, M-1\}$, $i \in I_{\text{int}}$, and the difference scheme is stable.

If $D \subset \mathbf{R}^3$, $x_i = (x_i^1, x_i^2) \in \mathbf{R}^2$, then the heat equation is

$$u_{,t} - c^2 (u_{,11} + u_{,22}) = 0$$

and (5) becomes

$$U(t_{n+1}, x_i^1, x_i^2) = \lambda[U(t_n, x_i^1+h, x_i^2)+U(t_n, x_i^1-h, x_i^2)+U(t_n, x_i^1, x_i^2+h) + U(t_n, x_i^1, x_i^2-h)]+(1-4\lambda)U(t_n, x_i^1, x_i^2).$$

Then inequalities (6) hold if $\lambda \leqslant 0.25$. Both above results are classical and given in fundamental monographs (cf. [4], [6]).

In the following we will consider any dimension N and an irregular grid in $\bar{\Omega}$. Let us replace (3) by

$$u_{,t}-\Big[\sum_{i,j=1}^{N} a_{ij}(t, x)u_{,ij}+\sum_{i=1}^{N} b_i(t, x)u_{,i}\Big] = f(t, x), \tag{7}$$

$(t, x)\in D \cup \partial D_T$, and the boundary condition (4) by

$$u(t, x) = g_1(t, x), \quad (t, x)\in \partial D_B, \tag{8}$$

$$\sum_{i=1}^{N} d_i(t, x)u_{,i}(t, x) = g_2(t, x), \quad (t, x)\in \partial D_2. \tag{9}$$

The boundary condition (9) may have the form

$$u_{,n} = g_2 \quad \text{or} \quad \sum_{i,j=1}^{N} a_{ij}n^i u_{,j} = g_2,$$

where $n = (n^1, \ldots, n^N)$ is the outward normal vector. A discrete form of (7) on an irregular net is

$$k^{-1}[U(t_{n+1}, x_i)-U(t_n, x_i)] - \sum_{j\in G(i)} \{[\alpha_j(n, i)h^{-2}+\beta_j(n, i)h^{-1}][U(t_n, x_j)-U(t_n, x_i)]\} = f(t_n, x_i), \tag{10}$$

and of (8), (9)

$$U(t_n, x_i) = g_1(t_n, x_i), \quad (t_n, x_i)\in \partial D_B, \tag{11}$$

$$h^{-1} \sum_{j\in G(i)} \delta_j(n, i)[U(t_n, x_i)-U(t_n, x_j)] = g_2(t_n, x_i) \tag{12}$$

for $(t_n, x_i)\in \partial D_2$, where α_j, β_j are coefficients of difference schemes. This boundary condition may describe the Dirichlet, Neumann or mixed BVP.

THEOREM 2.1. *Consider the boundary-value problem* (7)–(9) *and let U be the solution of the corresponding difference problem* (10)–(12). *Suppose that for all n, i the following assumptions hold*:

(13)
(i) $\alpha_j(n, i),\ \ \delta_j(n, i) > 0,\ \ \alpha_j(n, i)h^{-1}+\beta_j(n, i) > 0, \quad \forall j\in G(i)$,

(ii) $f(t_n, x_i),\ g_2(t_n, x_i) \leqslant 0$,

(iii) $0 < k \leqslant \min_{n,i}\{\sum_{j\in G(i)} [\alpha_j(n, i)h^{-2}+\beta_j(n, i)h^{-1}]\}^{-1}$

and let $m \in I$ *be such that*

(14) $$U(T, x_m) = \max_{i \in I} U(T, x_i).$$

Then there is a node $(t_n, x_k) \in \partial D_B$ *such that*

(15) $$U(T, x_m) \leqslant U(t_n, x_k).$$

Proof. For a fixed $n \in \{0, \ldots, M-1\}$ $(t_M = T)$, let $m = m(n)$ be such that

$$U(t_{n+1}, x_m) = \max_{i \in I} U(t_{n+1}, x_i).$$

Then either

1° $(t_{n+1}, x_m) \in \partial D_B$, or

2° $(t_{n+1}, x_m) \in D \cup \partial D_T$, or

3° $(t_{n+1}, x_m) \in \partial D_2$.

In the latter case, by (12),

$$U(t_n, x_m) \sum_{j \in G(m)} \delta_j = \sum_{j \in G(m)} \delta_j U(t_n, x_j) + hg_2(t_n, x_m) \leqslant \sum_{j \in G(m)} \delta_j U(t_n, x_j)$$

because of (2) and (13)(ii), and there is $r \in G(m) \subset I_{\text{int}}$ such that

(16) $$U(t_{n+1}, x_r) = \max_{i \in I_{\text{int}}} U(t_{n+1}, x_i) = U(t_{n+1}, x_m),$$

which implies 2°.

In that case $(m = r)$

$$U(t_{n+1}, x_m) = k \sum_{j \in G(m)} \{[\alpha_j h^{-2} + \beta_j h^{-1}][U(t_n, x_j) - U(t_n, x_m)]\}$$
$$+ U(t_n, x_m) + kf(t_n, x_m).$$

Therefore

$$U(t_{n+1}, x_m) \leqslant k \sum_{j \in G(m)} [\alpha_j h^{-2} + \beta_j h^{-1}] U(t_n, x_j)$$
$$+ [1 - k \sum_{j \in G(m)} (\alpha_j h^{-2} + \beta_j h^{-1})] U(t_n, x_m),$$

where k fulfils (13)(iii). For $p \in G(m)$ such that

$$U(t_n, x_p) = \max_{j \in G(m)} U(t_n, x_j),$$

we have

$$U(t_{n+1}, x_m) \leqslant U(t_n, x_p) \{ \sum_{j \in G(m)} [k(\alpha_j h^{-2} + \beta_j h^{-1}) - k(\alpha_j h^{-2} + \beta_j h^{-1})] + 1\}$$
$$= U(t_n, x_p).$$

We have thus proved that

(17) $$\max_{i \in I} U(t_{n+1}, x_i) = U(t_{n+1}, x_m) \leqslant U(t_n, x_p) \leqslant \max_{i \in I} U(t_n, x_i).$$

Repeating (17) at most M times from $t_{n+1} = T$ (to $t_n = 0$) we find $(t_n, x_p) \in \partial D_B$ and this is the point (t_n, x_k) which fulfils (15). ∎

For a more general equation

$$(18) \qquad Lu = u_{,t} - \Big[\sum_{i,j=1}^{N} a_{ij}(t, x) u_{,ij} + \sum_{i=1}^{N} b_i(t, x) u_{,i} + c(t, x) u\Big]$$
$$= f(t, x), \quad (t, x) \in D,$$

with the boundary conditions (8) and

$$(19) \qquad Bu = \sum_{i=1}^{N} d_i(t, x) u_{,i}(t, x) + e(t, x) u(t, x) = g_2(t, x),$$

$(t, x) \in \partial D_2$, the discrete equations (10)–(12) become

$$(20) \qquad L_h^e U = k^{-1}[U(t_{n+1}, x_i) - U(t_n, x_i)] - \sum_{j \in G(i)} \{[\alpha_j(n, i) h^{-2} + \beta_j(n, i) h^{-1}]$$
$$\times [U(t_n, x_j) - U(t_n, x_i)]\} - \gamma(n, i) U(t_n, x_i) = f(t_n, x_i),$$

$(t_n, x_i) \in D \cup \partial D_T$, and

$$(21) \qquad B_h U = h^{-1} \sum_{j \in G(i)} \delta_j(n, i)[U(t_n, x_i) - U(t_n, x_j)] + \varepsilon(n, i) U(t_n, x_i)$$
$$= g_2(t_n, x_i), \quad (t_n, x_i) \in \partial D_2.$$

THEOREM 2.2. *Consider the boundary-value problem* (8), (18), (19) *and let* U *be the solution of the corresponding difference problem* (11), (20), (21). *Suppose that* (i), (ii) *of Theorem* 2.1 *are satisfied and* $\forall (n, i)$

$$(22) \qquad \begin{array}{ll} \text{(iii)} & \varepsilon(n, i) \geqslant 0, \quad \gamma(n, i) \leqslant 0, \\ \text{(iv)} & 0 < k \leqslant \min\limits_{n,i} \{ \sum\limits_{j \in G(i)} [\alpha_j(n, i) h^{-2} + \beta_j(n, i) h^{-1}] - \gamma(n, i)\}^{-1}. \end{array}$$

Let m *be defined as in Theorem* 2.1, *with*

$$(23) \qquad U(T, x_m) \geqslant 0.$$

Then the conclusion of Theorem 2.1 *remains valid.*

The proof runs analogously to that of Theorem 2.1. (16) is a consequence of the inequality

$$U(t_n, x_m)\Big[\sum_{j \in G(m)} \delta_j + h\varepsilon\Big] = \sum_{j \in G(m)} \delta_j U(t_n, x_j) + h g_2(t_n, x_m)$$
$$\leqslant \sum_{j \in G(m)} \delta_j U(t_n, x_j)$$

and (17) follows from

$$U(t_{n+1}, x_m) = k \sum_{j\in G(m)} \{[\alpha_j h^{-2} + \beta_j h^{-1}][U(t_n, x_j) - U(t_n, x_m)]\}$$

$$+(1+k\gamma)U(t_n, x_m) + kf(t_n, x_m)$$

$$\leqslant k \sum_{j\in G(m)} [\alpha_j h^{-2} + \beta_j h^{-1}] U(t_n, x_j)$$

$$+\{1+k[\gamma - \sum_{j\in G(m)} (\alpha_j h^{-2} + \beta_j h^{-1})] U(t_n, x_m)$$

$$\leqslant U(t_n, x_p)(1+k\gamma) \leqslant U(t_n, x_p),$$

where k fulfils (22). ∎

COROLLARY 2.1. *If one replaces*: (13)(ii) *by* $f(t_n, x_i)$, $g_2(t_n, x_i) \geqslant 0$, $\forall n, i$ *and* (14), (23) *by* $U(T, x_m) = \min_{i\in I} U(T, x_i) \leqslant 0$, *then the conclusion of Theorem* 2.2 *remains valid with* (15) *replaced by* $U(T, x_m) \geqslant U(t_n, x_k)$.

COROLLARY 2.2. *If the assumptions of Theorem* 2.2 *hold and* $g_1(t_n, x_i) \leqslant 0$ $\forall (t_n, x_i) \in \partial D_B$, *then* $U(t_n, x_i) \leqslant 0$ $\forall (t_n, x_i) \in \partial D_T \cup D$.

COROLLARY 2.3. *If the assumptions of Corollary* 2.1 *hold and* $g_1(t_n, x_i) \geqslant 0$, $\forall (t_n, x_i) \in \partial D_B$, *then* $U(t_n, x_i) \geqslant 0$ $\forall (t_n, x_i) \in \partial D_T \cup D$.

Remark 2.1. Analogous corollaries are valid for Theorem 2.1.

Remark 2.2. If the coefficients of (7) depend strongly on time, a variable time step $k(n) = t_{n+1} - t_n$ may be considered. Then the stability conditions (13)(iii) and (22)(iv) should be replaced by

$$0 < k(n) \leqslant \min_i \{ \sum_{j\in G(i)} [\alpha_j(n, i)h^{-2} + \beta_j(n, i)h^{-1}]\}^{-1}$$

and

$$0 < k(n) \leqslant \min_i \{ \sum_{j\in G(i)} [\alpha_j(n, i)h^{-2} + \beta_j(n, i)h^{-1}] - \gamma(n, i)\}^{-1}\},$$

respectively, and Theorems 2.1, 2.2 remain valid.

Remark 2.3. By inspecting the proof of Theorem 2.1 it may easily be seen that the assumption $\alpha_j > 0$ ((13) (i)) is not used and can be omitted. However, the assumption $\alpha_j h^{-1} + \beta_j > 0$ depends on the mesh dimension h and can be fulfilled for bounded α_j, β_j and h tending to 0^+ if $\alpha_j > 0$. The same refers to Theorems 2.2, 3.1, 3.2.

3. Difference inequalities for the implicit method. In this method equation (7) is discretized as follows:

$$(24)\quad k^{-1}[U(t_n, x_i)-U(t_{n-1}, x_i)] - \sum_{j\in G(i)} \{[\alpha_j(n, i)h^{-2}+\beta_j(n, i)h^{-1}][U(t_n, x_j)-U(t_n, x_i)]\} = f(t_n, x_i).$$

Let the boundary conditions (8), (9) have the same form as before for $(t_n, x_i)\in\partial D_2$.

THEOREM 3.1. *Let us consider the boundary-value problem* (7)–(9) *and let U be the solution of the corresponding difference problem* (11), (12), (24) *of the implicit method. Suppose that* (i), (ii) *of Theorem* 2.1 *are satisfied and let m be defined as in that theorem. Then the conclusion of Theorem* 2.1 *remains valid.*

The proof is similar to that of Theorem 2.1. The difference consists in the way in which (17) is obtained. In the present case it is proved as follows: Let $U(t_n, x_m) = \max_{i\in I} U(t_n, x_i)$. Then

$$\begin{aligned} U(t_{n-1}, x_m) &= -k \sum_{j\in G(m)} \{[\alpha_j h^{-2}+\beta_j h^{-1}][U(t_n, x_j)-U(t_n, x_m)]\} \\ &\quad + U(t_n, x_m)-kf(t_n, x_m) \\ &\geqslant -k \sum_{j\in G(m)} [\alpha_j h^{-2}+\beta_j h^{-1}]\, U(t_n, x_j) \\ &\quad +\Big[1+k \sum_{j\in G(m)} (\alpha_j h^{-2}+\beta_j h^{-1})\Big]\, U(t_n, x_m) \\ &\geqslant U(t_n, x_m) \end{aligned}$$

and

$$\max_{i\in I} U(t_n, x_i) = U(t_n, x_m) \leqslant U(t_{n-1}, x_m) \leqslant \max_{i\in I} U(t_{n-1}, x_i). \ \blacksquare$$

For equation (18) with boundary conditions (8), (19) the discrete counterparts in the implicit method are

$$\begin{aligned} (25)\quad L_h^i U = {} & k^{-1}[U(t_n, x_p)-U(t_{n-1}, x_p)] \\ & - \sum_{j\in G(p)} \{[\alpha_j(n, p)h^{-2}+\beta_j(n, p)h^{-1}][U(t_n, x_j)-U(t_n, x_p)]\} \\ & -\gamma(n, p)\, U(t_n, x_p) = f(t_n, x_p), \qquad (t_n, x_p)\in D\cup\partial D_T, \end{aligned}$$

with boundary conditions (11), (21).

THEOREM 3.2. *Let us consider the boundary-value problem* (8), (18), (19) *and let U be the solution of the corresponding difference problem* (11), (21), (25). *Suppose that* (i), (ii) *of Theorem* 2.1 *and* (iii) *of Theorem* 2.2 *are satisfied and let m be defined as in Theorem* 2.1 *with* (23) *fulfilled. Then the conclusion of Theorem* 2.1 *remains valid.*

The proof is a modification of that of Theorem 3.1 to the same extent as the proof of Theorem 2.2 is a modification of that of Theorem 2.1. We omit the details.

COROLLARY 3.1. *If the assumptions of Theorem 3.2 hold and moreover g_1 is non-positive, thus U is non-positive too.*

Remark 3.1. Corollaries of Theorems 3.1 and 3.2 analogous to Corollaries 2.1 and 2.3 are also valid. The exact formulations are left to the reader.

4. Strong discrete maximum principle. The following two sections will use some ideas and notations of [2].

DEFINITION 4.1 ([4], [16]). A square $n \times n$ matrix $A = (a_{ij})$ is *irreducible* if there is no permutation $(i_1, \ldots, i_n)$ of the sequence $(1, \ldots, n)$ and $m \in \{1, \ldots, n)$ such that $a_{i_r i_s} = 0$, $\forall i_r < m$, $i_s \geqslant m$.

Let the nodes $X_j = (t_n, x_i)$ be numbered from 1 to L and let

$$X_j \in \begin{cases} \bar{D} \backslash \partial D_B & \text{if } j \leqslant L_0 < L, \\ \partial D_B & \text{if } j > L_0. \end{cases}$$

The matrix of the implicit method (11), (21), (25) has then the form

$$\bar{A} = \begin{array}{c} \begin{array}{cc} L_0 & L_1 \end{array} \\ \left[\begin{array}{c|c} A & A^{\partial} \\ \hline 0 & I \end{array}\right] \end{array} \begin{array}{l} L_0 \\ L_1 \end{array} \tag{26}$$

where $L_1 = L - L_0$. Next we set

$$\bar{f}(X_j) = \begin{cases} f(X_j), & X_j \in D \cup \partial D_T, \\ g_1(X_j), & X_j \in \partial D_B, \\ g_2(X_j), & X_j \in \partial D_2. \end{cases}$$

Let U be the solution of the equation

$$\bar{A}U = \bar{f} \tag{27}$$

replacing the system (11), (21), (25).

LEMMA 4.1. *Suppose that the matrix A of* (26) *is irreducible, the assumptions* (i), (ii) *of Theorem* 2.1 *and* (iii) *of Theorem* 2.2 *are fulfilled and*

$$\forall j > L_0 \; \exists i \leqslant L_0: \quad a_{ij} < 0, \tag{28}$$

in other words, the matrix A^{∂} has no column containing only zeros. Then

$$\bar{f} \leqslant 0 \quad in\ D, \quad f(X_r) < 0 \tag{29}$$

imply

$$U \leqslant 0 \quad \text{in } \bar{D}, \quad U < 0 \quad \text{on} \quad \bar{D} \backslash \partial D_B. \tag{30}$$

If $X_r \in \bar{D} \backslash \partial D_B$, *condition* (28) *is not necessary.*

Proof. Corollary 3.1 implies

$$U \leqslant 0. \tag{31}$$

Let $\bar{f}(X_m) < 0$. Then $\sum_j a_{mj} U(X_j) = \bar{f}(X_m)$ implies that $U(X_p) < 0$ for some p. Assume, contrary to (30), that there is an $X_k \in \bar{D} \backslash \partial D_B$ with $U(X_k) = 0$. Two cases are possible:

1° $X_p \in D \cup \partial D_T \cup \partial D_2$, i.e. $p \leqslant L_0$,

2° $X_p \in \partial D_B$, i.e. $p > L_0$.

If 1° holds, there is a sequence $k = r_0, r_1, \ldots, r_m = p$ contained in $\{1, \ldots, L_0\}$ such that

$$a_{r_{j-1} r_j} < 0, \quad i = 1, \ldots, m, \tag{32}$$

because A is irreducible [16]. In case 2°, there is a $p_0 \leqslant L_0$ for which $a_{p_0 p} < 0$ (Assumption (28)). Then there is a sequence $k = r_0, r_1, \ldots, r_{m-1} = p_0, r_m = p$ and satisfying (32). Let i be an element of this sequence for which

$$U(X_i) = 0, \quad U(X_{i+1}) < 0. \tag{33}$$

The ith equation of the system (27) has the form

$$\bar{f}(X_i) = k^{-1}[U(X_i) - U(X_s)] + \sum_j (\alpha_j h^{-2} + \beta_j h^{-1})[U(X_i) - U(X_j)] - \gamma U(X_i) > 0$$

when $X_i \in D \cup \partial D_T$, and

$$\bar{f}(X_i) = h^{-1} \sum_j \delta_j [U(X_i) - U(X_j)] + \varepsilon U(X_i) > 0$$

when $X_i \in \partial D_2$, because of (13), (22) (iii), (31)–(33). This contradicts (29). ∎

THEOREM 4.1 (strong discrete maximum principle). *Suppose that*

– *assumptions* (i), (ii) *of Theorem* 2.1, (iii) *of Theorem* 2.2 *and* (28) *are fulfilled,*

– *A is irreducible,*

– *U is the solution of* (27),

– *there is an* $X_m \in \bar{D} \backslash \partial D_B$ *such that*

$$U(X_m) = \max_{X_j \in \bar{D}} U(X_j) \geqslant 0. \tag{34}$$

Then $U(X_j) = U(X_m)$ $\forall j \in \bar{D} \backslash \partial D_B$.

Proof. Suppose, on the contrary, that there is an $X_k \in \bar{D} \backslash \partial D_B$ with $U(X_k) < U(X_m)$. Let $V(X_j) = U(X_j) - U(X_m)$, $\forall X_j \in \bar{D}$.

V fulfils

(35) $L_h^i V(X_j) = L_h^i U(X_j) + \gamma U(X_m) = f(X_j) + \gamma U(X_m) \leqslant 0 \quad \text{in } D \cup \partial D_T,$

(36) $B_h V(X_j) = B_h U(X_j) - \varepsilon U(X_m) = g_2(X_j) - \varepsilon U(X_m) \leqslant 0 \quad \text{on } \partial D_2,$

(37) $V(X_j) = U(X_j) - U(X_m) = g_1(X_j) - U(X_m) \leqslant 0 \quad \text{on } \partial D_B$

and

(38) $$V(X_k) < 0.$$

(35)–(38) imply (29). V is then negative in the whole set $\bar{D} \backslash \partial D_B$, thus at X_m, contrary to (34). ∎

Remark 4.1. If, in Theorem 4.1, U is assumed to be the solution of the system of equations (11), (12), (24), i.e. if $\gamma = \varepsilon = 0$, then the assertion of Theorem 4.1 remains valid even if

$$U(X_m) = \max_{X_j \in \bar{D}} U(X_j) < 0.$$

The proof is similar.

5. Discrete Green functions. For the fixed boundary-value problem (8), (18), (19) and its discrete counterparts (11), (20), (21) in the explicit method and (11), (21), (25) in the implicit method discrete Green functions $G_s^e(X_k;\ Y_j)$, $G_s^i(X_k;\ Y_j)$ $(s = 1, 2, 3)$ may be defined as the solutions of the following systems:

$$L_h^e G_1^e(X_k;\ Y_j) = L_h^i G_1^i(X_k;\ Y_j) = 0, \quad X_k \in D \cup \partial D_T,$$

$$G_1^e(X_k;\ Y_j) = G_1^i(X_k;\ Y_j) = \begin{cases} 1, & X_k = Y_j, \\ 0, & X_k \neq Y_j, \end{cases} \quad X_k \in \partial D_B,$$

$$B_h G_1^e(X_k;\ Y_j) = B_h G^i(X_k;\ Y_j) = 0, \quad X_k \in \partial D_2, \quad \forall Y_j \in \partial D_B;$$

$$L_h^e G_2^e(X_k;\ Y_j) = L_h^i G_2^i(X_k;\ Y_j) = 0, \quad X_k \in \bar{D} \cup \partial D_T,$$

$$G_2^e(X_k;\ Y_j) = G_2^i(X_k;\ Y_j) = 0, \quad X_k \in \partial D_B,$$

$$B_h G_2^e(X_k;\ Y_j) = B_h G_2^i(X_k;\ Y_j) = \begin{cases} 1, & X_k = Y_j, \\ 0, & X_k \neq Y_j, \end{cases} \quad X_k \in \partial D_2, \ \forall Y_j \in \partial D_2;$$

$$L_h^e G_3^e(X_k;\ Y_j) = L_h^i G_3^i(X_k;\ Y_j) = \begin{cases} 1, & X_k = Y_j, \\ 0, & X_k \neq Y_j, \end{cases} \quad X_k \in D \cup \partial D_T,$$

$$G_3^e(X_k;\ Y_j) = G_3^i(X_k;\ Y_j) = 0, \quad X_k \in \partial D_B,$$

$$B_h G_3^e(X_k;\ Y_j) = B_h G_3^i(X_k;\ Y_j) = 0, \quad X_k \in \partial D_2, \quad \forall Y_j \in D \cup \partial D_T,$$

where Y_j are parameters of the families G_k^e, G_k^i.

The operators L_h^e, L_h^i, B_h are linear, which implies

PROPOSITION 5.1. *Each solution U of* (11), (20), (21) *has the form*

$$U(X_k) = \sum_{Y_j \in \partial D_B} G_1^e(X_k;\, Y_j) g_1(Y_j) + \sum_{Y_j \in \partial D_2} G_2^e(X_k;\, Y_j) g_2(Y_j) + \sum_{Y_j \in D \cup \partial D_T} G_3^e(X_k;\, Y_j) f(Y_j).$$

PROPOSITION 5.2. *Each solution U of* (11), (21), (25) *has the form*

$$U(X_k) = \sum_{Y_j \in \partial D_B} G_1^i(X_k;\, Y_j) g_1(Y_j) + \sum_{Y_j \in \partial D_2} G_2^i(X_k;\, Y_j) g_2(Y_j) + \sum_{Y_j \in D \cup \partial D_T} G_3^i(X_k;\, Y_j) f(Y_j).$$

Theorems 2.2 & 3.2 imply the following

THEOREM 5.1. *If the assumptions* (13)(i), (22) *hold, then*

$$G_1^e(X_k;\, Y_j),\ G_2^e(X_k;\, Y_j),\ G_3^e(X_k;\, Y_j) \geqslant 0, \quad \forall X_k,\ Y_j \in \bar{D},$$

$$\sum_{Y_j \in \partial D_B} G_1^e(X_k;\, Y_j) \leqslant 1, \quad \forall X_k \in \bar{D}.$$

THEOREM 5.2. *If the assumptions* (13)(i), (22)(iii) *hold, then*

$$G_1^i(X_k;\, Y_j),\ G_2^i(X_k;\, Y_j),\ G_3^i(X_k;\, Y_j) \geqslant 0, \quad \forall X_k,\ Y_j \in \bar{D},$$

$$\sum_{Y_j \in \partial D_B} G_1^i(X_k;\, Y_j) \leqslant 1, \quad \forall X_k \in \bar{D}.$$

6. Concluding remarks. An extension of Theorems 1.2–1.4 to discrete equations, declared in the Introduction, has been made, yielding Theorems 2.1–3.2. Theorems 2.2, 3.2 are generalizations of the discrete maximum principle of P. G. Ciarlet ([2], Th. 3) to parabolic equations of second order. Theorems 2.1, 3.1 are somewhat stronger (maximum can be negative as well) but the zero-order components of the differential operators L_h^e, L_h^i and the boundary operator B_h have to disappear. In Theorems 2.1–3.2 the irreducibility of the coefficient matrix is not assumed. This is due to the discretization method used in which the difference quotient with respect to time is uniquely defined. Theorem 4.1, which is a counterpart of Theorem 1.1, has no equivalent in [2]. It is valid for the implicit method but not for the explicit one. Theorems 5.1, 5.2 extend the corresponding theorem of [2] (Th. 4) to a parabolic boundary-value problem. Generally, Theorems 2.1, 3.1, 4.1 can also be considered as detailed and extended formulations of the general discrete maximum principles of T. Styś [14]. We have also obtained a stability condition for the explicit method on irregular grids ((22)(iv)). It may be strengthened in particular cases (cf. [9], [10]) but there is no general formula for a better value of the time step length.

The above theorems may be applied to the proof of convergence of the solution of a parabolic BVP obtained by the finite difference method on regular and irregular grids. As far as the author knows, this is not done yet. The results of [2] were also applied in a similar way.

The regularity of the domain D, of the coefficients of (18) and of its solution are not considered nor is the parabolicity of the operator L. This is not needed in fact; an exact solution of (8), (18), (19) is not taken into account, a discrete solution U defined on the nodes X_l is discussed only. The assumption on the form of the domain $D = (0, T) \times \Omega$ can be easily replaced by a weaker one (a non-cylindrical domain).

References

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Acad. Press, N. Y. & London 1979.

[2] P. G. Ciarlet, *Discrete maximum principle for finite-difference operators*, Aequationes Math. 4 (3) (1970), 338–352.

[3] L. Collatz, *Aufgaben monotoner Art*, Arch. Math. (Basel) 3 (5) (1952), 366–376.

[4] —, *Numerische Behandlung von Differentialgleichungen*, Springer, Berlin 1955.

[5] A. Fitzke, *Method of difference inequalities for parabolic equations with mixed derivatives*, Ann. Polon. Math. 31 (1975), 121–129.

[6] G. E. Forsythe and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations*, Wiley, N. Y. & London 1960.

[7] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Engl. Cliffs 1964.

[8] Z. Kowalski, *Difference inequalities for the equation of diffusion*, Univ. Iagel. Acta Math. 24 (1984), 131–134.

[9] T. Liszka, *The finite difference method at arbitrary irregular meshes and advantages of its applications in mechanics*, Ph. D. Thesis, Politechnika Krakowska, Kraków 1978 (in Polish).

[10] T. Liszka and J. Orkisz, *The finite difference method at arbitrary irregular grids and its application in applied mechanics*, Comput. & Structures 11 (1980), 83–95.

[11] R. H. Mac Neal, *An assymetrical finite difference network*, Quart. Appl. Math. 11 (1953), 295–310.

[12] M. Malec, *Schéma explicite des différences finies pour un système d'équations non linéaires du type parabolique avec des conditions aux limites non linéaires*, Ann. Polon. Math. 41 (1983), 185–192.

[13] J. Orkisz, *Computer approach to the finite difference method*, Mechanika i Komputer 4 (1981), 7–69 (in Polish).

[14] T. Styś, *A discrete maximum principle*, Dissertationes Math. 183 (1981).

[15] J. Szarski, *Differential Inequalities*, PWN, Warszawa 1967.

[16] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, London 1962.

INSTITUTE OF BUILDING MECHANICS
TECHNICAL UNIVERSITY OF KRAKÓW
ul. Warszawska 24, 31-155 Kraków, Poland

Reçu par la Rédaction le 17.05.1988
Révise le 8.03.1989 et 15.02.1990