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Description of a class of multivalued differential equations with almost weakly stable trivial solution

by MICHAŁ KISIELEWICZ (Zielona Góra)

Abstract. In the present paper a multivalued differential relation of the form

$$\dot{x} \in F(t, x), \tag{1}$$

is considered. It is shown that the class $\mathscr{A}$ of all multivalued mappings F defined on $[0, \infty) \times \boldsymbol{R}^n$ for which the trivial solution $x_F = 0$ of (1) is almost weak stable is an $F_{\sigma\delta\sigma}$ set in any complete metric space.

1. Introduction. It is the purpose of this note to present a description the class of all multivalued differential equations of the form

$$\dot{x} \in F(t, x) \tag{1}$$

with the almost weakly stable trivial solution. It is shown that the class $\mathscr{A}$ of all multivalued mappings F defined on $[0, \infty) \times \boldsymbol{R}^n$ for which the trivial solution $x_F \equiv 0$ of (1) is almost weak stable, is an $F_{\sigma\delta\sigma}$ set in any complete metric space. The weak stability theorem for multivalued differential equations of form (1) has been proved in [5].

2. Notation and basic definitions. Let $\boldsymbol{R}^n$ be the n-dimensional Euclidean space, 0 its zero element, $\|x\|$ the norm of $x \in \boldsymbol{R}^n$. For non-empty sets $A, B \subset \boldsymbol{R}^n$ we use the Hausdorff distance $h(A, B) = \max[r(A, B), r(B, A)]$, where $r(A, B) = \sup\{a(x, B): x \in A\}$ and $a(x, B) = \inf\{\|x-y\|: y \in B\}$. The set of all non-empty compact convex subsets of $\boldsymbol{R}^n$ is denoted by Ω^n. Let $\mathscr{C}_T$ denote the Banach space of continuons mappings of $[0, T]$ into $\boldsymbol{R}^n$ and let $\mathscr{L}_T^1$ be the Banach space of (equivalence classes of) Lebesgue integrable mappings of $[0, T]$ into $\boldsymbol{R}^n$.

We say that $F: [0, \infty) \times \boldsymbol{R}^n \to \Omega^n$ satisfies Carathéodory type conditions if $F(\cdot, x)$ is measurable in $t \geqslant 0$ for every fixed $x \in \boldsymbol{R}^n$, $F(t, \cdot)$ is continuons in $x \in \boldsymbol{R}^n$ for fixed $t \geqslant 0$ and there is an $m_F \in \bigcap_{T>0} \mathscr{L}_T^1$ such that $\|F(t, x)\|_h \leqslant m_F(t)$ for a.e. $t \geqslant 0$ and $x \in \boldsymbol{R}^n$; $\|F(t, x)\|_h = h(F(t, x), \{0\})$.

It is known (see [1], [3]) that for every F satisfying Carathéodory type conditions and for every $T > 0$, $x_0 \in \boldsymbol{R}^n$ and $t_0 \geqslant 0$ there exists an absolutely continuons mapping $x(\cdot; t_0, x_0)$ of $[0, T]$ into $\boldsymbol{R}^n$ such that $x(t_0; t_0, x_0) = x_0$ and $\dot{x}(t; t_0, x_0) \in F\big(t, x(t; t_0, x_0)\big)$ for a.e. $t \in [0, T]$. If the above conditions are satisfied for each $T \geqslant 0$, then the function $x(\cdot; t_0, x_0) \in \bigcap_{T>0} \mathscr{C}_T$ will be called a *trajectory* of (1) corresponding to (t_0, x_0, F). The union of all trajectories of (1) corresponding to (t_0, x_0, F) will be denoted by $S(t_0, x_0, F)$.

Let $\mathscr{N}$ denote the set of all positive integers and let $\mathscr{C} = \bigcap_{T>0} \mathscr{C}_T$. By p_T we shall mean the seminorm of $x \in \mathscr{C}_T$ defined by $p_T(x) = \sup\{\|x(t)\|: 0 \leqslant t \leqslant T\}$. It is known (see Yosida, *Functional analysis*, 1968, p. 23) that the space $\mathscr{C}$ together with the metric d defined by

$$d(x, y) = \sum_{T=1}^{\infty} \frac{1}{2^T} \cdot \frac{p_T(x-y)}{1+p_T(x-y)}$$

for $x, y \in \mathscr{C}$ is a complete metric space.

In a similar way we construct the complete metric space $(\mathscr{F}, \varrho)$. For this, let us denote by $\mathscr{F}$ the quotient space of all mappings $F\colon [0, \infty) \times \times \boldsymbol{R}^n \to \Omega^n$ satisfying Carathéodory type conditions determined by an equivalence relation $\sim$ defined in the following way:

$$F \sim G \quad \text{iff } F(t, x) = G(t, x) \quad \text{for all } x \in \boldsymbol{R}^n \text{ and a.e. } t \geqslant 0.$$

Similarly as in [4] it can be verified that $\varrho\colon \mathscr{F} \times \mathscr{F} \to \boldsymbol{R}^1$ defined by

$$\varrho(F, G) = \sum_{T=1}^{\infty} \frac{\varrho_T(F, G)}{1+\varrho_T(F, G)},$$

where

$$\varrho_T(F, G) = \sup\Big\{\int_0^T h\big(F(t, x), G(t, x)\big)\,dt\colon\ x \in \boldsymbol{R}^n\Big\}$$

for $F, G \in \mathscr{F}$ is a metric in $\mathscr{F}$.

It is not difficult to verify that $(\mathscr{F}, \varrho)$ is a complete metric space.

3. Almost weak stability. We shall consider multivalued differential equations of form (1) under the assumption that the right-hand side $F \in \mathscr{F}$ is such that $0 \in F(t, 0)$ for a.e. $t \in [0, T]$ and all $T > 0$. Denoting by $\mathscr{F}_0$ the set of all such mappings $F \in \mathscr{F}$, we see that $\mathscr{F}_0$ is a closed subset of $\mathscr{F}$. Suppose that $F \in \mathscr{F}_0$. The trivial solution $x_F \equiv 0$ of (1) is called to be *almost weakly stable* if for each $t_0 \geqslant 0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x_0 \in \boldsymbol{B}(0, \delta)$(1) there exists a $x(\cdot, t_0, x_0) \in S(t_0, x_0, F)$ such that

(1) $\boldsymbol{B}(0, \delta)$ denotes the closed ball of $\boldsymbol{R}^n$.

$d(0, x(\cdot; t_0, x_0)) \leqslant \varepsilon$. In the sequel we shall use the following lemma (for the proof see [2], [3]).

LEMMA 1. *Let* $F \in \mathscr{F}$, $t_0 \geqslant 0$, $x_0 \in \boldsymbol{R}^n$. *Then* $x(\cdot, t_0, x_0) \in S(t_0, x_0, F)$ *if and only if*

1° $x(t_0; t_0, x_0) = x_0$,

2° *for every* $T > 0$ *and each* $t_1, t_2 \in [0, T]$, $t_1 < t_2$, *we have*

$$x(t_2; t_0, x_0) - x(t_1; t_0, x_0) \in \int_{t_1}^{t_2} F(s, x(s; t_0, x_0))\,ds,$$

where the integral is mean in Aumann's sense.

Now, let us introduce the following notation:

$$H(t_0, x_0, m) = \{F \in \mathscr{F}:\ \|F(t, x)\|_h \leqslant m(t) \text{ for a.e. } t \geqslant 0, x \in \boldsymbol{R}^n \text{ and } S(t_0, x_0, F) \neq \emptyset\},$$

where $m \in \bigcap_{T=1}^{\infty} \mathscr{L}_T^1$, $\|F(t, x)\|_h = h(F(\cdot), \{0\})$, $Q(t_0, x_0, m) = \mathscr{F}_0 \cap H(t_0, x_0, m)$ and

$$\mathscr{P}(t_0, x_0, m, \varepsilon) = \{F \in Q(t_0, x_0, m):\ d(0, x(\cdot; t_0, x_0)) \leqslant \varepsilon\},$$

where $x(\cdot; t_0, x_0) \in S(t_0, x_0, F)$.

We shall need the following lemma:

LEMMA 2. $H(t_0, x_0, m)$ *is a closed subset of* $\mathscr{F}$ *for each* $t_0 \geqslant 0$, $x_0 \in \boldsymbol{R}^n$ *and* $m \in \bigcap_{T>0} \mathscr{L}_T^1$.

Proof. For fixed $t_0 \geqslant 0$, $x_0 \in \boldsymbol{R}^n$ and $m \in \bigcap_{T>0} \mathscr{L}_T^1$ let $\{F_n\}$ be a sequence in $H(t_0, x_0, m)$ such that $\varrho(F_n, F_0) \to 0$ as $n \to \infty$. By the completeness of $\mathscr{F}$, we have $F_0 \in \mathscr{F}$. It remains to show that $F_0 \in H(t_0, x_0, m)$. It is not difficult to see that $\|F_0(t, x)\|_h \leqslant m(t)$ for a.e. $t \geqslant 0$ and $x \in \boldsymbol{R}^n$. Let $\{x_n(\cdot; t_0, x_0)\}$ be a sequence of trajectories of (1) corresponding to $F = F_n$. Then $x_n(t_0; t_0, x_0) = x_0$ and $\dot{x}_n(t; t_0, x_0) \in F_n(t; x_n(t; t_0, x_0))$ for a.e. $t \in [0, T]$ and each $T > 0$.

By the definition of the metric ϱ, from the convergence $\varrho(F_n, F) \to 0$ as $n \to \infty$ it follows that there exists a subsequence $\{F_k\}$ of $\{F_n\}$ such that $\lim_{k\to\infty} h(F_k(t, x), F_0(t, x)) = 0$ for every $T > 0$ and a.e. $t \in [0, T]$, uniformly with respect to $x \in \boldsymbol{R}^n$. In virtue of Lemma 1, 1°, for every $T > 0$ we obtain

$$x_k(t_2; t_0, x_0) - x_k(t_1; t_0, x_0) \in \int_{t_1}^{t_2} F_k(s, x_k(s; t_0, x_0))\,ds$$

for $t_1, t_2 \in [0, T]$, $t_1 < t_2$ and $k \in \mathscr{N}$. Therefore

$$\|x_k(t_2; t_0, x_0) - x_k(t_1; t_0, x_0)\| \leqslant \int_{t_1}^{t_2} m(t)\,dt$$

Hence it follows that for any $T > 0$ there is a subsequence of $\{x_k(\cdot; t_0, x_0)\}$ uniformly converging on $[0, T]$, call it again $\{x_k(\cdot; t_0, x_0)\}$. Let $x_0(\cdot; t_0, x_0) = \lim_{k\to\infty} x_k(\cdot; t_0, x_0)$. Of course, we have $x_0(t_0; t_0, x_0) = x_0$.

It is not difficult to see that we can further select a subsequence of $\{x_k(\cdot; t_0, x_0)\}$, keeping notation $\{x_k(\cdot; t_0, x_0)\}$, such that $d\big(x_k(\cdot; t_0, x_0), x_0(\cdot; t_0, x)\big) \to 0$ as $k \to \infty$.

For every $T > 0$, $k \in \mathcal{N}$, and a.e. $t \in [0, T]$ we have

$$a\Big([x_0(t_2; t_0, x_0) - x_0(t_1; t_0, x_0)], \int_{t_1}^{t_2} F_0\big(s, x_0(s; t_0, x)\big)\, ds\Big) \leqslant \|x_0(t_2; t_0, x_0) - x_k(t_2; t_0, x_0)\| + \|x_0(t_1; t_0, x_0) - x_k(t_1; t_0, x_0)\| + \int_0^T h\Big(F_k\big(t, x_k(t; t_0, x_0)\big), F_0\big(t, x_k(t; t_0, x_0)\big)\Big) dt + \int_0^T h\Big(F_0\big(t, x_k(t; t_0, x_0)\big), F_0\big(t, x_0(t, t_0 x_0)\big)\Big) dt$$

for $t_1, t_2 \in [0, T]$ and $t_1 < t_2$. Hence it follows that $\dot{x}_0(t; t_0, x_0) \in F_0\big(t, x_0(t; t_0, x_0)\big)$ for a.e. $t \in [0, T]$, $T > 0$; i.e. $S(t_0, x_0, F_0) \neq \emptyset$.

In a similar way we obtain

LEMMA 3°. *$\mathcal{P}(t_0, x_0, m, \varepsilon)$ is a closed subset of $\mathcal{F}$ for each $\varepsilon > 0$, $t_0 \geqslant 0$, $x_0 \in \boldsymbol{R}^n$ and $m \in \bigcap_{T>0} \mathcal{L}_T^1$.*

Let

$$\mathcal{R}(t_0, \varepsilon, \delta, m) = \bigcap_{x_0 \in \boldsymbol{B}(0,\delta)} \mathcal{P}(t_0, x_0, m)$$

and let W^+ denote the set of all positive rational numbers. It is easy to prove that for every $t_0 \geqslant 0$ and $m \in \bigcap_{T>0} \mathcal{L}_T^1$

$$\bigcap_{\varepsilon>0} \bigcup_{\delta>0} \mathcal{R}(t_0, \varepsilon, \delta, m) = \bigcap_{\varepsilon \in W^+} \bigcup_{\delta \in W^+} \mathcal{R}(t_0, \varepsilon, \delta, m).$$

Hence and by Lemma 2° we obtain:

THEOREM 1°. *The set $\mathcal{A}$ of all $F \in \mathcal{F}_0$ for which the trivial solution $x_F \equiv 0$ of (1) is almost weakly stable is an $F_{\sigma\delta\sigma}$ set in $(\mathcal{F}, \varrho)$.*

THEOREM 2°. *The set β of all $F \in \mathcal{F}_0$ for which the trivial solution $x_F \equiv 0$ of (1) is almost weakly unstable is a $G_{\delta\sigma\delta}$ set in $(\mathcal{F}, \varrho)$.*

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