

Continuous solutions of the functional equation $\varphi(f(x)) = G(x, \varphi(x))$ for vector-valued functions φ

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Introduction. The object of this paper is to study the functional equation

$$(1) \quad \varphi(f(x)) = G(x, \varphi(x)),$$

where f and G denote the known functions and φ the unknown function. The functions φ and G are vector-valued functions with ranges contained in R^k . We define the norm of an element $u \in R^k$ as usual by

$$\|u\| = \sqrt{\sum_{i=1}^k u_i^2}.$$

We shall prove a theorem which is true for $k \geq 2$. This paper is a generalization of the results of papers [1] and [4]. In [4] the function G is a complex-valued function of the special form

$$G(x, z) = -A(x) - B(x)/z$$

and the equation considered there has the form

$$\varphi(f(x))\varphi(x) + A(x)\varphi(x) + B(x) = 0.$$

In [1] Kuczma and Kordylewski consider a solution of the functional equation

$$(2) \quad F(x, \varphi(x), \varphi(f(x))) = 0,$$

where $\varphi(x)$ denotes the required function, while $f(x)$ and $F(x, y, z)$ are known functions. They suppose that the sets Γ_x and $\Omega_{f(x)}$ (the definition is given in § 2) are identical. In our paper we assume that the symmetric difference of these sets is countable set.

The discussion of equation (1) is carried out with the aid of a certain lemma and a theorem which are proved by means of some considerations of topological nature.

§ 1. Let $v(x)$ be a continuous vector-valued function defined in the closed interval $\langle 0, 1 \rangle$. We define the set S as follows:

$$S = \{(x, u): \|u - v(x)\| \leq r, x \in \langle 0, 1 \rangle\}, \quad r > 0,$$

and we put

$$\delta[(x, u), (x', u')] = \max \left(\max_{1 \leq i \leq k} |u_i - u'_i|, |x - x'| \right).$$

Evidently δ is a metric which makes S a metric space.

For every $x_0 \in \langle 0, 1 \rangle$ we denote by U_{x_0} the set

$$U_{x_0} = \{(x, u) \in S, x = x_0\}.$$

We shall consider equation (1) under the assumption that $k \geq 2$.

LEMMA. Let Z be a compact set, $Z \subset S$, and let $(0, a_0) \in U_0$, $(1, a'_1) \in U_1$. Further suppose that for every $x \in \langle 0, 1 \rangle$ the set $Z \cap U_x$ contains at most one element and that $(0, a_0) \notin Z$, $(1, a'_1) \notin Z$.

Then there exists a vector-valued function $\varphi \in C^0(\langle 0, 1 \rangle)$ ⁽¹⁾ such that $\varphi(0) = a_0$, $\varphi(1) = a'_1$ and $(x, \varphi(x)) \in S \setminus Z$ for every $x \in \langle 0, 1 \rangle$.

Proof. We shall consider the set $S \setminus Z$. For every $p \in S \setminus Z$ there exists an $\varepsilon_p > 0$ such that for every $q \in S$ for which $\delta(q, p) < \varepsilon_p$ we have $q \in S \setminus Z$. For $p = (x_p, u) \in U_{x_p}$, $x_p \in (0, 1)$, we denote by V_p the interval $(x_p - \varepsilon_p, x_p + \varepsilon_p)$; in the case where $p \in U_0$ we put $V_p = \langle 0, \varepsilon_p \rangle$, and for $p \in U_1$ we put $V_p = (1 - \varepsilon_p, 1]$.

The class of all $\{V_p\}_{p \in S \setminus Z}$ forms an open cover of the closed interval $\langle 0, 1 \rangle$. So we can choose from $\{V_p\}_{p \in S \setminus Z}$ a finite open cover $\{V_{p_i}\}_{i=0, \dots, n}$ of the interval $\langle 0, 1 \rangle$ such that no more than two of the sets V_{p_i} have common points, and

$$V_{p_0} = V_{(0, a_0)}, \quad V_{p_n} = V_{(1, a'_1)}, \quad V_{p_{i-1}} \cap V_{p_i} \neq \emptyset, \quad V_{p_{i+1}} \cap V_{p_{i-1}} = \emptyset \\ \text{for } i = 1, \dots, n.$$

To the sets V_{p_i} there corresponds a sequence of points

$$p_0 = (0, a_0), \quad p_i = (x'_i, a_i) \quad \text{for } i = 1, \dots, n-1, \quad p_n = (1, a'_1).$$

We can find numbers x_0, \dots, x_{n+1} such that $x_0 = 0$, $x_{n+1} = 1$, $x_i \in V_{p_{i-1}} \cap V_{p_i}$ for $i = 1, 2, \dots, n$. Let

$$C_i = \{(x, a_i): x_i \leq x \leq x_{i+1}\}_{i=0, \dots, n}.$$

Since $U_{x_i} \cap Z$ contains no more than one point for every i , $1 \leq i \leq n$, there exists a set $D_i \subset U_{x_i}$ homeomorphic with the interval $\langle 0, 1 \rangle$ such

⁽¹⁾ $C^0(\langle a, b \rangle)$ denotes the set of continuous functions on the interval $\langle a, b \rangle$.

that $D_i \cap Z = \emptyset$, i.e. there exist continuous vector-valued functions $\mathbf{a}_i(t)$, $i = 1, 2, \dots, n$, such that

$$(x, \mathbf{u}) \in D_i \Leftrightarrow \mathbf{u} = \mathbf{a}_i(t), \quad x = x_i, \quad t \in \langle 0, 1 \rangle,$$

and, moreover,

$$\mathbf{a}_i(0) = \mathbf{a}_{i-1}, \quad \mathbf{a}_i(1) = \mathbf{a}_i, \quad i = 1, 2, \dots, n.$$

The set $D = \bigcup_{i=0}^n C_i \cup \bigcup_{i=1}^n D_i$ is a compact set and $Z \cap D = \emptyset$. Hence it follows that there exists an $\varepsilon > 0$ such that $\delta(Z, D) > \varepsilon$.

Now we shall define the vector-valued function $\varphi(x)$, $x \in \langle 0, 1 \rangle$:

$$\varphi(x) = \begin{cases} \mathbf{a}_i & \text{for } x \in \langle x_i, x_{i+1} - \varepsilon \rangle, \quad i = 0, \dots, n-1, \\ \mathbf{a}'_1 & \text{for } x \in \langle x_n, 1 \rangle, \\ \mathbf{a}_i \left(\frac{x - x_i + \varepsilon}{\varepsilon} \right) & \text{for } x \in \langle x_i - \varepsilon, x_i \rangle, \quad i = 1, \dots, n. \end{cases}$$

It is evident that the function $\varphi(x)$ is continuous, $\varphi(0) = \mathbf{a}_0$, $\varphi(1) = \mathbf{a}'_1$ and, for every x , $\delta[(x, \varphi(x)), D] < \varepsilon$. On the other hand, we know that $\delta(Z, D) > \varepsilon$, so $(x, \varphi(x)) \in S \setminus Z$ for every $x \in \langle 0, 1 \rangle$.

Now we shall prove the following

THEOREM 1. *We assume that:*

1. *Functions $\mathbf{u}_n(x)$, $n = 1, 2, \dots$, are vector-valued functions defined and continuous in closed sets $F_n \subset \langle 0, 1 \rangle$, respectively.*
2. *For every $x \in F_n$ we have $\|\mathbf{u}_n(x)\| \leq r$, $n = 1, 2, \dots$, $r > 0$.*
3. *The points $(0, \mathbf{a}_0)$ and $(1, \mathbf{a}_1)$ fulfil the conditions*

$$\|\mathbf{a}_0\| < r, \quad \|\mathbf{a}_1\| < r$$

and, moreover,

$$\mathbf{u}_n(0) \neq \mathbf{a}_0, \quad \mathbf{u}_n(1) \neq \mathbf{a}_1$$

(if $0 \in F_n$ or $1 \in F_n$).

Then there exists a vector-valued function $\mathbf{u}_0 \in C^0(\langle 0, 1 \rangle)$ fulfilling the conditions:

$$\mathbf{u}_0(0) = \mathbf{a}_0, \quad \mathbf{u}_0(1) = \mathbf{a}_1,$$

$$\|\mathbf{u}_0(x)\| \leq r \quad \text{for every } x \in \langle 0, 1 \rangle,$$

$$\mathbf{u}_0(x) \neq \mathbf{u}_n(x) \quad \text{for every } x \in F_n, \quad n = 1, 2, \dots$$

Proof. Let us put

$$E = \{\mathbf{u}(x): \mathbf{u}(x) \in C^0(\langle 0, 1 \rangle), \mathbf{u}(0) = \mathbf{a}_0, \mathbf{u}(1) = \mathbf{a}_1, \|\mathbf{u}(x)\| \leq r\}.$$

If we introduce in space E the metric

$$\varrho(u, u') = \max_{1 \leq i \leq k} \left(\sup_{0 \leq x \leq 1} |u_i(x) - u'_i(x)| \right),$$

then E becomes a complete metric space. (Convergence in E is equivalent to uniform convergence.)

Now we define sets E_n as follows

$$E_n = \{u \in E: \bigcap_{x \in F_n} u(x) = u_n(x)\}.$$

To prove that sets E_n are closed it is sufficient to show the following implication:

$$(u_r \rightarrow u, u_r \in E_n) \Rightarrow u \in E_n, \quad n = 1, 2, \dots$$

The function u belongs to $C^0(\langle 0, 1 \rangle)$ as a limit of a uniformly convergent sequence of continuous functions, and $u(0) = a_0$, $u(1) = a_1$. It follows from the definition of the sets E_n that if $u_r \in E_n$, then there exists an $x_r \in F_n$ such that $u_r(x_r) = u_n(x_r)$, $n = 1, 2, \dots$

We can choose from the sequence $\{x_r\}$ a subsequence which is convergent. To avoid new symbols we can denote this subsequence by $\{x_r\}$; thus $\lim_{r \rightarrow \infty} x_r = \bar{x} \in F_n$. In turn $u_r(x) \rightarrow u(x)$, and, since the convergence is uniform, $u_r(x_r) \rightarrow u(\bar{x})$. On the other hand, $u_r(x_r) = u_n(x_r) \rightarrow u_n(\bar{x})$, whence $u(\bar{x}) = u_n(\bar{x})$. Thus we have proved that the sets E_n are closed. Moreover, they contain no interior points. In fact, supposing the contrary, let $\bar{u}(x)$ be an interior point of a set E_n . Then there exists an $\varepsilon > 0$ such that

$$\{u, \varrho(u, \bar{u}) < \varepsilon\} \subset E_n.$$

We define the set

$$Z_n = \{(x, u_n(x)): x \in F_n\}, \quad n = 1, 2, \dots$$

It is evident that Z_n is a compact set and $(0, a_0) \notin Z$, $(1, a_1) \notin Z$. But from the lemma it follows that there exists a continuous vector-valued function $\varphi \in E$ such that $\varrho(\varphi, \bar{u}) < \varepsilon$, $(x, \varphi(x)) \neq (x, u_n(x))$ for every $x \in F_n$, i.e. $\varphi \notin E_n$. This contradicts the hypothesis that \bar{u} is an interior point of E_n .

Thus the sets E_n have no interior points and $\bigcup_{n=1}^{\infty} E_n$ is a set of the first category. By Baire's theorem there exists a vector-valued function $u_0 \in E \setminus \bigcup_{n=1}^{\infty} E_n$.

§ 2. In this section we shall construct continuous solutions of the functional equation (1), where f and G are given functions and $\varphi(x)$ denotes the required function. For this purpose we shall prove a certain theorem.

Let $\Omega \subset R^{k+1}$ ($k \geq 2$) be a connected region, and suppose that for every $x \in \langle a, b \rangle$ there exists a $y \in R^k$ such that $(x, y) \in \Omega$. Let

$$G: \Omega \rightarrow R^k.$$

For an arbitrary x we denote by Ω_x the x -section of the set Ω , i.e.

$$\Omega_x = \{y: (x, y) \in \Omega\}$$

and we put

$$\Gamma_x = G(x, \Omega_x).$$

We shall prove the following

THEOREM 2. *We assume the following hypotheses:*

(i) $f \in C^0(\langle a, b \rangle)$, where a and b are two consecutive roots of the equation $f(x) = x$, $f(x)$ is strictly increasing in $\langle a, b \rangle$, and $f(x) > x$ for every $x \in (a, b)$.

(ii) $G(x, y) \in C^0(\Omega)$ and for every fixed $x \in \langle a, b \rangle$ G is invertible with respect to y in Ω_x and the inverse function $H(x, z)$ is continuous in Γ_x .

(iii) $\Gamma_x \div \Omega_{f(x)} = \bigcup_{i=1}^{\infty} \{u^{(i)}(x)\} \cup \bigcup_{j=1}^{\infty} \{v^{(j)}(x)\}$, where

$$u^{(i)}(x) \in \Gamma_x \setminus \Omega_{f(x)}, \quad v^{(j)}(x) \in \Omega_{f(x)} \setminus \Gamma_x,$$

and

$$u^{(i)}(x) \in C^0(\langle a, b \rangle), \quad v^{(j)}(x) \in C^0(\langle a, b \rangle).$$

(iv) There exists $(x_0, \eta) \in \Omega$, $x_0 \in (a, b)$ such that $\eta = G(x_0, \eta)$.

If hypotheses (i)-(iv) are fulfilled, then for every $\varepsilon > 0$ there exists a vector-valued function $\varphi(x)$ with the following properties:

1° $\varphi(x) \in C^0((a, b))$.

2° $\varphi(x)$ satisfies equation (1).

3° $\|\varphi(x) - \eta\| < \varepsilon$ for every $x \in \langle x_0, f(x_0) \rangle$.

Proof. Let us write

$$x_m = f^m(x_0), \quad m = 0, \pm 1, \pm 2, \dots,$$

where $f^m(x)$ denotes the m -th iterate of the function $f(x)$, i.e.

$$f^0(x) = x,$$

$$f^{m+1}(x) = f(f^m(x)), \quad f^{m-1}(x) = f^{-1}(f^m(x)), \quad m = 0, \pm 1, \pm 2, \dots$$

From hypotheses (i) and from the lemmas and the corollary proved in [2] it follows that

$$f(\langle a, b \rangle) = \langle a, b \rangle,$$

the sequence $\{f^m(x_0)\}$ is strictly increasing, and

$$\lim_{m \rightarrow +\infty} f^m(x_0) = b.$$

Similarly, the sequence $\{f^{-m}(x_0)\}$ is strictly decreasing and

$$\lim_{m \rightarrow +\infty} f^{-m}(x_0) = a.$$

Thus we may write

$$(a, b) = \bigcup_{-\infty}^{\infty} \langle x_{i-1}, x_i \rangle,$$

and we have

$$f(\langle x_{i-1}, x_i \rangle) = \langle x_i, x_{i+1} \rangle.$$

Now we define sequences of vector-valued functions $\overset{(i)}{u}_n(x)$ and $\overset{(j)}{v}_n(x)$. We put

$$\begin{aligned} \overset{(i)}{u}_0(x) &= \overset{(i)}{u}(f^{-1}(x)), & \overset{(j)}{v}_0(x) &= \overset{(j)}{v}(f^{-1}(x)), \\ \dots\dots\dots, & \dots\dots\dots, \\ \overset{(i)}{u}_{n+1}(x) &= \overset{(i)}{H}(x, \overset{(i)}{u}_n(f(x))), & \overset{(j)}{v}_{n+1}(x) &= \overset{(j)}{G}(f^{-1}(x), \overset{(j)}{v}_n(f^{-1}(x))), \\ \dots\dots\dots, & \dots\dots\dots, \\ n &= 0, 1, 2, \dots; \quad i = 1, 2, \dots; \quad j = 1, 2, \dots \end{aligned}$$

We consider the interval $\langle x_0, f(x_0) \rangle \subset (a, b)$ and we define the sets $\overset{(i)}{F}_n$ of all points $x \in \langle x_0, f(x_0) \rangle$ for which the functions $\overset{(i)}{u}_n(x)$ are defined and the inequalities $\|\overset{(j)}{u}_n(x) - \boldsymbol{\eta}\| \leq \varepsilon$ hold for $i = 1, 2, \dots, n = 0, 1, \dots$ and similarly the sets $\overset{(j)}{F}'_n$ of all points $x \in \langle x_0, f(x_0) \rangle$ for which the functions $\overset{(j)}{v}_n(x)$ are defined and $\|\overset{(j)}{v}_n(x) - \boldsymbol{\eta}\| \leq \varepsilon$ for $j = 1, 2, \dots; n = 0, 1, \dots$, where ε is a fixed positive number.

The sets $\overset{(i)}{F}_n, \overset{(j)}{F}'_n$ are closed and from hypotheses (i) and (ii) it follows that $\overset{(i)}{u}_n(x) \in C^0(\overset{(i)}{F}_n), \overset{(j)}{v}_n(x) \in C^0(\overset{(j)}{F}'_n)$. There is a countable number of sets $\overset{(i)}{F}_n$ and $\overset{(j)}{F}'_n$.

The vector $\boldsymbol{\eta}$ is a fixed point of the transformation $\bar{\mathbf{y}} = \mathbf{G}(x_0, \mathbf{y})$, since $\mathbf{G}(x_0, \boldsymbol{\eta}) = \boldsymbol{\eta}$.

Thus there exists a neighbourhood $U_{\boldsymbol{\eta}}$ of the point $\boldsymbol{\eta}$, $U_{\boldsymbol{\eta}} \subset \{\mathbf{y}: \|\mathbf{y} - \boldsymbol{\eta}\| < \varepsilon\}$, such that its image by the transformation $\bar{\mathbf{y}} = \mathbf{G}(x_0, \mathbf{y})$ is a neighbourhood $U'_{\boldsymbol{\eta}}$ of the point $\boldsymbol{\eta}$ contained in the set $\{\mathbf{y}: \|\mathbf{y} - \boldsymbol{\eta}\| < \varepsilon\}$.

Since there are only countably many values $\bar{\eta} \in U_{\eta}$ such that one of the equalities

$$\begin{aligned} {}^{(i)}u_n(x_0) &= \bar{\eta}, & {}^{(i)}u_n(f(x_0)) &= G(x_0, \bar{\eta}), \\ {}^{(j)}v_n(x_0) &= \bar{\eta}, & {}^{(j)}v_n(f(x_0)) &= G(x_0, \bar{\eta}), \end{aligned}$$

$$i, j = 1, 2, \dots; n = 0, 1, \dots$$

holds, then there exists a value $\eta^* \in U_{\eta}$ such that $G(x_0, \eta^*) \in G(x_0, U_{\eta})$ and

$$\begin{aligned} {}^{(i)}u_n(x_0) &\neq \eta^*, & {}^{(i)}u_n(f(x_0)) &\neq G(x_0, \eta^*), \\ {}^{(j)}v_n(x_0) &\neq \eta^*, & {}^{(j)}v_n(f(x_0)) &\neq G(x_0, \eta^*), \end{aligned}$$

$$i, j = 1, 2, \dots; n = 0, 1, 2, \dots$$

Thus, on account of Theorem 1 (the sequence $\{u_n(x)\}$ in Theorem 1 is the sequence of all the functions $\{u_n(x)\}$ and $\{v_n(x)\}$, the points $(0, a_0)$ and $(1, a_1)$ are the points (x_0, η^*) and $(f(x_0), G(x_0, \eta^*))$) there exists a vector-valued function $u(x)$ fulfilling the following conditions:

$$(3) \quad u(x) \in C^0(\langle x_0, f(x_0) \rangle),$$

$$(4) \quad u(x_0) = \eta^*, \quad u(f(x_0)) = G(x_0, \eta^*)$$

and

$$(5) \quad \begin{aligned} u(x) &\neq {}^{(i)}u_n(x), & i &= 1, 2, \dots, \\ u(x) &\neq {}^{(j)}v_n(x), & j &= 1, 2, \dots, \text{ for } n = 0, 1, \dots, \end{aligned}$$

$$(6) \quad \|u(x) - \eta\| < \varepsilon \quad \text{for every } x \in \langle x_0, f(x_0) \rangle.$$

Now we are able to define the function $\varphi(x)$, the existence of which has been stated in this theorem. We put

$$(7) \quad \varphi(x) = \begin{cases} u(x) & \text{for } x \in \langle x_0, x_1 \rangle, \\ G(f^{-1}(x), \varphi(f^{-1}(x))) & \text{for } x \in \langle x_n, x_{n+1} \rangle, n = 1, 2, \dots, \\ H(x, \varphi(f(x))) & \text{for } x \in \langle x_{-n}, x_{-n+1} \rangle, n = 1, 2, \dots \end{cases}$$

We have to prove that function (7) possesses the required properties. At first we shall prove that the function $\varphi(x)$ is defined for every $x \in (a, b)$. We have

$$(a, b) = (a, x_0) \cup \langle x_0, x_1 \rangle \cup (x_1, b).$$

We take an arbitrary $\bar{x} \in (x_1, b)$. There exist a positive integer n and an $x \in \langle x_0, x_1 \rangle$ such that $\bar{x} = f^n(x)$. According to (7)

$$\varphi(f^n(x)) = G\{f^{-1}[f^n(x)], \varphi[f^{-1}(f^n(x))]\} = G\{f^{n-1}(x), \varphi[f^{n-1}(x)]\}.$$

In order that the above expression be defined we must have

$$\varphi(f^{n-1}(x)) \in \Omega_{f^{n-1}(x)},$$

and on account of (iii) it is sufficient to have the inequalities:

$$\varphi[f^{n-1}(x)] \neq \mathbf{u}^{(i)}[f^{n-2}(x)], \quad i = 1, 2, \dots,$$

i.e., in view of (7),

$$G\{f^{n-2}(x), \varphi[f^{n-2}(x)]\} \neq \mathbf{u}^{(i)}[f^{n-2}(x)], \quad i = 1, 2, \dots$$

In virtue of (ii), these conditions are equivalent to

$$\varphi[f^{n-2}(x)] \neq \mathbf{H}[f^{n-2}(x), \mathbf{u}^{(i)}(f^{n-2}(x))], \quad i = 1, 2, \dots$$

Making again use of (7) and of assumption (ii) we get

$$G[f^{n-3}(x), \varphi(f^{n-3}(x))] \neq \mathbf{H}[f^{n-2}(x), \mathbf{u}^{(i)}(f^{n-2}(x))] = \mathbf{u}_1^{(i)}[f^{n-2}(x)],$$

whence

$$\varphi[f^{n-3}(x)] \neq \mathbf{H}\{f^{n-3}(x), \mathbf{u}_1^{(i)}[f^{n-2}(x)]\} = \mathbf{u}_2^{(i)}[f^{n-3}(x)], \quad i = 1, 2, \dots$$

Repeating this argument we arrive after n steps at the condition

$$\varphi(x) \neq \mathbf{H}\{x, \mathbf{u}_{n-2}^{(i)}[f(x)]\} = \mathbf{u}_{n-1}^{(i)}(x), \quad i = 1, 2, \dots$$

But this is true on account of (5).

For the interval (a, x_0) the proof is quite analogous. (One must appeal to the condition

$$\varphi(x) \neq \mathbf{v}_{n-1}^{(j)}(x), \quad j = 1, 2, \dots)$$

Thus the function $\varphi(x)$ is defined in the whole interval (a, b) .

The continuity of the function $\varphi(x)$ in the intervals (x_n, x_{n+1}) , $n = 0, \pm 1, \pm 2, \dots$, follows from the continuity of the function $\mathbf{u}(x)$ in the interval $\langle x_0, x_1 \rangle$ and from the assumptions regarding the functions G and H . Thus in order to prove that the function $\varphi(x)$ is continuous in the whole interval (a, b) we need only to prove that

$$(8) \quad \lim_{x \rightarrow x_n} \varphi(x) = \varphi(x_n), \quad n = 0, \pm 1, \pm 2, \dots$$

The proof runs by induction. For $n = 1$ we have by (4) on account of the continuity of the function $u(x)$ in the interval $\langle x_0, x_1 \rangle$:

$$\lim_{x \rightarrow x_1 - 0} \varphi(x) = \lim_{x \rightarrow x_1 - 0} u(x) = u(x_1) = G(x_0, \eta^*).$$

By (7) and (4) we have

$$\begin{aligned} \lim_{x \rightarrow x_1 + 0} \varphi(x) &= \lim_{x \rightarrow x_1 + 0} G\{f^{-1}(x), \varphi[f^{-1}(x)]\} = \lim_{x \rightarrow x_0 + 0} G(x, \varphi(x)) \\ &= \lim_{x \rightarrow x_0 + 0} G(x, u(x)) = G(x_0, u(x_0)) = G(x_0, \eta^*), \end{aligned}$$

and

$$\varphi(x_1) = G\{f^{-1}(x_1), \varphi[f^{-1}(x_1)]\} = G(x_0, \varphi(x_0)) = G(x_0, u(x_0)) = G(x_0, \eta^*),$$

so relation (8) holds.

Now, let us suppose that relation (8) holds for a certain index $N > 1$. We have, since by the induction hypothesis $\varphi(x)$ is continuous at x_N ,

$$\lim_{x \rightarrow x_{N+1}} \varphi(x) = \lim_{x \rightarrow x_N} \varphi(f(x)) = \lim_{x \rightarrow x_N} G[x, \varphi(x)] = G[x_N, \varphi(x_N)] = \varphi(x_{N+1}).$$

Thus, by means of induction we have proved that relation (8) holds for every $n > 0$.

The proof that it holds also for $n \leq 0$ is quite analogous. In fact, for $n = 0$

$$\begin{aligned} \lim_{x \rightarrow x_0 - 0} \varphi(x) &= \lim_{x \rightarrow x_0 - 0} H\{x, \varphi[f(x)]\} = \lim_{x \rightarrow x_1 - 0} H[f^{-1}(x), \varphi(x)] \\ &= \lim_{x \rightarrow x_1 - 0} H[f^{-1}(x), u(x)] = H[x_0, u(x_1)] \\ &= H[x_0, G(x_0, \eta^*)] = \eta^*. \end{aligned}$$

On the other hand,

$$\lim_{x \rightarrow x_0 + 0} \varphi(x) = \lim_{x \rightarrow x_0 + 0} u(x) = u(x_0) = \eta^*$$

and

$$\varphi(x_0) = u(x_0) = \eta^*,$$

so relation (8) holds.

Now, let us suppose that relation (8) holds for a certain index $-N \leq 0$. We have

$$\begin{aligned} \lim_{x \rightarrow x_{-(N+1)}} \varphi(x) &= \lim_{x \rightarrow x_{-N}} \varphi(f^{-1}(x)) = \lim_{x \rightarrow x_{-N}} H[f^{-1}(x), \varphi(x)] \\ &= H\{x_{-(N+1)}, \varphi[f(x_{-(N+1)})]\} = \varphi(x_{-(N+1)}). \end{aligned}$$

Thus by means of induction we have proved that relation (8) holds for every $n = 0, \pm 1, \pm 2, \dots$, so the function $\varphi(x)$ belongs $C^0(a, b)$.

Property 2° follows easily from the form of relations (7) and from the fact that the function $\mathbf{H}(x, z)$ is inverse to the function $\mathbf{G}(x, y)$.

Property 3° follows immediately from (6).

This completes the proof.

References

- [1] J. Kordylewski and M. Kuczma, *On the functional equation $F(x, \varphi(x), \varphi(f(x))) = 0$* , Ann. Polon. Math. 7 (1959/60), pp. 21-32.
- [2] M. Kuczma, *On the functional equation $\varphi(x) + \varphi[f(x)] = F(x)$* , ibidem 6 (1959), pp. 281-287.
- [3] — *General solution of the functional equation $\varphi(f(x)) = G(x, \varphi(x))$* , ibidem 9 (1960/61), pp. 275-284.
- [4] — and P. Vopěnka, *On the functional equation $\lambda(f(x))\lambda(x) + A(x)\lambda(x) + B(x) = 0$* , Ann. Univ. Sci. Budapestensis III-IV (1960/61), pp. 123-133.

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