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Integral representations for even positive definite functions

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1. Statement of the results. We denote by $x = (x_1, \ldots, x_n)$ a variable point in the real n-dimensional euclidean space R^n. A function $g(x)$ is said to be *even* if $g(\sigma_1 x_1, \ldots, \sigma_n x_n) = g(x_1, \ldots, x_n)$ for all $\sigma_j = \pm 1$. We denote by C^∞ the set of all infinitely differentiable functions on R^n, and by C_c^∞ the set of all functions in C^∞ having a compact support. A continuous function $f(x)$ on R^n is called an *even positive definite* (e.p.d.) function if $f(x)$ is even and if for every even function φ in C_c^∞,

$$\int_{R^n}\int_{R^n} f(x-y)\varphi(x)\overline{\varphi(y)}\,dx\,dy \geqslant 0 . \tag{1}$$

Let $\mathfrak{M}_j = \{z_j = x_j + iy_j;\ x_j = 0 \text{ or } y_j = 0\}$, $\mathfrak{M} = \mathfrak{M}_1 \times \ldots \times \mathfrak{M}_n$. For $n = 1$ Krein [13] proved that if f is e.p.d. then there exist positive measures $d\sigma_1, d\sigma_2$ such that

$$f(x) = \int_0^\infty \cos(xt)\,d\sigma_1(t) + \int_0^\infty \cosh(xt)\,d\sigma_2(t) \tag{2}$$

where $\int_0^\infty d\sigma_1(t) < \infty$ and $d\sigma_2$ is such that the second integral on the right-hand side of (2) is convergent for all x. The measures $d\sigma_1, d\sigma_2$ are not unique, in general. (2) can also be written in the form

$$f(x) = \int_{\mathfrak{M}} e^{ixt}\,d\sigma(t) \tag{2'}$$

where $d\sigma(t)$ is an even measure. It is not known whether such a representation holds for all e.p.d. f, if $n > 1$. If however f is assumed to be bounded by

$$f(x) = O(e^{a|x|^2}) \quad \text{for some } a > 0 , \tag{3}$$

then the measures $d\sigma_1, d\sigma_2$ are unique ([2], [3], [9], [14], [15], [16], [18]). Furthermore, the existence and uniqueness of the representation (2′) is

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then also known for $n \geqslant 1$ (this was first proved by Vilenkin [17] who extended to $n \geqslant 1$ the proof, for $n = 1$, of [9]).

More recently, Kostuchenko and Mityagin [12] developed Krein's method [13] and obtained integral representations for various classes of positive definite generalized functions as well as even positive definite generalized functions. Their method is different from that of [9], [17] (who also considered representations of some generalized functions).

The main purpose of the present paper is to establish the existence and uniqueness of the representation (2′) (for $n \geqslant 1$) under the following assumption:

(4) for each $1 \leqslant j \leqslant n$, the function $f(0, \ldots, 0, x_j, 0, \ldots, 0)$ has at most one integral representation (of the form (2)).

Thus, in particular, if for some $a > 0$

$$(4') \qquad f(0, \ldots, 0, x_j, 0, \ldots, 0) = O\big(\exp(ax_j^2)\big) \qquad (j = 1, \ldots, n)$$

then (2′) is valid with a unique (positive) measure $d\sigma$.

For the sake of future references we state our main result in the following theorem.

THEOREM 1. *Let $f(x)$ be a continuous even positive definite (e.p.d.) function and assume that* (4) *holds. Then $f(x)$ has a unique integral representation of the form* (2′), *where $d\sigma(t)$ is even and $\int\limits_{\mathfrak{M}} |e^{ixt}|\, d\sigma(t) < \infty$ for all x.*

We shall also give a new proof to the effect that (4′) implies (4), namely we shall prove:

THEOREM 2. *Let $f(x)$ be a continuous e.p.d. function of one real variable and assume that* (3) *holds. Then $f(x)$ has a unique integral representation of the form* (2).

From Theorem 1 we obtain the following corollary.

COROLLARY. *If $f(x)$ is an e.p.d. function satisfying* (4′) *then $f(x)$ satisfies* (3) (*with another a*).

Indeed, substituting $x_k = 0$ for all $k \neq j$ into (2′) and using (4′), we get

$$\int\limits_{\mathfrak{M}_j^*} e^{|x_j t_j|}\, d\sigma(t) = O(e^{ax_j^2}),$$

where $\mathfrak{M}_j^* = \mathfrak{M}_1 \times \ldots \times \mathfrak{M}_{j-1} \times \overline{\mathfrak{M}}_j \times \mathfrak{M}_{j+1} \times \ldots \times \mathfrak{M}_n$, $\overline{\mathfrak{M}}_j = \{z_j = x_j + iy_j;\ x_j = 0\}$. Hence

$$|f(x)| \leqslant \int\limits_{\mathfrak{M}} \prod_{j=1}^{n} |e^{ix_j t_j}|\, d\sigma(t) \leqslant \sum_{j=1}^{n} \int\limits_{\mathfrak{M}} |e^{inx_j t_j}|\, d\sigma(t) = O\Big(\sum_{j=1}^{n} e^{nax_j^2}\Big) = O(e^{na|x|^2}).$$

It is not known whether the existence part of Theorem 1 remains true if the assumption (4) is omitted and $n > 1$. For an analogous problem of moments, Zarhina [19] gave an example where there is no existence. Theorem 2 does not remain true if (3) is replaced by $f(x) = O\big(\exp(a|x|^{2+\varepsilon})\big)$ for some $\varepsilon > 0$; see [9], [12].

Our method of proving Theorem 1 is entirely different from both the method of [9], [17] and the method of [13], [12]. It relies upon a method of Devinatz [4], [5], [6], [7] which he developed in solving some problems of moments and problems of extension of positive definite functions. In particular, some of his arguments in [7] will be closely followed. As will be indicated in § 8, some of the results of [12] concerning representation of positive definite generalized functions can also be proved by the present method.

The proof of Theorem 1 is given in §§ 2-6. Theorem 2 is proved in § 7. In § 8, various extensions are indicated.

2. Reproducing kernels. A complex-valued function $K(x, y)$ defined on $R^n \times R^n$ is called a *positive definite kernel* if for any finite set of points $x^k \in R^n$ and complex numbers ξ_k,

$$\sum K(x^j, x^k)\xi_j\bar{\xi}_k \geqslant 0 .$$

We associate with K a Hilbert space $\mathcal{F}$ which is the completion of the linear set consisting of functions $g(x) = \sum \xi_k K(x, x^k)$ (finite sum), the norm given by $\|g\|^2 = (g, g) = \sum K(x^j, x^k)\xi_j\bar{\xi}_k$. K is called the *reproducing kernel* of $\mathcal{F}$ since $g(y) = \big(g(x), K(x, y)\big)$ holds for all $g \in \mathcal{F}$, $y \in R^n$. K is uniquely determined by the last property and the requirement that, for each y, $K(x, y)$ belongs to $\mathcal{F}$.

The theory of reproducing kernels was extensively studied by Aronszajn [1]; for a summary of those properties that will be used later on, see [7], pp. 112-113.

For simplicity of notation we shall henceforth restrict ourselves to the case $n = 2$; the extension to $n > 2$ will be rather obvious. By approximating an even complex measure $d\mu(x)$ with a compact support by a sequence of even functions in C_c^∞ we find that (1) holds also when $\varphi(x)dx$ is replaced by $d\mu(x)$. Taking measures concentrated at $4h$ points $x^j_{(a)}$ ($j = 1, \ldots, h$; $a = 1, 2, 3, 4$), where

$$x^j_{(1)} = (x^j_1, x^j_2), \quad x^j_{(2)} = (x^j_1, -x^j_2), \quad x^j_{(3)} = (-x^j_1, x^j_2), \quad x^j_{(4)} = (-x^j_1, -x^j_2)$$

and where $\mu(x^j_{(a)}) = \xi_j$, we find that

$$K(x, y) \equiv f(x+y)+f(x-y)+f(x+\hat{y})+f(x-\hat{y}) \tag{5}$$

is a positive definite kernel; here $\hat{y} = (y_1, -y_2)$ if $y = (y_1, y_2)$.

Conversely, if $K(x, y)$ is a positive definite kernel then $f(x)$ can easily be shown to be an e.p.d. function.

Analogously, in the case $n = 1$, $f(x)$ is e.p.d. if and only if $f(x+y)+f(x-y)$ is a positive definite kernel.

Let $\mathcal{F}$ be the Hilbert space associated with the kernel (5), and let $\mathcal{F}'$ be the linear set consisting of the functions $g(x) = \sum \xi_k K(x, y^k)$ (finite sum). Note that the elements of $\mathcal{F}$ are even continuous functions on R^2. For any t, τ, r, ϱ in R^2 $(r_1 r_2 \varrho_1 \varrho_2 \neq 0)$ set $g_t(x) = \sum \xi_k K(x, y^k + t)$,

$$\tilde{g}_{r,\varrho} = \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{r_1} \int_0^{\varrho_2} \int_0^{\varrho_1} g_{t+\tau}\, d\tau_1\, d\tau_2\, dt_1\, dt_2 . \tag{6}$$

The integrand is a continuous function from (t, τ) into $\mathcal{F}$. Since whenever $g_m \to 0$ in the $\mathcal{F}$-norm also $g_m(x) \to 0$ uniformly for x in bounded sets of R^2, we conclude that $\tilde{g}_{r,\varrho}(x)$ is the value of the right-hand side of (6) with $g_{t+\tau}$ replaced by $g_{t+\tau}(x)$.

Let $\tilde{D}$ be the linear set spanned by all the $\tilde{g}_{r,\varrho}$ when g varies in $\mathcal{F}'$ and r, ϱ vary in R^2. Since $\tilde{g}_{r,\varrho} \to g$ (in the $\mathcal{F}$-norm) as $r, \varrho \to 0$, $\tilde{D}$ is dense in $\mathcal{F}$. Let A_1 be the operator

$$A_1 g(x) = \frac{\partial^2 g(x)}{\partial x_1^2} ;$$

its domain $d(A_1)$ consists (by definition) of all $g \in \mathcal{F}$ such that $\partial g(x)/\partial x_1$, $\partial^2 g(x)/\partial x_1^2$ exist and are continuous functions on R^2, and $\partial^2 g/\partial x_1^2$ belongs to $\mathcal{F}$. It is easily seen that A_1 is a closed operator.

3. Preliminary lemmas. We shall prove several lemmas concerning A_1 and its adjoint A_1^*.

LEMMA 1. *$d(A_1)$ is dense in $\mathcal{F}$.*

It follows that A_1^* exists and $A_1^{**} = A_1$.

LEMMA 2. *$A_1^* \subseteq A_1$ and $\tilde{D} \subseteq d(A_1^*)$.*

LEMMA 3. *Let B_1 be the restriction of A_1^* to $\tilde{D}$. Then the closure of B_1 coincides with A_1^*.*

Lemma 1 follows by showing that for every $\tilde{g}_{r,\varrho}$

$$\frac{\partial^2}{\partial x_1^2} \tilde{g}_{r,\varrho} \tag{7}$$

$$= \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{\varrho_2} [g_{(r_1+\varrho_1)u+(t_2+\tau_2)v} - g_{r_1 u+(t_2+\tau_2)v} - g_{\varrho_1 u+(t_2+\tau_2)v} + g_{(t_2+\tau_2)v}]\, d\tau_2\, dt_2$$

where $u = (1, 0)$, $v = (0, 1)$. The calculation is analogous to that in [7], p. 115.

To prove Lemma 2, we introduce

$$(8) \qquad \tilde{f}_{y;r,\varrho} = \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{r_1} \int_0^{\varrho_2} \int_0^{\varrho_1} K(\cdot, y+t+\tau)\, d\tau_1 d\tau_2 dt_1 dt_2$$

where $g(\cdot)$ means the function whose value at each point x is $g(x)$. Now, if $g \in d(A_1^*)$ then

$$(A_1^* g, \tilde{f}_{y;r,\varrho}) = (g, A_1 \tilde{f}_{y;r,\varrho}).$$

Using (7) in evaluating $A_1 \tilde{f}_{y;r,\varrho}$ and then using the reproducing property of K we find, after taking $\varrho_2 \to 0$, $r_2 \to 0$,

$$(A_1^* g, \lim_{r_2 \to 0, \varrho_2 \to 0} \tilde{f}_{y;r,\varrho})$$

$$= \frac{1}{r_1 \varrho_1} \big[g\big(y+(r_1+\varrho_1)u\big) - g(y+r_1 u) - g(y+\varrho_1 u) + g(y)\big].$$

Taking $r_1 \to 0$, $\varrho_1 \to 0$ and observing that the left-hand side has a limit, namely, $A_1^* g(y)$, and that this limit is a continuous function, we conclude that $g(y)$ has continuous two derivatives with respect to y_1 and $\partial^2 g(y)/\partial y_1^2 = A_1^* g(y)$. Hence, $A_1^* \subseteq A_1$.

In order to prove that $\tilde{D} \subseteq d(A_1^*)$, it suffices to show that each $\tilde{f}_{y;r,\varrho}$ belongs to $d(A_1^*)$. Let $g \in d(A_1)$. Then

$$(A_1 g, \tilde{f}_{y;r,\varrho}) = \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{r_1} \int_0^{\varrho_2} \int_0^{\varrho_1} \frac{\partial^2}{\partial x_1^2} g(y+t+\tau)\, d\tau_1 d\tau_2 dt_1 dt_2$$

$$= \lim_{h \to 0, k \to 0} \frac{1}{hk r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{r_1} \int_0^{\varrho_2} \int_0^{\varrho_1} \big[g\big(y+(h+k)u+t+\tau\big) -$$

$$- g(y+hu+t+\tau) - g(y+ku+t+\tau) + g(y+t+\tau)\big]\, d\tau_1 d\tau_2 dt_1 dt_2$$

$$= \lim_{h \to 0, k \to 0} \Big(g, \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{\varrho_2} \frac{1}{hk} \int_0^{r_1} \int_0^{\varrho_1} \big[K\big(\cdot, y+(h+k)u+t+\tau\big) -$$

$$- K(\cdot, y+hu+t+\tau) - K(\cdot, y+ku+t+\tau) + K(\cdot, y+t+\tau)\big]\, d\tau_1 dt_1 d\tau_2 dt_2\Big).$$

For $f_y(x) \equiv f(x-y)$ we have

$$\int_0^{r_1} \int_0^{\varrho_1} (f_{y+(h+k)u+t+\tau} - f_{y+hu+t+\tau} - f_{y+ku+t+\tau} + f_{y+t+\tau})\, d\tau_1 dt_1$$

$$= \int_0^{h} \int_0^{k} (f_{y+t+\tau+(r_1+\varrho_1)u} - f_{y+t+\tau+r_1 u} - f_{y+t+\tau+\varrho_1 u} + f_{y+t+\tau})\, d\tau_1 dt_1.$$

Proceeding similarly with the other three terms of $K(x, y)$ and then taking $h \to 0$, $k \to 0$, we find that

$$(A_1 g, \tilde{f}_{y;r,\varrho}) = \Big(g, \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \int_0^{r_2} \int_0^{\varrho_2} [K(\cdot, y + (r_1 + \varrho_1)u + (t_2 + \tau_2)) -$$
$$-K(\cdot, y + r_1 u + (t_2 + \tau_2)v) - K(\cdot, y + \varrho_1 u + (t_2 + \tau_2)v) +$$
$$+K(\cdot, y + (t_2 + \tau_2)v)] \, d\tau_2 \, dt_2\Big).$$

Since the right-hand side is a continuous functional of g in the $\mathcal{F}$-topology, $\tilde{f}_{y;r,\varrho} \in d(A_1^*)$.

To prove Lemma 3, observe that the method by which it was proved that $A_1^* \subseteq A_1$ also yields $B_1^* \subseteq A_1$. On the other hand, $B_1 \subseteq A_1^*$ implies $B_1^* \supseteq A_1^{**} = A_1$. Hence $B_1^* = A_1$ and, consequently, the closure B_1^{**} of B_1 coincides with A_1^*.

Let $\{r_m\}$, $\{\varrho_m\}$ be two fixed sequences of positive numbers such that $r_m \to 0$, $\varrho_m \to \infty$ as $m \to \infty$. Set

$$g_{r_m,\varrho_m} \equiv \frac{1}{r_m \varrho_m} \int_0^{r_m} \int_0^{\varrho_m} g_{(t_1+\tau_1)u} \, d\tau_1 \, dt_1 \tag{9}$$

and let $\tilde{D}_0$ be the linear set spanned by the g_{r_m,ϱ_m} as g varies in $\mathcal{F}'$.

LEMMA 4. *The closure of the restriction of A_1^* to $\tilde{D}_0$ coincides with A_1^* and*

$$A_1^* g_{r_m,\varrho_m} = \frac{1}{r_m \varrho_m} [g_{(r_m+\varrho_m)u} - g_{r_m u} - g_{\varrho_m u} + g].$$

The proof is obtained by slightly modifying the proof of Lemmas 1, 2, 3, replacing $\tilde{D}$ by $\tilde{D}_0$.

We next introduce A_2, A_2^*, etc., with respect to the variable x_2. Analogues of Lemmas 1-4 hold for A_2. We also have:

LEMMA 5. *$\tilde{D} \subset d(A_1^* A_2^*) \cap d(A_2^* A_1^*)$ and for any $\tilde{g}_{r,\varrho} \in \tilde{D}$,*

$$\begin{aligned} A_1^* A_2^* \tilde{g}_{r,\varrho} &= A_2^* A_1^* \tilde{g}_{r,\varrho} \\ &= \frac{1}{r_1 r_2 \varrho_1 \varrho_2} \{[g_{(r_1+\varrho_1)u+(r_2+\varrho_2)v} - g_{(r_1+\varrho_1)u+r_2 v} - \\ &\quad - g_{(r_1+\varrho_1)u+\varrho_2 v} + g_{(r_1+\varrho_1)u}] - [g_{r_1 u+(r_2+\varrho_2)v} - g_{r_1 u+r_2 v} - g_{r_1 u+\varrho_2 v} + g_{r_1 u}] - \\ &\quad - [g_{\varrho_1 u+(r_2+\varrho_2)v} - g_{\varrho_1 u+r_2 v} - g_{\varrho_1 u+\varrho_2 v} + g_{\varrho_1 u}] + [g_{(r_2+\varrho_2)v} - g_{r_2 v} - g_{\varrho_2 v} + g]\}. \end{aligned} \tag{10}$$

The proof of (10) is obtained by evaluating $A_2^* h$ as in the proof of Lemma 1, where $h = A_1^* \tilde{g}_{r,\varrho}$.

4. Formulas of integral representations. We first establish:

LEMMA 6. *A_1^* has a self-adjoint extension.*

Indeed, since A_1^* is closed and symmetric it remains to show that its deficiency indices are equal, or, equivalently, that the dimensions of the eigenspaces of $(A_1^*)^* = A_1$ corresponding to the eigenvalues $\pm i$ are the same. This however follows by noting that $A_1 g = ig$ implies $A_1\bar{g} = -i\bar{g}$.

Let H_1 be a self-adjoint extension of A_1^* and let $\{E_1(t)\}$ be the (unique) spectral family associated to A_1^* $(E_1(t-0) = E_1(t))$. For any $c > 0$, set $E_{1c} = E_1(c) - E_1(-c)$, $\mathcal{F}_{1c} = E_{1c}\mathcal{F}$. The operator

$$U_1(y_1 u)g(x) = 2\int_{-\infty}^{\infty} \cosh(y_1\sqrt{t})\, dE_1(t) g(x) \tag{11}$$

is a bounded operator on each $\mathcal{F}_{1c}$.

LEMMA 7. *For any $g \in \mathcal{F}_{1c}$*

$$U_1(y_1 u)g(x) = g(x+y_1 u) + g(x-y_1 u)\,. \tag{12}$$

Proof. $g \in \bigcap_{m=1}^{\infty} d(H_1^m)$. Now, $A_1^* \subseteq H_1$ implies $A_1 = A_1^{**} \subseteq H_1^* = H_1$; hence, $g \in \bigcap_{m=1}^{\infty} d(A_1^m)$. Thus, $\partial^{2m} g(x)/\partial x_1^{2m}$ exists and $= H_1^m g(x)$. g is moreover an entire analytic function with respect to x_1, with infinite radius of convergence about each point x; indeed,

$$\left|\frac{\partial^{2m} g(x)}{\partial x_1^{2m}}\right| = |H_1^m g(x)| = \left|\left(H_1^m g, K(\cdot, x)\right)\right| \leqslant \|H_1^m g\|\sqrt{K(0,0)} \leqslant c^m \|g\| \sqrt{K(0,0)}$$

and

$$\begin{aligned}\left|\frac{\partial^{2m+1} g(x)}{\partial x_1^{2m+1}}\right| &\leqslant \text{const}\left\{\max_{|y_1 - x_1|\leqslant 1}\left|\frac{\partial^{2m+2} g(y_1, x_2)}{\partial y_1^{2m+2}}\right| + \max_{|y_1 - x_1|\leqslant 1}\left|\frac{\partial^{2m} g(y_1, x_2)}{\partial y_1^{2m}}\right|\right\}\\ &\leqslant \text{const}\cdot c^m.\end{aligned}$$

We then get, from (11),

$$\begin{aligned}U_1(y_1 u)g(x) &= 2\sum_{m=0}^{\infty}\frac{y_1^{2m}}{(2m)!} H_1^m g(x) = 2\sum_{m=0}^{\infty}\frac{y_1^{2m}}{(2m)!}\cdot\frac{\partial^{2m} g(x)}{\partial x_1^{2m}}\\ &= g(x+y_1 u) + g(x-y_1 u)\,.\end{aligned}$$

Let H_2 be a self-adjoint extension of A_2^* and let $\{E_2(t)\}$ be the spectral family corresponding to H_2. We say that H_1 and H_2 *commute* if

$$E_1(t_1)E_2(t_2) = E_2(t_2)E_1(t_1) \quad \text{for all } t_1, t_2\,. \tag{13}$$

Assume that (13) holds and introduce $E_{2c}=E_2(c)-E_2(-c)$, $E_c=E_{1c}E_{2c}$, $\mathcal{F}_c=E_c\mathcal{F}$. Then, if $g\in\mathcal{F}_c$,

$$(14)\quad U(y)g(x)\equiv U_1(y_1u)\,U_2(y_2v)g(x)$$
$$=g(x+y_1u+y_2v)+g(x+y_1u-y_2v)+g(x-y_1u+y_2v)+$$
$$+g(x-y_1u-y_2v)\,,$$

where $U_2(y_2v)$ is defined analogously to $U_1(y_1u)$. We also have,

$$(15)\quad \big(U(y)g,g\big)=4\int_{-c}^{c}\int_{-c}^{c}\cosh\big(y_1\sqrt{t_1}\big)\cosh\big(y_2\sqrt{t_2}\big)\big(dE_1(t_1)dE_2(t_2)g,g\big)\,.$$

Using (14), (15) for the special case $g(x)=E_cK(x,0)$ and noting that $E_cK(x,y)$ is the reproducing kernel of the Hilbert space $\mathcal{F}_c$, we get

$$(16)\quad E_cK(y_1u+y_2v,0)+E_cK(y_1u-y_2v,0)+E_cK(-y_1u+y_2v,0)+$$
$$+E_cK(-y_1u-y_2v,0)=\int_{-c}^{c}\int_{-c}^{c}\cosh\big(y_1\sqrt{t_1}\big)\cosh\big(y_2\sqrt{t_2}\big)d\sigma_c(t)$$

where

$$d\sigma_c(t)=4\big(dE_1(t_1)dE_2(t_2)E_cK(x,0),E_cK(x,0)\big)$$
$$=4\big(dE_1(t_1)dE_2(t_2)K(x,0),K(x,0)\big)$$
$$=64\big(dE_1(t_1)dE_2(t_2)f(x),f(x)\big)\equiv 16d\sigma(t)\,.$$

As $c\to\infty$, the left-hand side of (16) converges to

$$K(y_1u+y_2v,0)+K(y_1u-y_2v,0)+K(-y_1u+y_2v,0)+$$
$$+K(-y_1u-y_2v,0)=16f(y)\,.$$

Since $d\sigma(t)\geqslant 0$, by taking $y_1=0$, $c\to\infty$ we conclude from (16) that

$$\int_{-\infty}^{0}\int_{0}^{\infty}\cosh\big(y_2\sqrt{t_2}\big)d\sigma(t)<\infty\,.$$

Similarly, taking $y_2=0$, $c\to\infty$, we get

$$\int_{0}^{\infty}\int_{-\infty}^{0}\cosh\big(y_1\sqrt{t_1}\big)d\sigma(t)\,.$$

Finally, taking $c\to\infty$, we find that also

$$\int_{0}^{\infty}\int_{0}^{\infty}\cosh\big(y_1\sqrt{t_1}\big)\cosh\big(y_2\sqrt{t_2}\big)d\sigma(t)<\infty$$

and, furthermore,

$$(17)\qquad f(y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cosh\left(y_1\sqrt{t_1}\right)\cosh\left(y_2\sqrt{t_2}\right)d\sigma(t)$$

where

$$(18)\qquad d\sigma(t)=4\left(dE_1(t_1)\,dE_2(t_2)f,f\right).$$

To see that (17) generalizes (2) to $n=2$, write

$$f(y)=\int_{-\infty}^{0}\int_{-\infty}^{0}+\int_{-\infty}^{0}\int_{0}^{\infty}+\int_{0}^{\infty}\int_{-\infty}^{0}+\int_{0}^{\infty}\int_{0}^{\infty}.$$

Substituting, in each of the integrals, $\tau_1=\sqrt{|t_1|}$, $\tau_2=\sqrt{|t_2|}$, we get

$$(19)\qquad f(y)=\int_{0}^{\infty}\int_{0}^{\infty}\cos(y_1t_1)\cos(y_2t_2)\,d\sigma_1(t)+\int_{0}^{\infty}\int_{0}^{\infty}\cos(y_1t_1)\cosh(y_2t_2)\,d\sigma_2(t)+$$

$$+\int_{0}^{\infty}\int_{0}^{\infty}\cosh(y_1t_1)\cos(y_2t_2)\,d\sigma_3(t)+\int_{0}^{\infty}\int_{0}^{\infty}\cosh(y_1t_1)\cosh(y_2t_2)\,d\sigma_4(t),$$

where $d\sigma_j(t)$ are positive measures satisfying

$$(20)\qquad \begin{gathered}\int_{0}^{\infty}\int_{0}^{\infty}d\sigma_1(t)<\infty,\quad \int_{0}^{\infty}\int_{0}^{\infty}\cosh(y_2t_2)\,d\sigma_2(t)<\infty,\\ \int_{0}^{\infty}\int_{0}^{\infty}\cosh(y_1t_1)\,d\sigma_3(t)<\infty,\quad \int_{0}^{\infty}\int_{0}^{\infty}\cosh(y_1t_1)\cosh(y_2t_2)\,d\sigma_4(t)<\infty\end{gathered}$$

for all y_1, y_2.

Formula (19) can also be written in the form (2′) where $d\sigma(t)$ (now different from the $d\sigma$ given in (18)) is an even measure satisfying $\int_{\mathfrak{M}}|e^{ixt}|\,d\sigma(t)<\infty$ for all $x\in R^2$.

5. Criterion for uniqueness. We shall establish in this section a useful criterion for uniqueness. It will be used both in completing the proof of Theorem 1 in § 6 and in proving Theorem 2. In § 2 we introduced $\mathcal{F}$, A_1 in case $n=2$. Similarly we now introduce these notions for $n=1$. For $n=1$, $K(x,y)=f(x+y)+f(x-y)$.

LEMMA 8. *A continuous e.p.d. function $f(x)$ of one real variable (i.e., $n=1$) has a unique representation of the form* (2) (*with* $\int d\sigma_1(t)<\infty$, $\int\cosh(xt)\,d\sigma_2(t)<\infty$ *for all* x) *if and only if A_1^* is a self-adjoint operator in $\mathcal{F}$.*

Proof. Suppose $f(x)$ has a unique representation and let H_1, H_2 be two self-adjoint extensions of A_1^*. We need to prove $H_1=H_2$, or equivalently, $F_1(t) = F_2(t)$ where $\{F_1(t)\}$, $\{F_2(t)\}$ are the spectral families corresponding to H_1 and H_2 respectively. From the proof of (17), (18) specialized to $n = 1$ we obtain:

$$f(y) = 2\int_0^\infty \cosh\left(y\sqrt{t}\right) d\left(F_1(t)f, f\right) = 2\int_0^\infty \cosh\left(y\sqrt{t}\right) d\left(F_2(t)f, f\right).$$

Hence, by the uniqueness assumption, $d\left(F_1(t)f, f\right) = d\left(F_2(t)f, f\right)$. Next, for any Borel set Δ,

$$\int_\Delta \cosh\left(y\sqrt{t}\right)\cosh\left(z\sqrt{t}\right) d\left(F_1(t)f, f\right) = \int_\Delta \cosh\left(y\sqrt{t}\right)\cosh\left(z\sqrt{t}\right) d\left(F_2(t)f, f\right),$$

i.e.

$$\left(F_1(\Delta)V_1(z)K(x,0), V_1(y)K(x,0)\right) = \left(F_2(\Delta)V_2(z)K(x,0), V_2(y)K(x,0)\right) \tag{21}$$

where $V_j(z) = 2\int_{-\infty}^{\infty} \cosh\left(z\sqrt{t}\right) dF_j(t)$.

If Δ is contained in $(-c, c)$ we may replace the $V_1(z)K(x, 0)$ in (21) by

$$\begin{aligned} F_{1c}V_1(z)K(x,0) &= V_1(z)F_{1c}K(x,0) \\ &= F_{1c}K(x+z,0) + F_{1c}K(x-z,0) = 2F_{1c}K(x,z), \end{aligned}$$

where Lemma 7 specialized to $n = 1$ has been used. Proceeding similarly with $V_1(y)K(x, 0)$, $V_2(z)K(x, 0)$, $V_2(y)K(x, 0)$, (21) takes the form

$$\left(F_1(\Delta)K(x,z), K(x,y)\right) = \left(F_2(\Delta)K(x,z), K(x,y)\right),$$

from which it follows (as the $K(x, y)$ span a linear set dense in $\mathcal{F}$) that $F_1(\Delta) = F_2(\Delta)$, i.e., $F_1(t) = F_2(t)$ for all t.

Conversely, let A_1^* be self-adjoint and suppose that $f(x)$ can be represented in the form

$$f(x) = \int_{-\infty}^{\infty} \cosh\left(x\sqrt{t}\right) d\sigma(t). \tag{22}$$

We wish to prove that $d\sigma(t)$ is uniquely determined. Let Λ_0 be the linear subset of $L^2(d\sigma)$ consisting of all functions $G(t)$ satisfying

$$\int_{-\infty}^{\infty} \cosh\left(x\sqrt{t}\right) G(t)\, d\sigma(t) = 0.$$

Since, by (22),

$$K(x,y) \equiv f(x+y) + f(x-y) = 2\int_{-\infty}^{\infty} \cosh\left(x\sqrt{t}\right)\cosh\left(y\sqrt{t}\right) d\sigma(t), \tag{23}$$

we can apply Theorem 4 of [4] and conclude that there exists a unitary mapping W between Λ_0^+ (the orthogonal complement of Λ_0 in $L^2(d\sigma)$) and $\mathcal{F}$, given by

$$h(x) = \sqrt{2} \int_{-\infty}^{\infty} \cosh\left(x\sqrt{t}\right) H(t)\, d\sigma(t) \qquad (H \in \Lambda_0^+,\ h = WH)\,,$$

$$\|h\|^2 = \int_{-\infty}^{\infty} |H(t)|^2 d\sigma(t)\,.$$

Let Λ_1 be the set of all functions $H(t)$ such that $H(t) \in \Lambda_0^+$, $tH(t) \in \Lambda_0^+$. Define an operator T on $W\Lambda_1$ by

$$Th(x) = \sqrt{2} \int_{-\infty}^{\infty} \cosh\left(x\sqrt{t}\right) tH(t)\, d\sigma(t)\,.$$

Clearly $A_1 h = Th$; hence, $T \subseteq A_1$. T is also easily seen to be closed and symmetric.

If $g = \sum \xi_k K(x, y^k)$ (finite sum) then for each of the functions g_{r_m, ϱ_m}, defined in (9), the corresponding $H(t)$ (which is calculated by using (23)) is bounded by $|t|^{-1} O\left(\exp\left(a\sqrt{|t|}\right)\right)$ for all t; this can be verified by directly evaluating the two integrations in the expression for $H(t)$. It follows that $g_{r_m, \varrho_m} \in d(T)$. By Lemma 4, then, $A_1^* \subseteq T$. Hence $T \supseteq A_1^* = (A_1^*)^* \supseteq T^* \supseteq T$, i.e., $A_1^* = T$.

Now, for any Borel set Δ and $H \in \Lambda_0^+$, set

$$B(\Delta)h(x) = \int_{-\infty}^{\infty} \cosh\left(x\sqrt{t}\right) \chi_\Delta(t) H(t)\, d\sigma(t)$$

where χ_Δ is the characteristic function of Δ. The $B(\Delta)$ form a spectral resolution of the identity and $2\left(B(\Delta)f, f\right) = \sigma(\Delta)$ (since $H(t) \equiv 1$ if $h = \sqrt{2} f$). Also,

$$(A_1^* g, h) = (Tg, h) = \int_{-\infty}^{\infty} d\left(B(t)g, h\right).$$

Thus, $\{B(t)\}$ coincides with the spectral family $\{E(t)\}$ of A_1^* and, consequently, $d\sigma(t) = 2\left(E(t)f, f\right)$, i.e., $d\sigma$ is uniquely determined.

The proof of the second part of Lemma 8 can easily be extended to yield:

LEMMA 9. *If $f(x)$ is a continuous e.p.d. function and $n = 2$, and if A_1^*, A_2^* are self-adjoint operators which commute with each other, then f has a unique integral representation of the form* (19) *such that* (20) *holds.*

6. Completion of the proof. In view of Lemma 9, it remains to prove:

LEMMA 10. *Under the assumptions of Theorem* 1, *the operators* A_1^*, A_2^* *are self-adjoint and commute with each other.*

Proof. To prove that A_1^* is self-adjoint in suffices to show that the deficiency indices are zero, or equivalently, that if $(A_1^*)^*g = ig$, $(A_1^*)^*h = -ih$ then $g = 0$, $h = 0$. Since $A_1^{**} = A_1$ we have $\partial^2 g/x_1^2 = ig$. For any fixed x_2^0 the function $g_0(x_2) \equiv g(x_1, x_2^0)$ is a restriction of $g(x)$ to $x_2 = x_2^0$ and therefore (see [1], p. 351) it belongs to the Hilbert space $\mathcal{F}^*$ associated with the kernel

$$K_0(x_1, y_1) = K(x, y)|_{y_2=x_2=x_2^0} = 2f(x_1+y_1, 0) + 2f(x_1-y_1, 0)\,.$$

By our assumptions on f and by Lemma 8 it follows that A_1^*, in $\mathcal{F}^*$, is self-adjoint. Since, however, $d^2g_0/dx_1^2 = ig_0$, we must have $g_0(x_1) \equiv 0$. We have thus proved $g(x) \equiv 0$. Similarly, $h(x) \equiv 0$.

The proof that A_2^* is self-adjoint is similar. It thus remains to show that A_1^* commutes with A_2^*, i.e., that (13) holds.

Let λ be any imaginary number and let $\Omega = (A_1^* - \lambda I)\tilde{D}$, I being the identity operator. Denote by C_2 the restriction of A_2^* to Ω. If we prove that

$$C_2^* = A_2 \tag{24}$$

then (13) follows easily. Indeed, we then conclude that $C_2^{**} = A_2^* = A_2$, i.e., the closure of C_2 is A_2. Hence, for any imaginary number μ, $\Phi \equiv (A_2^* - \mu I)\Omega$ is dense in $\mathcal{F}$. But in view of (10),

$$(A_2^* - \mu I)(A_1^* - \lambda I)g = (A_1^* - \lambda I)(A_2^* - \mu I)g \equiv h$$

for all $g \in \tilde{D}$. Hence, the resolvents $R_{j\lambda}$ of A_j^* satisfy

$$R_{1\lambda}R_{2\mu}h = R_{2\mu}R_{1\lambda}h$$

for all $h \in \Phi$. It follows that $R_{1\lambda}R_{2\mu} = R_{2\mu}R_{1\lambda}$. (13) now follows by using the formula

$$E(\lambda)h = \frac{1}{2\pi i}\lim_{\tau\to 0}\int_{-\infty}^{\lambda}(R_{\varrho+i\tau} - R_{\varrho-i\tau})h\,d\varrho\,,$$

where $\{E(\lambda)\}$ and R_μ are the spectral family and the resolvent of a self-adjoint operator.

Proof of (24). For every $g \in d(C_2^*)$

$$\big(C_2^* g, (A_1^* - \lambda I)\tilde{f}_{y;r,\varrho}\big) = \big(g, A_2^*(A_1^* - \lambda I)\tilde{f}_{y;r,\varrho}\big)\,. \tag{25}$$

Setting

$$h = g_{(r_2+\varrho_2)v} - g_{r_2 v} - g_{\varrho_2 v} + g - \int_0^{r_2}\int_0^{\varrho_2} C_2^* g_{(t_2+\tau_2)v}\,d\tau_2\,dt_2 \tag{26}$$

and using (7), (10), we find that (25) is equivalent to

$$\frac{h\big(y+(r_1+\varrho_1)u\big)-h(y+r_1u)-h(y+\varrho_1u)+h(y)}{r_1\varrho_1}$$
$$=-\frac{\bar{\lambda}}{r_1\varrho_1}\int\limits_0^{r_1}\int\limits_0^{\varrho_1}h\big(y+(t_1+\tau_1)u\big)\,d\tau_1\,dt_1 .$$

As $r_1\to 0$, $\varrho_1\to 0$ the right-hand side converges to $-\bar{\lambda}h(y)$ which is a continuous function. Hence, $\partial h/\partial x_1$, $\partial^2 h/\partial x_1^2$ exist and

$$\frac{\partial^2 h(x)}{\partial x_1^2}=-\bar{\lambda}h(x)\,. \tag{27}$$

The most general solution of (27) which is an even function is

$$c(x_2)\cosh(\mu x_1)\qquad\text{where}\qquad \mu=\sqrt{-\bar{\lambda}}\,. \tag{28}$$

Since $h\in\mathcal{F}$, it is an even function and thus has the form (28), i.e., $h(x)=h(x_2v)\cosh(\mu x_1)$. Substituting h from (26) into the last relation we obtain (taking $r_2\to 0$, $\varrho_2\to 0$)

$$\frac{\partial^2}{\partial x_2^2}\{g(x)-g(x_2v)\cosh(\mu x_1)\}=C_2^*g(x)-\cosh(\mu x_1)\,C_2^*g(x_2v)\,. \tag{29}$$

Consider next the positive definite kernel

$$A(x,y)=K(x,y)+K_0(x,y)$$

where $K_0(x,y)=\cosh(\mu x_1)\overline{\cosh(\mu y_1)}k(x_2,y_2)$, $k(x_2,y_2)=2f(0,x_2+y_2)+2f(0,x_2-y_2)$. Let $\mathcal{F}_2$ be the Hilbert space associated with $K_0(x,y)$ and let $\mathcal{F}_3$ be the Hilbert space associated with $A(x,y)$. The elements of $\mathcal{F}_3$ have the form

$$w(x)=h(x)+\cosh(\mu x_1)g(x_2v)\qquad (h,g\in\mathcal{F})\,.$$

If $w(x)\equiv 0$ then, for any fixed $x_2=x_2^0$, $h(x_1,x_2^0)$ satisfies: $d^2h/dx_1^2=-\bar{\lambda}h$. Since $h(x_1,x_2^0)$ is the restriction to $x_2=x_2^0$ of $h(x_1,x_2)$ which belongs to $\mathcal{F}$, it belongs to the Hilbert space $\mathcal{F}^*$ associated with the kernel

$$K(x,y)|_{x_2=y_2=x_2^0}=2f(x_1+y_1,0)+2f(x_1-y_1,0)\,.$$

As $f(x_1,0)$ has a unique integral representation, A_1^* is self-adjoint and, consequently, $h(x_1,x_2^0)\equiv 0$. It follows that $h(x)\equiv 0$, $g(x_2v)\equiv 0$. We conclude (using [2], pp. 352-353) that $\mathcal{F}_3=\mathcal{F}\oplus\mathcal{F}_2$ (orthogonal sum).

Denote by A_2', A_2'' the operator A_2 corresponding to $\mathcal{F}_2$ is $\mathcal{F}_3$, respectively. Since $f(0,x_2)$ has a unique representation, A_2' is easily seen to be self-adjoint; hence also $A_2\oplus A_2'$ is self-adjoint. Clearly,

$$A_2\oplus A_2'\subseteq A_2''\,. \tag{30}$$

We next show that $(A_2'')^*$ is self-adjoint. It suffices to show: if $A_2''g = \pm ig$ then $g \equiv 0$. Now for each fixed $x_1 = x_1^0$ g satisfies $\partial^2 g/\partial x_2^2 = \pm ig$ and it belongs to the Hilbert space associated with the kernel

$$A(x, y)|_{x_1=y_1=x_1^0} = \gamma K(x, y)|_{x_1=y_1=x_1^0} \quad \text{where} \quad \gamma = 1 + |\cosh(\mu x_1^0)|^2 .$$

Since $f(0, x_2)$ has a unique representation, $g(x_1^0, x_2) \equiv 0$.

Having proved that $(A_2'')^*$ is self-adjoint it follows from (30) that

$$A_2'' = A_2 \oplus A_2' . \tag{31}$$

Now, if $g \in d(C_2^*)$ then (29) shows that $g(x) - g(x_2 v)\cosh(\mu x_1)$ belongs to $d(A_2'')$. Hence, by (31), $g \in d(A_2)$ and $A_2 g = \partial^2 g/\partial x_2^2 = C_2^* g$. This completes the proof of (24).

7. Proof of Theorem 2. In view of Lemma 8 it suffices to show that A_1^* is self-adjoint, i.e., if H_1, H_2 are two self-adjoint extensions of A_1^* then $H_1 = H_2$. Let $\{F_1(t)\}$, $\{F_2(t)\}$ be the spectral families associated with H_1 and H_2 respectively, and consider the functions

$$\varphi_j(x, t) = \int_{-\infty}^{\infty} e^{i\lambda t} dF_j(\lambda) \tilde{f}_{y;r,\varrho}(x) \qquad (j = 1, 2) .$$

Since $\tilde{f}_{y;r,\varrho} \in \tilde{D} \subset d(A_1^*)$,

$$\int_{-\infty}^{\infty} e^{i\lambda t} \lambda dF_j(\lambda) \tilde{f}_{y;r,\varrho}(x) = A_1^* \varphi_j(x, t) .$$

It follows that

$$\frac{\partial \varphi_j(x, t)}{\partial t} = i \frac{\partial^2 \varphi_j(x, t)}{\partial x^2} \qquad (j = 1, 2) .$$

Also, $\varphi_1(x, 0) = \tilde{f}_{y;r,\varrho}(x) = \varphi_2(x, 0)$. Using (3) we find that $\tilde{f}_{y;r,\varrho}(x) = O(\exp(bx^2))$ for some $b > 0$ and, consequently,

$$\varphi_j(x, t) = O(e^{bx^2}) \qquad (j = 1, 2) .$$

We can therefore employ a uniqueness theorem for the Cauchy problem (see [8], p. 180) and conclude that $\varphi_1(x, t) \equiv \varphi_2(x, t)$. Thus, for any $g \in \mathcal{F}$,

$$\int_{-\infty}^{\infty} e^{it\lambda} d(F_1(\lambda) \tilde{f}_{y;r,\varrho}, g) = \int_{-\infty}^{\infty} e^{it\lambda} d(F_2(\lambda) \tilde{f}_{y;r,\varrho}, g) ;$$

both integrals are absolutely convergent. From the uniqueness of the Fourier transform it follows that

$$d(F_1(\lambda) \tilde{f}_{y;r,\varrho}, g) = d(F_2(\lambda) \tilde{f}_{y;r,\varrho}, g) .$$

Since the linear space spanned by the $\tilde{f}_{y;r,\varrho}$ is dense in $\mathcal{F}$, $dF_1(\lambda) = dF_2(\lambda)$, i.e., $H_1 = H_2$.

8. Further results. 8.1. Our proof of Theorems 1 and 2 can be modified to yield a proof of Bochner's theorem (which states that every continuous positive definite function is the Fourier transform of a unique positive measure). We now take $A_1 = -i(\partial/\partial x_1)$. In proving the analogue of Theorem 2 we use a uniqueness theorem of the Cauchy problem for hyperbolic equations (see [8], p. 181).

8.2. In [12] Kostuchenko and Mityagin obtained integral representations of positive definite generalized functions f (f is *positive definite* (p.d.) if for every test function φ $(f, \varphi * \varphi^*) \geqslant 0$ where $\varphi^*(x) = \overline{\varphi(-x)}$). Even though the method of the present paper is valid only for continuous functions, one can proceed without difficulty to obtain representations for various generalized functions (for instance, for distributions), by the following scheme:

If f is a p.d. generalized function and ψ a test function then $T = f * \psi * \psi^*$ is a continuous p.d. function. But then our method yields

$$T(x) = \int e^{i\lambda x} d\sigma_\psi(\lambda).$$

We next observe that

$$d\sigma(\lambda) = \frac{d\sigma_\psi(\lambda)}{|\widetilde{\psi}(\lambda)|^2}$$

is independent of ψ, where $\widetilde{\psi}$ is the Fourier transform of ψ. Hence $T(0) = \int d\sigma_\psi(\lambda)$ takes the form

$$(f, \psi * \psi^*) = \int |\widetilde{\psi}(\lambda)|^2 d\sigma(\lambda). \tag{32}$$

It follows that if φ is another test function then

$$(f, \varphi * \psi^*) = \int \widetilde{\varphi}(\lambda)\overline{\widetilde{\psi}(\lambda)} d\sigma(\lambda). \tag{33}$$

From (32) and the fact that f is bounded on some neighbourhood of the origin of the test space one can obtain a bound on $d\sigma(\lambda)$. Next one takes a sequence $\psi = \psi_m \to \delta$ (δ = Dirac's measure) in (33) and gets, by justifying the passage to limit inside the integral, $(f, \varphi) = \int \widetilde{\varphi}(\lambda) d\sigma(\lambda)$, i.e., f is the Fourier transform of σ. This scheme was accomplished in several instances in [12]; see also [11]. The basic formulas (32), (33) were derived in [12] by using a spectral theorem of Gelfand and Kostuchenko [10], [11].

Uniqueness of the representation for generalized functions can also be inferred from uniqueness for continuous functions.

8.3. Remark 8.2 holds also for even positive definite generalized functions.

8.4. If we define $\varphi^*(x) = \overline{\varphi(x)}$ (x is n-dimensional) then the relation $(f, \varphi * \varphi^*) \geqslant 0$ is equivalent (for continuous f) to

$$\sum f(x^j + x^k)\xi_j \bar{\xi}_k \geqslant 0, \tag{34}$$

i.e., $K(x, y) = f(x+y)$ is a positive definite kernel. We wish to obtain a representation of the form

$$f(x) = \int_{R^n} e^{xt} d\sigma(t). \tag{35}$$

If such a representation is valid, then $f(x)$ can be extended into the n-dimensional complex space as an entire function. Assuming that f is such a function, we define

$$U_1(y_1 u) = \int e^{y_1 t} dE_1(t)$$

where $\{E_1(t)\}$ is the spectral family corresponding to A_1^*, and $A_1 g$ is defined by $-i(\partial g/\partial x_1)$. We find that

$$U_1(y_1 u) g(x) = g(x + iy_1 u).$$

Proceeding similarly to the proofs of Theorems 1 and 2 and noting that for $n = 1$ A_1^* is self-adjoint (without any assumption on f) we conclude that there exists a unique representation of the form (35).

8.5. If $f(x)$ is assumed to satisfy $f(-x) = f(x)$ and if (1) holds for all $\varphi \in C_c^\infty$ satisfying $\varphi(-x) = \varphi(x)$ then $f(x+y)+f(x-y)$ is a positive definite kernel. The proof of Theorem 1 can be modified and we then find that, for $n = 2$, $f(x_1, y_1)+f(x_1, -y_1)$ has a representation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cosh\left(x_1\sqrt{t_1}\right)\cosh\left(y_1\sqrt{t_2}\right) d\varrho_1(t).$$

There seems to be some difficulty in deriving a representation for $f(x_1, y_1) - f(x_1, -y_1)$ which should be of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sinh\left(x_1\sqrt{t_1}\right)\sinh\left(y_1\sqrt{t_2}\right) d\varrho_2(t).$$

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