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On sufficient conditions of optimality of second order

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Dedicated to the memory of Jacek Szarski

Let $f, g_1, \ldots, g_m$ be real valued continuously differentiable functions defined on a domain $\Omega \subset \boldsymbol{R}^n$. We consider a following optimization problem

$$f(x) \to \inf, \quad g_i(x) \leqslant 0, \quad i = 1, \ldots, m. \tag{1}$$

Under hypotheses that at $x_0 \in \Omega$ all constraints g_i are active (i.e. $g_i(x_0) = 0$, $i = 1, \ldots, m$), that the gradients of g_i taken at x_0, ∇g_i, are linearly independent, that the necessary Kuhn–Tucker conditions holds (i.e. there are $\lambda_i \geqslant 0$ such that the gradient of the Lagrangian

$$L(x) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \tag{2}$$

taken at the point x_0 is equal to 0

$$\nabla L(x_0) = 0) \tag{3}$$

it was shown in [5] that x_0 is a local solution of problem (1) if and only if it is a local solution of a following reduced problem

$$\begin{aligned} &f(x) \to \inf, \\ &g_i(x) = 0 \quad \text{if} \quad \lambda_i > 0, \\ &g_i(x) \leqslant 0 \quad \text{if} \quad \lambda_i = 0. \end{aligned} \tag{4}$$

In the present note it will be shown how the classical second order conditions of sufficiency for problem (1) (see [1], [4], a historical discussion of the subject is given in [2]) follows from the above mentioned result of [5].

Let

$$\begin{aligned} T_i &= \{x\colon (\nabla g_i, x) = 0\} \quad \text{if} \quad \lambda_i > 0, \\ T_i &= \{x\colon (\nabla g_i, x) \leqslant 0\} \quad \text{if} \quad \lambda_i = 0. \end{aligned} \tag{5}$$

A following theorem extends the classical one.

THEOREM. *Let $f, g_1, \ldots, g_m$ be k-time continuously differentiable functions. Let at x_0 all constraints g_i be active (i.e. $g_i(x_0) = 0$, $i = 1, \ldots, m$). Suppose that there are $\lambda_1, \ldots, \lambda_m \geqslant 0$ such that the differentials to the order $k-1$ of the Lagrangian* (2) *are equal to* 0 *on the whole space $\boldsymbol{R}^n$,*

$$d^i L(x_0, h) = 0, \quad h \in \boldsymbol{R}^n, \quad i = 1, \ldots, k-1, \tag{6}$$

and that

$$d^k L(x_0, h) > 0 \quad \text{for} \quad h \neq 0, \ h \in T \tag{7}$$

where

$$T = \bigcap_{i=1}^{m} T_i. \tag{8}$$

Then x_0 is a local minimum of problem (1).

Proof. At the begining we shall prove the theorem under an additional hypothesis that the gradients of g_i taken at the point x_0, ∇g_i, are linearly independent. In this case, basing on the result of [5], it is enough to prove that x_0 is a local minimum of problem (4).

Using compactness arguments it is easy to show that there is $c > 0$ such that

$$d^k L(x_0, h) \geqslant c\|h\|^k \quad \text{for} \quad h \in T. \tag{9}$$

Since $d^k L(x_0, h)$ is a homogeneous polynomial of order k, there is an $\varepsilon > 0$ such that

$$d^k L(x_0, h) \geqslant \tfrac{1}{2} c\|h\|^k \quad \text{for} \quad h \in T_\varepsilon, \tag{10}$$

where

$$T_\varepsilon = \{h \in \boldsymbol{R}^n \colon \operatorname{dist}(h, T) \leqslant \varepsilon\|h\|\} \tag{11}$$

(compare [3]).

Continuity of gradients of $f, g_1, \ldots, g_m$ imply that for each $\varepsilon > 0$ there is a neighbourhood of zero Q such that

$$Q \cap \{h \colon g_i(x_0+h) = 0 \text{ for } \lambda_i > 0, \ g_i(x_0+h) \leqslant 0, \text{ for } \lambda_i = 0\} \subset T_\varepsilon. \tag{12}$$

Formulae (10) and (12) imply the theorem under the additional hypothesis that the gradients ∇g_i are linearly independent.

Now we shall show that this hypothesis is superfluous. Let

$$p = \dim\big(\operatorname{span}(\nabla g_i)\big) \tag{13}$$

and let

$$r = \dim\big(\operatorname{span}(\nabla g_i, \lambda_i > 0)\big). \tag{14}$$

We can choose the functions

$$\bar{g}_j(x) = g_{i_j}(x), \quad j = 1, 2, \ldots, p, \tag{15}$$

in such a way that the gradients of $\bar{g}_j$ taken at the point x_0 are linearly independent and $\lambda_{i_j} > 0$, $j = 1, \ldots, r$.

Now we shall consider a following problem

$$(16)\qquad \begin{aligned} &f(x) \to \inf, \\ &\bar{g}_j(x) = 0, \quad j = 1, \ldots, r, \\ &\bar{g}_j(x) \leqslant 0, \quad j = r+1, \ldots, p. \end{aligned}$$

Any local solution of problem (16) is a local solution of problem (4). It is easy to verify that the set

$$(17)\qquad T' = \{x\colon (\nabla\bar{g}_i, x) = 0,\ j = 1, \ldots, r,\ (\nabla\bar{g}_j, x) \leqslant 0,\ j = r+1, \ldots, p\}$$

is precisely equal to the set T. It finishes the proof of the theorem.

For $k = 2$ we obtain as a particular case the classical theorem [1], [4].

The condition that the differentials $d^i L(x_0, h)$ vanish on the whole space $\boldsymbol{R}^n$ cannot be replaced by the weaker condition that they vanish on T.

EXAMPLE. Let

$$(18)\qquad f(x, y, z) = x + 2y + y^2 - 2x^2 - z^2.$$

Let

$$(19)\qquad g_1(x, y, z) = -(x+y) + z^2, \quad g_2(x, y, z) = -y + z^4.$$

Let $x_0 = (0, 0, 0)$. It is easy to verify that the space T is of the form $T = \{(0, 0, z), z \in \boldsymbol{R}\}$. The both Lagrange multipliers are equal to one, $\lambda_1 = \lambda_2 = 1$, and the Lagrangian $L(x)$ is equal to

$$(20)\qquad L(x) = z^4 - 2x^2 + y^2.$$

Hence for $h \in T$

$$(21)\qquad d^i L(x_0, h) = 0, \quad i = 1, 2, 3, 5, 6, \ldots$$

and

$$(22)\qquad d^4 L(x_0, h) = 4!\,\|h\|^4.$$

On the other hand, by the result of [5], $(0, 0, 0)$ is not a local minimum of function (18) with constraints (19).

References

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[4] E. J. McShane, *Sufficient conditions for a weak relative minimum in the problem of Bolza*, Trans. Amer. Math. Soc. 52 (1942), p. 344–379.

[5] S. Rolewicz, *On sufficient conditions of optimality in mathematical programming*, Operation Research Verf. 40 (1981), p. 149–152.

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