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A remark on regularly separated closed sets

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Dedicated to the memory of Jacek Szarski

Abstract. The notion of regularly separated closed sets, due to S. Łojasiewicz, is given a characterization from the viewpoint of the theory of locally convex spaces.

Let E and F be real Hausdorff locally convex spaces. A semi-norm map on (E, F) is a map p whose values p_E and p_F are continuous semi-norms on E and F respectively.

A set Γ of semi-norm maps on (E, F) is called a *calibration for* (E, F) if its *components*

$$\Gamma_E = \{p_E\colon p \in \Gamma\} \quad \text{and} \quad \Gamma_F = \{p_F\colon p \in \Gamma\}$$

induce the locally convex topologies on E and F respectively.

It has been shown in [4], p. 11, that, for every continuous linear map $u\colon E \to F$, there exists a calibration Γ for (E, F) such that the following condition is satisfied: for each $p \in \Gamma$ there exists $\gamma_p > 0$ such that

$$p_F[u(x)] \leqslant \gamma_p p_E(x) \quad \text{for every } x \in E.$$

When the calibration is chosen well, the system of these inequalities will reflect the full nature of the map u.

In the proof of his inverse mapping theorem in locally convex spaces, S. Łojasiewicz, Jr. [2] has used this method when $E = F$, equipped with a calibration with the identical components. We refer to [4] for a detailed account of this method and to [5] and [6] for further developments.

In this note, we shall give a remark on the notion of *regular separation* due to Łojasiewicz [1]. We shall show that closed subsets X and Y of an Euclidean space are regularly separated if and only if there exists a particular calibration for a pair of locally convex spaces which consist of some Whitney functions of class C^∞ defined on $X \cup Y$ and X respectively.

1. Spaces of Whitney functions. For a closed subset X of a finite-dimensional Euclidean space, we denote by $C^\infty(X)$ the set of all Whitney functions of class C^∞ defined on X. Therefore, each element f of $C^\infty(X)$ is a sequence $(f^k\colon |k| = 0, 1, 2, \ldots)$ of real-valued continuous functions on X which satisfy the following condition: for any compact subset K of X and all k, m such that $|k| \leqslant m$,

$$(R_x^m f)^k(y) = o(|x-y|^{m-|k|}),$$

when $x, y \in K$ and $|x-y| \to 0$, where

$$(R_x^m f)^k(y) = f^k(y) - \sum_{|i| \leqslant m-|k|} (i!)^{-1}(y-x)^i f^{k+i}(x).$$

We refer to [3], Chapter I, for these definitions and notations.

The space $C^\infty(X)$ is a vector space with the coordinatewise addition. To define a locally convex topology on $C^\infty(X)$, we take a sequence (K_n) of compact subsets of X such that $K_n \subset K_{n+1}$, $\bigcup K_n = X$ and any compact subset of X is contained in some K_n. Then, for $f = (f^k) \in C^\infty(X)$, we set

$$q_{m,K_n}(f) = \sup\{|f^k(x)|\colon x \in K_n,\ |k| \leqslant m\},$$

$$r_{m,K_n}(f) = \sup\left\{\frac{|(R_x^l f)^k(y)|}{|y-x|^{l-|k|}}\colon x \neq y,\ x, y \in K_n,\ |k| \leqslant l \leqslant m\right\}$$

and

$$p_{m,K_n}(f) = q_{m,K_n}(f) + r_{m,K_n}(f).$$

Then, $\{p_{m,K_n}\colon m \geqslant 0, n \geqslant 1\}$ defines a locally convex topology on $C^\infty(X)$ by which it is a Fréchet space.

Let Z be a closed subset of X. An element f of $C^\infty(X)$ is said to be *flat on* Z if $f^k(x) = 0$ for all $k \geqslant 0$ and $x \in Z$. We denote by $C^\infty(Z, X)$ the set of all elements of $C^\infty(X)$ which are flat on Z. Then, $C^\infty(Z, X)$ is clearly a closed subspace of $C^\infty(X)$ and hence itself is a Fréchet space with respect to $\{p_{m,K_n}\}$ defined above.

2. Regularly separated closed sets. Closed subsets X and Y of a finite-dimensional Euclidean space are called by Łojasiewicz [1] *regularly separated* (or *regularly situated* by Malgrange [3]) if $X \cap Y = \emptyset$ or the following condition is satisfied: for every compact subset K of X and every compact subset L of Y there exist constants $\alpha \geqslant 1$ and $\beta > 0$ such that

$$d(x, L) > \beta d(x, X \cap Y)^\alpha \quad \text{if} \quad x \in K,$$

where d is the Euclidean metric of the Euclidean space.

Since this condition is symmetric with respect to K and L, we may assume that we also have the following inequality:

$$d(y, K) > \beta d(y, X \cap Y)^\alpha \quad \text{if} \quad y \in L$$

for the same constants. Then, it is easy to see that X and Y are regularly separated if and only if the following condition is satisfied: for compact subsets K of X and L of Y there exist constants $a = a(K, L)$, $\beta = \beta(K, L)$ and a compact subset $\sigma(K, L)$ of $X \cap Y$ such that, for every $(x, y) \in K \times L$ there exists $z \in \sigma(K, L)$ which satisfies

$$|x-y| \geqslant \beta \max(|x-z|^{a}, |y-z|^{a}).$$

In other words, regularly separated closed subsets X and Y determine two essential functions: a and σ. We shall emphasize this fact by saying that X and Y are *(a, σ)-regularly separated.*

3. A characterization. We consider two closed subsets X and Y of a finite-dimensional Euclidean space. Let $\{M_n\}$ be a sequence of compact subsets of $X \cup Y$ such that $M_n \subset M_{n+1}$, $\bigcup M_n = X \cup Y$ and every compact subset of $X \cup Y$ is contained in some M_n. We set

$$K_n = M_n \cap X \quad \text{and} \quad L_n = M_n \cap Y.$$

When X and Y are (a, σ)-regularly separated, we find compact subsets $\sigma(K_n, L_n)$ of $X \cap Y$ and, therefore, there exist $\sigma(n)$ such that $\sigma(K_n, L_n)$ and $\sigma(L_n, K_n)$ are contained in $K_{\sigma(n)}$.

We now consider two Fréchet spaces $C^\infty(Y, X \cup Y)$ and $C^\infty(X \cap Y, X)$, equipped with the sets of semi-norms $\{p_{m,M_n}: m \geqslant 0,\ n \geqslant 1\}$ and $\{p_{m,K_n}: m \geqslant 0,\ n \geqslant 1\}$ respectively, using the same notations as in Section 1.

For each $f = (f^k) \in C^\infty(Y, X \cup Y)$, we define the restriction:

$$u(f) = (f^k/X).$$

Then, u is a linear map of $C^\infty(Y, X \cup Y)$ into $C^\infty(X \cap Y, X)$. The following proposition shows the way the regular separation determines a correspondence between (p_{m,M_n}) and (p_{m,K_n}) with respect to this map u.

PROPOSITION. *When X and Y are (a, σ)-regularly separated, there are constants $\gamma_{m,n}$ such that*

$$(*) \qquad p_{m,M_n}(f) \leqslant \gamma_{m,n} p_{a_n m, K_{\sigma(n)}}\big(u(f)\big)$$

for every $f \in C^\infty(Y, X \cup Y)$, $m \geqslant 0$ and $n \geqslant 1$, where $a_n = a(K_n, L_n)$.

Proof. For $f \in C^\infty(Y, X \cup Y)$, since $M_n = K_n \cup L_n$ and

$$f^k(x) = 0 \quad \text{for every } x \in L_n,$$

we always have

$$q_{m,M_n}(f) = q_{m,K_n}\big(u(f)\big) \leqslant q_{a_n m, K_{\sigma(n)}}\big(u(f)\big),$$

because $m \leqslant a_n m$ and $K_n \subset K_{\sigma(n)}$. Hence, we need to show that there exist $\gamma_{m,n} > 0$ such that

$$r_{m,M_n}(f) \leqslant \gamma_{m,n} r_{a_n m, K_{\sigma(n)}}\big(u(f)\big).$$

Let $|k| \leqslant l \leqslant m$ and $x, y \in M_n$. By the definition, it is obvious that $(*)$ holds if $x, y \in K_n$ or $x, y \in L_n$.

Assume that $x \in K_n$ and $y \in L_n$. Then,

$$(R_x^l f)^k(y) = -\sum_{|i| \leqslant l-|k|} (i!)^{-1}(y-x)^i f^{k+i}(x).$$

We take $z \in \sigma(K_n, L_n)$ such that

$$|x-y| \geqslant \beta_n |x-z|^{a_n},$$

where $\beta_n = \beta(K_n, L_n)$. Then, for

$$\lambda = a_n l - (a_n - 1)(|k| + |i|),$$

we have

$$f^{k+i}(x) = (R_z^\lambda f)^{k+i}(x)$$

and

$$\frac{|(R_x^l f)^k(y)|}{|y-x|^{l-|k|}} \leqslant \sum_{|i| \leqslant l-|k|} (i!)^{-1} |y-x|^{|i|-l+|k|} |x-z|^{\lambda-|k|-|i|} \frac{|(R_z^\lambda f)^{k+i}(x)|}{|z-x|^{\lambda-|k|-|i|}}$$

$$\leqslant \Big(\sum_{|i| \leqslant l-|k|} (i!)^{-1} \beta_n^{|k|+|i|-l} \Big) r_{a_n m, K_{\sigma(n)}}\big(u(f)\big).$$

When $x \in L_n$ and $y \in K_n$, we have

$$(R_x^l f)^k(y) = f^k(y).$$

We take $z \in \sigma(K_n, L_n)$ such that

$$|x-y| \geqslant \beta_n |y-z|^{a_n}.$$

Then, for $\mu = a_n l - (a_n - 1)|k|$, we have

$$\frac{|(R_x^l f)^k(y)|}{|y-x|^{l-|k|}} = \frac{|(R_z f)^k(y)|}{|y-z|^{\mu-|k|}} \cdot |y-z|^{\mu-|k|} \cdot |y-x|^{|k|-l}$$

$$\leqslant \beta_n^{|k|-l} r_{a_n m, K_{\sigma(n)}}\big(u(f)\big).$$

Thus, we have inequality $(*)$.

The converse of this proposition is also true, i.e., if inequality $(*)$ with $m = 1$ holds, X and Y are regularly separated. To see this, let K and L be compact subsets of X and Y respectively and choose M_n which contains $K \cup L$. Then, for $f \in C^\infty(Y, X \cup Y)$,

$$p_{1,M_n}(f) \leqslant \beta_{1,n} p_{a_n, K_{\sigma(n)}}\big(u(f)\big).$$

Hence, if $x \in K$ and $y \in L$,

$$|f(x)| = |f(x) - f(y) - (x-u) f^1(y)| \leqslant \beta_{1,n} p_{a_n, K_{\sigma(n)}}\big(u(f)\big) |x-y|,$$

or

$$|f(x)| \leqslant \beta_{1,n} p_{a_n, K_{\sigma(n)}}\big(u(f)\big) d(x, L).$$

Thus, the same argument as [3], p. 15, proves the converse.

4. Remark. Let Γ be a calibration for (E, F) and $u\colon E \to F$ be a continuous linear map which satisfies the following condition: for each $p \in \Gamma$ there exists a constant $\beta_p > 0$ such that

$$(\#) \qquad p_E(x) \leqslant \beta_p p_F\big(u(x)\big) \quad \text{for every } x \in E.$$

Then, it is obvious that u is injective and open. We shall show that the converse is also true.

PROPOSITION. *If $u\colon E \to F$ is a continuous linear map which is injective and open, then there exists a calibration Γ for (E, F) for which condition $(\#)$ is satisfied.*

Proof. Let Γ' be a calibration for (E, F) such that, for each $p' \in \Gamma'$, there exists a constant $\Gamma_{p'} > 0$ such that

$$p'_F\big(u(x)\big) \leqslant \gamma_{p'} p'_E(x) \quad \text{for every } x \in E.$$

We set

$$U_E(p') = \{x \in E\colon p'(x) < 1\}.$$

Then, since u is open, $u\big(U_E(p')\big)$ is an absolutely convex neighborhood of zero in the space $u(E)$. Hence, the Minkowski functional $\mu_{p'}$ of $u\big(U_E(p')\big)$ is a continuous semi-norm on $u(E)$ and

$$u\big(U_E(p')\big) = U_{u(E)}(\mu_{p'}).$$

There exists a continuous extension of $\mu_{p'}$ over F, which we shall denote by the same $\mu_{p'}$. We define a new calibration Γ for (E, F) by

$$\Gamma = \{p = (p'_E, p'_F \cup \mu_{p'})\colon p' \in \Gamma'\}.$$

In other words, for each $p \in \Gamma$, we have

$$p_E = p'_E \quad \text{and} \quad p_F = p'_F \cup \mu_{p'}.$$

It is now easy to see that

$$p_E(x) \leqslant p_F\big(u(x)\big) \quad \text{for every } x \in E.$$

Now, we consider the case when $E = C^\infty(Y, X \cup Y)$ and $F = C^\infty(X \cap \cap Y, X)$, and assume that there is a continuous linear map $u\colon E \to F$ which is injective and open. Then, it determines a calibration Γ for (E, F), for which condition $(\#)$ holds. Since $\Gamma_E = \{p_E\colon p \in \Gamma\}$ induces the topology of E, there exist $p \in \Gamma$ and a constant λ_n such that

$$p_{1, M_n} \leqslant \lambda_n p_E.$$

Since $\{p_{m,K_n}\}$ induces the topology of F, there exist a_n, $\sigma(n)$ and a constant μ_n such that

$$p_F \leqslant \mu_n p_{a_n, K_{\sigma(n)}}.$$

Then, for $\beta_n = \beta_p \lambda_n \mu_n$, we have

$$p_{1,M_n}(f) \leqslant \beta_n p_{a_n,K_{\sigma(n)}}\big(u(f)\big) \quad \text{for every } f \in E,$$

which will then imply that X and Y are regularly separated.

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